WEIGHT FUNCTIONS ON NON-ARCHIMEDEAN SPACES AND THE KONTSEVICH-SOIBELMAN SKELETON

LECTURE NOTES FOR THE 2013 SIMONS SYMPOSIUM ON NON-ARCHIMEDEAN AND TROPICAL GEOMETRY

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Abstract. These are the notes for a lecture given at the Simons Symposium on non-Archimedean and tropical geometry (Virgin Islands, March 31 – April 6, 2013). The material for the lecture was based on my joint paper with Mircea Mustaţă with the same title (arXiv:1212.6328).

1. Introduction

1.1. The work of Kontsevich and Soibelman.

(1.1.1) In [KS06], Kontsevich and Soibelman proposed a new interpretation of homological mirror symmetry based on non-archimedean geometry over the field of complex Laurent series $\mathbb{C}((t))$. Their main idea was to encode the geometry of a one-parameter degeneration of complex Calabi-Yau varieties into a topological manifold endowed with a $\mathbb{Z}$-affine structure with singularities, and to interpret mirror symmetry as a certain combinatorial duality between such manifolds. They worked out in detail the case of degenerations of $K3$-surfaces. These ideas were further developed by Gross and Siebert in their theory of *toric degenerations*. Gross and Siebert replaced the use of non-Archimedean geometry by methods from tropical and logarithmic geometry in order to extend the results of Kontsevich and Soibelman to higher-dimensional degenerations.

(1.1.2) An essential ingredient of the construction of Kontsevich and Soibelman is the following. We denote by Δ a small disc around the origin of the complex plane and we set $\Delta^* = \Delta \setminus \{0\}$. We denote by t a local coordinate on Δ centered at 0. Let X be a smooth projective family of varieties over Δ^* and let ω be a relative differential form of maximal degree on the family $X \to \Delta^*$. Kontsevich and Soibelman associated to these data a skeleton $\mathrm{Sk}(X, \omega)$, which is a topological subspace of the Berkovich analytfication of the $\mathbb{C}((t))$-variety obtained from X by base change. If X is a family of Calabi-Yau varieties, then we set $\mathrm{Sk}(X, \omega) = \mathrm{Sk}(X)$ where ω is any relative volume form on X. This definition does not depend on the choice of ω.

(1.1.3) Kontsevich and Soibelman proved that $\mathrm{Sk}(X, \omega)$ can explicitly be computed on any strict normal crossings model $\mathscr{X}$ for X over Δ: it is a union of faces of the Berkovich skeleton $\mathrm{Sk}(\mathscr{X})$ of the model $\mathscr{X}$. Their proof relied on the Weak Factorization Theorem. It is interesting to note that, even though the Berkovich skeleton $\mathrm{Sk}(\mathscr{X})$ heavily depends on the chosen model $\mathscr{X}$, the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X,\omega)$ only depends on X and ω. Thus it singles out certain faces of $\mathrm{Sk}(\mathscr{X})$ that must appear in the skeleton of *every* strict normal crossings model.

(1.1.4) We extended this construction to varieties over complete discretely valued fields K of arbitrary characteristic. Our approach does not use the Weak Factorization Theorem but only relies on basic computations on valuations and canonical sheaves. Moreover, we proved that the skeleton of a Calabi-Yau variety over $\mathbb{C}((t))$ is always connected. An interesting gadget that appears in our work is the *weight function*

$$\mathrm{wt}_\omega : X^{\mathrm{an}} \to \mathbb{R} \cup \{+\infty\}$$

associated to a smooth and proper K-variety X and a differential form of maximal degree ω on X. This weight function is piecewise affine on the Berkovich skeleton of any strict normal crossings model of X and strictly increasing as one moves away from the Berkovich skeleton. The Kontsevich-Soibelman skeleton is precisely the set of points where wt_ω reaches its minimal value; see Section 3.

1.2. **Log discrepancies in birational geometry.**

(1.2.1) Our approach is inspired by interesting analogies with some fundamental invariants in birational geometry. Let X be a smooth complex variety and $\mathcal{I}$ a coherent ideal sheaf on X. Let v be a divisorial valuation on X, that is, a positive real multiple of the discrete valuation on the function field $\mathbb{C}(X)$ associated to a prime divisor E on a smooth birational modification Y of X. We denote by N the multiplicity of the scheme $Z(\mathcal{I}\mathcal{O}_Y)$ along E and by $\nu - 1$ the multiplicity of E in the relative canonical divisor $K_{Y/X}$. We set

$$\mathrm{wt}_{\mathcal{I}}(v) = \frac{\nu}{N}$$

and we call this positive rational number the weight of $\mathcal{I}$ at v. Then the infimum of the set of weights of $\mathcal{I}$ at all divisorial valuations v on X is called the log-canonical threshold of the pair $(X,\mathcal{I})$ and denoted by $\mathrm{lct}(X,\mathcal{I})$. This is a measure for the singularities of the zero locus $Z(\mathcal{I})$ of $\mathcal{I}$ on X, and one of the most important invariants in birational geometry. We refer to [Ko95] for more background.

(1.2.2) It is a fundamental fact that the log-canonical threshold of $(X,\mathcal{I})$ can be computed on a single log-resolution of $(X,\mathcal{I})$, i.e., a proper birational morphism $h : Y \to X$ such that Y is smooth, h is an isomorphism over the complement of $Z(\mathcal{I})$, and $K_{Y/X}$ and $Z(\mathcal{I}\mathcal{O}_Y)$ are strict normal crossing divisors on Y. Namely, we have

$$\mathrm{lct}(X,\mathcal{I}) = \min\{\mathrm{wt}_{\mathcal{I}}(v)\}$$

where v runs over the divisorial valuations associated to the prime components of $Z(\mathcal{I}\mathcal{O}_Y)$. If this minimum is reached on a prime component E of $Z(\mathcal{I}\mathcal{O}_Y)$, then we say that E computes the log-canonical threshold of $(X,\mathcal{I})$. If we denote by $\mathcal{E}$ the union of such prime components E, then the Connectedness Theorem of Shokurov and Kollár states that for every point x of $Z(\mathcal{I})$ and every sufficiently small open neighbourhood U of x in $Z(\mathcal{I})$, the topological space $h^{-1}(U)\cap\mathcal{E}$ is connected. This was the main source of inspiration for our theorem on the connectedness of the skeleton of a Calabi-Yau variety over $\mathbb{C}((t))$ (Theorem 3.3.2).

(1.2.3) In [BFJ08] and [JM11], a closely related variant of the weight function $\mathrm{wt}_{\mathcal{I}}$ was extended from the set of divisorial valuations on X to the non-archimedean link of $Z(\mathcal{I})$ in X, that is, the analytic space over the field $\mathbb{C}$ with the trivial absolute value that we obtain by removing the generic fiber of the $\mathbb{C}$-variety $Z(\mathcal{I})$ from the generic fiber of the formal completion of X along $Z(\mathcal{I})$. We have made a similar construction to define weight functions on analytic spaces over discretely valued fields; see Section 3.

2. The Kontsevich-Soibelman skeleton

2.1. Birational and divisorial points.

(2.1.1) We fix some notation for the remainder of these notes. Let R be a complete discrete valuation ring with maximal ideal $\mathfrak{m}$, residue field k and quotient field K. We denote by v_K the discrete valuation $K^* \twoheadrightarrow \mathbb{Z}$. We define an absolute value on K by setting $|x| = \exp(-v_K(x))$ for each element x of K^*. We denote by X a connected, smooth and proper K-variety of dimension n and by ω a non-zero differential form of degree n on X.

(2.1.2) We denote by X^{an} the Berkovich analytification of X and by $i : X^{\mathrm{an}} \to X$ the analytification morphism. The set $\mathrm{Bir}(X)$ of birational points of X^{an} is defined as the inverse image under i of the generic point of X. It can be identified in a natural way with the set of real valuations on the function field $K(X)$ that extend the discrete valuation v_K on K: the valuation associated to $x \in \mathrm{Bir}(X)$ is given by

$$K(X)^* \to \mathbb{R} : h \mapsto -\ln|h(x)|.$$

We endow $\mathrm{Bir}(X)$ with the topology induced by the Berkovich topology on X^{an}.

(2.1.3) An R-model of X is a flat separated R-scheme of finite type $\mathscr{X}$ endowed with an isomorphism of K-schemes $\mathscr{X}_K \to X$. Note that we do not impose any properness condition on the model. Let $\mathscr{X}$ be a regular R-model of X and denote by $\widehat{\mathscr{X}}$ its formal $\mathfrak{m}$-adic completion. Then the generic fiber $\widehat{\mathscr{X}_\eta}$ of $\widehat{\mathscr{X}}$ is an analytic domain in X^{an} and we can consider the reduction map

$$\mathrm{red}_{\mathscr{X}} : \widehat{\mathscr{X}_\eta} \to \mathscr{X}_k.$$

If E is an irreducible component of the special fiber $\mathscr{X}_k$ with generic point ξ, then the fiber $\mathrm{red}_{\mathscr{X}}^{-1}(\xi)$ consists of a unique point, which we call the divisorial point associated to $(\mathscr{X}, E)$. It is the birational point on X that corresponds to the discrete valuation on $K(X)$ with valuation ring $\mathcal{O}_{\mathscr{X},\xi}$, normalized in such a way that it extends the discrete valuation v_K on K. We will denote the set of divisorial points (associated to all couples $(\mathscr{X}, E)$ as above) by $\mathrm{Div}(X) \subset \mathrm{Bir}(X)$.

2.2. Definition of the skeleton.

(2.2.1) Let $\mathscr{X}$ be a regular R-model of X, E an irreducible component of $\mathscr{X}_k$ and x the divisorial point on X^{an} associated to $(\mathscr{X}, E)$. The relative canonical sheaf $\omega_{\mathscr{X}/R}$ is a line bundle on $\mathscr{X}$ that extends to canonical line bundle $\omega_{X/K}$ on X. The differential form ω on X defines a rational section of $\omega_{\mathscr{X}/R}$ and thus a divisor $\mathrm{div}_{\mathscr{X}}(\omega)$ on $\mathscr{X}$. We denote by N the multiplicity of E in $\mathscr{X}_k$ and by $\nu - 1$ the multiplicity of E in $\mathrm{div}_{\mathscr{X}}(\omega)$. We define the weight of ω at x by the formula

$$\mathrm{wt}_{\omega}(x) = \nu/N$$

and the weight of X with respect to ω by

$$\mathrm{wt}_{\omega}(X) = \inf\{\mathrm{wt}_{\omega}(x) \mid x \in \mathrm{Div}(X)\}.$$

(2.2.2) A divisorial point x on X^{an} is called ω-essential if the weight function wt_{ω} reaches its minimal value at x, that is,

$$\mathrm{wt}_{\omega}(X) = \mathrm{wt}_{\omega}(x).$$

The skeleton $\mathrm{Sk}(X, \omega)$ of the pair (X, ω) is defined as the closure in $\mathrm{Bir}(X)$ of the set of ω-essential divisorial points. If the geometric genus of X is equal to one, then we set $\mathrm{Sk}(X) = \mathrm{Sk}(X, \omega)$ where ω is any non-zero differential form of degree n on X. This definition is independent of the choice of ω, since multiplying ω by an element λ of K^* shifts the weight function by $v_K(\lambda)$.

(2.2.3) It is obvious from the definition that $\mathrm{Sk}(X, \omega)$ is a birational invariant of the pair (X, ω), since the spaces $\mathrm{Bir}(X)$ and $\mathrm{Div}(X)$ and the weight function wt_{ω} are birational invariants.

3. The weight function and the Connectedness Theorem

3.1. **Definition and properties of the weight function.**

(3.1.1) Without some suitable assumptions on the existence of resolutions of singularities, we cannot say much more about the skeleton $\mathrm{Sk}(X, \omega)$; for instance, we cannot prove that $\mathrm{Sk}(X, \omega)$ is non-empty. Therefore, we will assume from now on that k has characteristic zero or X is a curve. With the current state of affairs, these are the cases where resolution of singularities is known in the form that we need. In particular, it is known that every proper R-model of X can be dominated by an *sncd*-model of X, that is, a regular proper R-model $\mathscr{X}$ whose special fiber $\mathscr{X}_k$ is a divisor with strict normal crossings.

(3.1.2) To any *sncd*-model $\mathscr{X}$ of X, one can associate its Berkovich skeleton $\mathrm{Sk}(\mathscr{X})$ in X^{an}. It is the set of all birational points on X such that the associated valuation is monomial with respect to the strict normal crossings divisor $\mathscr{X}_k$ on $\mathscr{X}$. This skeleton is canonically homeomorphic to the dual intersection complex of the divisor $\mathscr{X}_k$. Thus the vertices of $\mathrm{Sk}(\mathscr{X})$ correspond to the irreducible components of $\mathscr{X}_k$ (they are precisely the divisorial points associated to these components), and the faces of $\mathrm{Sk}(\mathscr{X})$ correspond to the connected components of intersections of sets of irreducible components of $\mathscr{X}_k$. The embedding of $\mathrm{Sk}(\mathscr{X})$ in X^{an} has a canonical continuous retraction

$$\rho_{\mathscr{X}} : X^{\mathrm{an}} \to \mathrm{Sk}(\mathscr{X}).$$

If f is a non-zero rational function on X, then the function

$$\mathrm{Sk}(\mathscr{X}) \to \mathbb{R} : x \mapsto -\ln|f(x)|$$

is continuous and piecewise affine.

Theorem 3.1.3. *There exists a unique smallest function*

$$\mathrm{wt}_\omega : X^{\mathrm{an}} \to \mathbb{R} \cup \{+\infty\}$$

with the following properties.

(1) *The function* wt_ω *is lower semi-continuous.*

(2) *Let* $\mathscr{X}$ *be an sncd-model for* X *and* x *a point of the Berkovich skeleton* $\mathrm{Sk}(\mathscr{X})$*. Let* f *be a rational function on* $\mathscr{X}$ *such that, locally at* $\mathrm{red}_{\mathscr{X}}(x)$*, we have*

$$\mathrm{div}(f) = \mathrm{div}_{\mathscr{X}}(\omega) + (\mathscr{X}_k)_{\mathrm{red}}.$$

Then

$$\mathrm{wt}_\omega(x) = -\ln|f(x)|.$$

In particular, wt_ω *is continuous and piecewise affine on* $\mathrm{Sk}(\mathscr{X})$*, and we get the same value as before on divisorial points. Moreover, for all* x *in* X^{an}*, we have*

$$\mathrm{wt}_\omega(x) \geq \mathrm{wt}_\omega(\rho_{\mathscr{X}}(x))$$

with equality if and only if $x \in \mathrm{Sk}(\mathscr{X})$*.*

(3) *The restriction of* wt_ω *to* $\mathrm{Bir}(X)$ *is a birational invariant of* (X, ω)*.*

3.2. **Computation of the skeleton.**

(3.2.1) We can use the properties of the weight function in Theorem 3.1.3 to compute the Kontsevich-Soibelman skeleton $\mathrm{Sk}(X, \omega)$ on an *sncd*-model $\mathscr{X}$ of X. The theorem immediately implies that $\mathrm{Sk}(X, \omega)$ is the subspace of the compact space $\mathrm{Sk}(\mathscr{X})$ consisting of the points where the continuous function $\mathrm{wt}_\omega|_{\mathrm{Sk}(\mathscr{X})}$ reaches its minimal value. In particular, $\mathrm{Sk}(X, \omega)$ is a non-empty compact topological space. We can make this description much more explicit, as follows.

(3.2.2) We write $\mathscr{X}_k = \sum_{i \in I} N_i E_i$. For each $i \in I$, we denote by $\nu_i - 1$ the multiplicity of E_i in the divisor $\mathrm{div}_{\mathscr{X}}(\omega)$. Recall that each face of $\mathrm{Sk}(\mathscr{X})$ corresponds to a connected component C of an intersection $\cap_{j \in J} E_j$ where J is a non-empty subset of I. We say that the face is ω-essential if

$$\frac{\mu_j}{N_j} = \min\{\frac{\mu_i}{N_i} \,|\, i \in I\}$$

and C is not contained in the closure in $\mathscr{X}$ of the canonical divisor $\mathrm{div}_X(\omega)$ (the divisor of zeroes of ω on the K-variety X).

Theorem 3.2.3. *The weight of* X *with respect to* ω *is given by*

$$\mathrm{wt}_\omega(X) = \min\{\frac{\mu_i}{N_i} \,|\, i \in I\}$$

and the skeleton $\mathrm{Sk}(X, \omega)$ *is the union of the* ω*-essential faces of* $\mathrm{Sk}(\mathscr{X})$*.*

(3.2.4) Here's an interesting example. Suppose that $R = \mathbb{C}[[t]]$ and that X is a $K3$-surface over K. Assume that X has an *sncd*-model $\mathscr{X}$ such that $\mathscr{X}_k$ is reduced and $\omega_{\mathscr{X}/R}$ is trivial. Such models played an important role in the classification of semi-stable degenerations of $K3$-surfaces by Kulikov and Persson-Pinkham. They have the special property that $\mathrm{Sk}(X) = \mathrm{Sk}(\mathscr{X})$, since all multiplicities N_i are equal to one and all ν_i are equal, for any choice of volume form ω on X, by the triviality of $\omega_{\mathscr{X}/R}$.

3.3. **The Connectedness Theorem.**

(3.3.1) Our main result is the following variant of the Shokurov-Kollár Connectedness Theorem. Like the proofs of Shokurov and Kollár, our proof is based on vanishing theorems: we deduced generalizations of Kawamata-Viehweg Vanishing and Kollár's Torsion-free Theorem for varieties over power series rings in characteristic zero by means of Greenberg approximation.

Theorem 3.3.2 (Connectedness Theorem)**.** *If X is a geometrically connected, smooth and proper K-variety of geometric genus one, then $Sk(X)$ is connected.*

References

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