# WILDLY RAMIFIED COVERS, II: SWAN CONDUCTORS, THE BERKOVICH DIFFERENT, AND LIFTING

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We assume familiarity with the terminology from Stefan Wewers's talk (Hurwitz trees, Artin characters, depth characters, thicknesses of annuli) as well as Michael Temkin's talk (the Berkovich different). We set some notation throughout.

- $k$ is an algebraically closed field of characteristic $p$.
- $G$ is a finite group.
- By convention, the image of an uppercase letter in a ring under a quotient map is the corresponding lowercase letter. So, if $R$ is a DVR, then we write $k[[t]]$ for the quotient of the power series ring $R[[T]]$ by the maximal ideal of $R$.

The reader who is interested in an in-depth introduction to these topics might consider reading the expository articles [Obu12] and [Obu17].

## 1. Motivation

In this talk, we are motivated by the following *lifting problem* for Galois covers of curves. Let $f : Y \to X$ be a $G$-Galois branched cover of smooth, proper, connected curves. Does there exist a characteristic zero DVR $R$ with residue field $k$, along with a $G$-Galois cover $f_R : Y_R \to X_R$ of smooth relative $R$-curves whose special fiber (*including the $G$-action*) is $f$?

If there is such an $f_R$, we say that *$f$ lifts to characteristic zero.*

**Remark 1.1.** If $G$ is trivial, then the lifting problem can be solved with $R = W(k)$, using standard techniques from deformation theory. In general, as long as $R$ is complete, one can first specify a lift $X_R$ of $X$ and this does not change whether the lifting problem can be solved.

**Remark 1.2.** One can think of the lifting problem as the "trivial" case of the lifting problem for harmonic morphisms of metrized complexes of curves (as studied by Amini, Baker, Brugallé, and Rabinoff), while keeping track of a group action. For our lifting problem, the underlying graphs of these complexes consist of a single vertex.

It turns out that there is a *local-global principle* for the lifting problem, see, e.g., [BM00], [Gar96], [GM98]. Namely, let $f : Y \to X$ be a $G$-Galois cover of smooth projective connected $k$-curves. At each ramification point $y \in Y$, the extension of complete local rings $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{X,f(y)}$ is an $I_y$-extension of power series rings $k[[z]]/k[[t]]$, where $I_y$ is the inertia group at $y$.

**Theorem 1.3** (Local-Global Principle)**.** *The cover $f$ lifts to characteristic zero if and only if each $I_y$-extension $\hat{\mathcal{O}}_{Y,y}/\hat{\mathcal{O}}_{X,f(y)} \cong k[[z]]/k[[t]]$ lifts to characteristic zero. That is, if and only if there exists a characteristic zero DVR $R$ and an $I_y$-extension*

*$R[[Z]]/R[[T]]$ such that the $I_y$-action on $R[[Z]]$ reduces to the given $I_y$-action on $k[[z]]$ modulo the maximal ideal of $R$.*

A $G$-Galois extension $k[[z]]/k[[t]]$ is called a *local $G$-extension*. The question of whether a local $G$-extension lifts to characteristic zero is called the *local lifting problem.*

**Remark 1.4.** In fact, if one can solve all the relevant local lifting problems for a cover $f : Y \to X$ over a particular *complete* DVR $R$, then one can solve the original lifting problem over the same $R$.

**Remark 1.5.** If $k[[z]]/k[[t]]$ is a local $G$-extension, then $G$ is of the form $P \rtimes \mathbb{Z}/m$, where $P$ is a $p$-group and $p \nmid m$. Thus one of the advantages of the local lifting problem is that one need only deal with solvable groups.

**Remark 1.6.** If $k[[z]]/k[[t]]$ is a local $G$-extension with $G$ prime-to-$p$, then, up to a change of variable, the extension is just the $\mathbb{Z}/m$-extension $k[[t^{1/m}]]/k[[t]]$. This clearly lifts to characteristic zero over any DVR $R$ containing $m$th roots of unity, and the lift is given by $R[[T^{1/m}]]/R[[T]]$ with the standard Galois action.

Combining this with the local-global principle proves the result, originally due to Grothendieck, that tame covers of curves lift from characteristic $p$ to characteristic zero.

It is possible for a local $G$-extension to have a *Hurwitz tree obstruction* to lifting, as discussed in Stefan Wewers's lecture. In fact, the Hurwitz tree obstruction is quite powerful. For instance, let us call a group $G \cong P \rtimes \mathbb{Z}/m$ a *local Oort group* for $p$ if every local $G$-extension in characteristic $p$ lifts to characteristic zero. As a consquence of the Hurwitz tree obstruction, one can prove the following result.

**Theorem 1.7** (See [CGH11] and [BW09])**.** *If $G$ is a local Oort group for $p$, then either*

- *$G$ is cyclic,*
- *$G \cong D_{p^n}$,*
- *or $G \cong A_4$ and $p = 2$.*

Partial converses of this theorem are known.

- That cyclic groups are local Oort groups is a result of Obus–Wewers and Pop ([OW14], [Pop14]), following results of Sekiguchi–Oort–Suwa ([SOS89]) and Green–Matignon ([GM98]), who proved the cases $\mathbb{Z}/p$ and $\mathbb{Z}/p^2$, respectively. This is the so-called *Oort conjecture.*
- That $D_p$ is a local Oort group is due to Bouw–Wewers ([BW06]) for $p$ odd and Pagot ([Pag02a] and [Pag02b]) for $p = 2$. The group $D_9$ was shown to be local Oort for 3 by Obus ([Obu15]), and $D_4$ was shown to be local Oort for 2 by Weaver ([Wea17]), building on work of Brewis ([Bre08]). The other dihedral cases remain open.
- The group $A_4$ was claimed to be local Oort by Bouw (see [BW06]), but the proof was not written down. A proof was written down by Obus ([Obu16]).

A major current research question is to determine the list of local Oort groups explicitly.

We know from Wewers's talk that any lift of a local $G$-extension gives rise to a Hurwitz tree with depth 0 at the root, whose Artin character at the root is equal to the Artin character of the local $G$-extension coming from the higher ramification filtration. Now, in all cases of Theorem 1.7 except for $D_{2^n}$, $n \geq 2$, we know how to construct a valid Hurwitz tree from scratch corresponding to any possible Artin conductor.

**Question 1.8.** Can we somehow use the information encoded in a Hurwitz tree to determine a lift of a local $G$-extension to characteristic zero?

The Hurwitz tree corresponding to a lift of a local $G$-extension determines the basic geometry of the branch locus of the lift, that is, the distances between the branch points. In order to use Hurwitz trees to construct lifts, we will need to enhance them with more data.

Namely, if $\Gamma$ is a Hurwitz tree coming from a lift, then on each vertex $v$ of $\Gamma$ with depth not equal to 0 (corresponding to a curve $C_v/k$), we can construct a "differential Swan conductor" $\omega_v$. The differential Swan conductor is a function from irreducible characters $\chi$ of $G$ to $\Omega_{k(C_v)/k}^{\otimes \dim(\chi)}$. The idea is that this retains even more of the characteristic zero information that is lost after reduction to characteristic $p$.

This construction of differential Swan conductors is done in great generality by Kato ([Kat87]), but simplifies when the group is cyclic. For simplicity, this is the case we focus on.

## 2. Differential Swan conductors

For simplicity, we assume $G = \mathbb{Z}/p^n$. Thus all irreducible characters $\chi$ are 1-dimensional, so $\omega_v(\chi) \in \Omega_{k(C_v)/k}$. In this case, the depth $\delta_v(\chi)$ and the divisor of the differential Swan conductor $[\omega_v]$ at a vertex $v$, as well as the Artin character $a_e(\chi)$ at an edge $e$ depend only on the order of $\chi$. By abuse of notation, when we write $\delta_v$, $[\omega_v]$, or $a_e$, we mean that we have plugged in a *faithful* character.

The differential Swan conductors satisfy certain compatibilities. The key one is the following.

**Proposition 2.1** ([BW09, Theorem 2.4.5])**.** *Let $\Gamma$ be the Hurwitz tree associated to a lift of a $\mathbb{Z}/p^n$-extension. Let $v$ be a vertex of $\Gamma$ corresponding to a curve $C_v$, and let $e$ be an edge of $\Gamma$ emanating from $v$, corresponding to a point $x \in C_v$. Then $a_e = -ord_{x_e}(\omega_v)$.*

If we further assume that $n = 1$, then $[\omega_v]$ is always the divisor of a *logarithmic* or *exact* differential form (see, e.g. [Obu12, Appendix B]). For general $n$, the divisor $[\omega_v]$ can be constructed from the divisors of the differential Swan conductors corresponding to intermediate $\mathbb{Z}/p$-extensions.

So far, we have only discussed differential Swan conductors on Hurwitz trees associated to lifts of local $G$-extensions. But one can try to define a general "differential Hurwitz tree," which is a Hurwitz tree augmented with functions from irreducible characters $\chi$ of $G$ to $\Omega_{k(C_v)/k}^{\otimes \dim(\chi)}$ for each vertex $v$ with depth not equal to zero, where these functions satisfy all the requisite compatibilities and requirements (e.g., as in Proposition 2.1 and the paragraph immediately after). This definition has only been written down in the case that $G$ has a $p$-Sylow subgroup of order $p$ (see [BW06], where it is simply called a *Hurwitz tree*), but we have a reasonable understanding of what it should be when $G$ has a general cyclic $p$-Sylow subgroup.

**Remark 2.2.** If we know how to define a differential Hurwitz tree for a given group $G$, then we can say that a local $G$-extension has a "differential Hurwitz tree obstruction" if there is no *differential* Hurwitz tree having depth zero and the correct Artin character at the root. This is a stronger obstruction than the regular Hurwitz tree obstruction, because one might be able to construct a Hurwitz tree, but unable to construct the necessary differential forms.

**Question 2.3.** Given a local $G$-extension and a differential Hurwitz tree with depth zero and correct Artin character at the root, can one construct a lift of the local $G$-extension?

## 3. Lifting

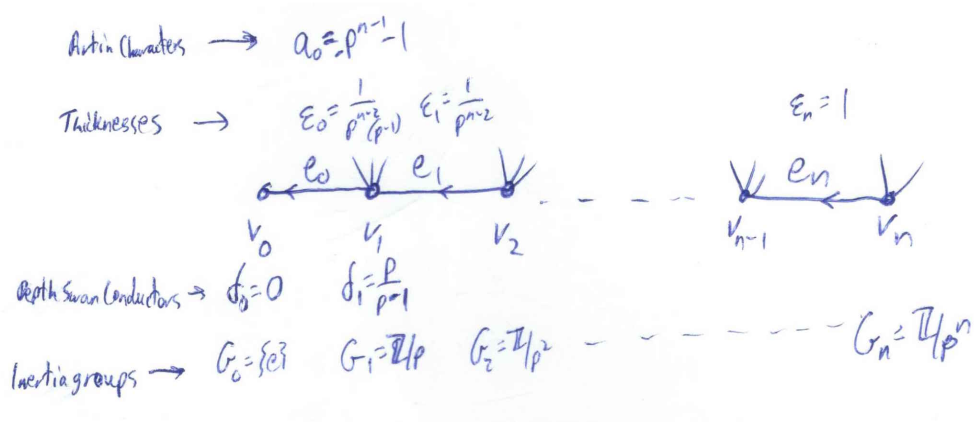
3.1. $\mathbb{Z}/p$**-case.** The answer to Question 2.3 is "yes" when $G$ has $p$-Sylow subgroup of order $p$ (see [Hen00] for $G \cong \mathbb{Z}/p$ and [BW06] for $G \cong \mathbb{Z}/p \rtimes \mathbb{Z}/m$). We sketch the idea for $\mathbb{Z}/p$. Let $\Gamma$ be the differential Hurwitz tree.

- **Step 0.** Recall that a lift $R[[Z]]/R[[T]]$ of a local $G$-extension $k[[z]]/k[[t]]$ corresponds to a cover of the open non-archimedean unit disk $\mathcal{D}$. We choose $R$ complete, and in all steps below we freely take finite extensions of $R$ as necessary.
- **Step 1.** The Hurwitz tree breaks $\mathcal{D}$ up into open annuli and "non-archimedean pants with arbitrarily many legs," i.e., affinoids that are complements of disjoint unions of open disks inside a closed disk.
- **Step 2.** For each non-root vertex $v$ of $\Gamma$, construct a $\mathbb{Z}/p$-cover of the corresponding pair of pants using $\omega_v$ and $\delta_v$.
- **Step 3.** For each edge $e$ of $\Gamma$, construct a $\mathbb{Z}/p$-cover of the corresponding annulus using $a_e$, $\delta_v$ for the $v$ incident to $e$, and $\varepsilon_e$ (the thickness of the annulus).
- **Step 4.** The Hurwitz tree compatibility requirements ensure that all these covers glue together, thus giving a cover $\mathcal{E} \to \mathcal{D}$.
- **Step 5.** In fact, there is only one isomorphism class of $\mathbb{Z}/p$-extension $k[[z]]/k[[t]]$ for any given Artin character[1]. Thus the reduction of $\mathcal{E} \to \mathcal{D}$ to characteristic $p$ gives the original extension $k[[z]]/k[[t]]$.

**Remark 3.1.** If one simply wants to solve the local lifting problem for $\mathbb{Z}/p$, rather than show how to get a lift corresponding to a given differential Hurwitz tree, then this is easy to do explicitly (see, e.g., [Obu12, Theorem 6.8]). However, to solve the local lifting problem for, say, dihedral groups of order $2p$ when $p$ is odd, there is no alternative known to the method outlined above.

3.2. $\mathbb{Z}/p^n$**-case.** If we take $G \cong \mathbb{Z}/p^n$, it is not known how to complete the analogs of Steps 3 and 4 above, and Step 5 is false. Solving the local lifting problem in this case requires a different idea. The key step is to find a way to lift a *single* $\mathbb{Z}/p^n$-extension $k[[z]]/k[[t]]$ with the *minimal possible* jumps in the higher ramification filtration $(1, p, \ldots, p^{n-1})$. In this case, one can write down a differential Hurwitz tree.

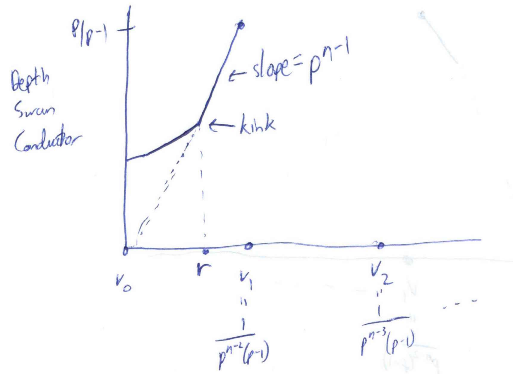
[1]The Artin character can be read off from the ramification jump, and the unique isomorphism class of extension with upper jump $j$ is given on the level of fields by $k((t))[y]/(y^p - y - t^{-j})$




In the picture above, vertex $v_i$ is attached to $p^{n-i} - p^{n-i-1}$ leaves for $1 \leq i \leq n$, and 2 leaves for $v_n$. The differential Swan conductors are somewhat difficult to write down explicitly, but it can be done. The vertex $v_0$ corresponds to the Gauss point (unit disc centered at $T = 0$), and and the other vertices $v_i$ are type II points corresponding to disks centered at $T = 0$. Other relevant information is on the diagram.

Now we sketch the proof of this key step. Let $\mathcal{D}$ be the open unit disk and, for $r > 0$, let $D_r$ be the closed unit disk, centered at 0, of radius $|p|^r$.

- **Step 1.** We attempt to use Kummer theory. That is, we assume $\zeta_p \in R$ and we seek a lift $R[[Z]]/R[[T]]$ by finding $g \in \mathrm{Frac}R[[T]]$ and taking the integral closure of $R[[T]]$ in $(\mathrm{Frac}R[[T]])[\sqrt[p^n]{g}]$.
- **Step 2.** Proceed by induction on $n$. We know the base case. If $k[[s]]/k[[t]]$ is the $\mathbb{Z}/p^{n-1}$-subextension of $k[[z]]/k[[t]]$, then suppose we have $g \in \mathrm{Frac}R[[T]]$ such that extracting a $p^{n-1}$st root of $g$ suffices to lift $k[[s]]/k[[t]]$.
- **Step 3.** Adjust $g$ so that it gives rise to a cover of $\mathcal{E}_{r_1}$ of the closed disk $\mathcal{D}_{r_1}$ lifting the part of the differential Hurwitz tree at $v_1$ and to the left.
- **Step 4.** The fact that $\mathcal{E}_{r_1}/\mathcal{D}_1$ yields the correct differential Swan conductor at $v_1$ means that we in fact automatically get a lift of a (bigger) disk $D_r$ for some $r < r_1$. In fact, the depth function of a faithful character (which is more or less Temkin's Berkovich different) can be extended to a piecewise linear function on the real line, as in the diagram below. It turns out that we can choose $r$ to be the the right-most kink in the diagram, and the slope of the depth to the right of $r$ is $p^{n-1}$.




- **Step 5.** After making a further adjustment to $g$, we can push this kink to the left, thus giving us a smaller $r$.
- **Step 6.** The location of the leftmost kink can be locally expressed as the negative of the valuation of an analytic function on an affinoid (the affinoid is a parameter space for "admissible" choices of $g$). This takes a minimum by the maximum principle for analytic functions.[2] By Step 5, this minimum must be 0! Note that this implies that the depth function at 0 is 0. Thus, we in fact get an extension $R[[Z]]/R[[T]]$ yielding the entire differential Hurwitz tree. So the reduction has the desired minimal ramification jumps.

**Remark 3.2.** As was mentioned earlier, the local lifting problem for $D_{p^n}$ is still open when $n > 1$, even for odd $p$. The method sketched above works perfectly well for $D_{p^n}$. The only obstacle is that we do not know how to write down a differential Hurwitz tree in general.

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[2]For a generalization of this result, see [OW16].

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