BRILL-NOETHER THEORY FOR CURVES OF FIXED GONALITY (AFTER JENSEN AND RANGANATHAN)

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Abstract. This talk, presented at the 2017 Simon Symposium on Nonarchimedean and Tropical Geometry, reports on recent work of Dave Jensen and Dhruv Ranganathan, giving an analogue of the Brill-Noether Theorem for the general curve of fixed genus and gonality. See Section 6 for the statement of their main result and an outline of the proof.

1. Background from classical Brill-Noether theory

Let X be a smooth projective curve of genus g over an algebraically closed nonarchimedean field K of pure characteristic zero. The *rank* of a divisor D on X is the dimension of its complete linear series. Equivalently,

$$r(D) = h^0(X, \mathcal{O}(D)) - 1.$$

The subscheme $W^r_d(X) \subset \mathrm{Pic}_d(X)$ parametrizes divisor classes of degree d and rank at least r. Its scheme structure comes from a natural description as the degeneracy locus of a map of vector bundles over $\mathrm{Pic}_d(X)$.

For simplicity, throughout this talk we will assume $0 \leq d \leq g-1$ and $r \geq 1$. By Riemann-Roch, these cases are enough to determine $W^r_d(X)$ for all r and d.

Recall that the Brill-Noether number associated to the tuple (g, r, d) is

$$\rho(g, r, d) = g - (r+1)(g - d + r).$$

Brill-Noether Theorem ([GH80]). *Suppose X is general. If $\rho(g, r, d) \geq 0$ then $W^r_d(X)$ is of pure dimension $\rho(g, r, d)$. Otherwise, $W^r_d(X)$ is empty.*

In fact, we now know much more about the geometry of $W^r_d(X)$ when X is sufficiently general, thanks to the power of various now-classical techniques in algebraic geometry, such as intersection theory, connectedness theorems for degeneracy loci, and limit linear series. For instance, the singular locus of $W^r_d(X)$ is precisely $W^{r+1}_d(X)$ [Gie82], $W^r_d(X)$ is irreducible when $\rho(g, r, d) > 0$ [FL81], and monodromy acts by the full symmetric group when $\rho(g, r, d) = 0$ [EH87]. The class of $W^r_d(X)$ in the Chow ring $A^*(\mathrm{Pic}_d(X))_{\mathbf{Q}}$ is given by the Kleiman-Laksov formula [KL72]:

$$[W^r_d(X)] = \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!} \cdot \theta^{g - \rho(g,r,d)}.$$

2. Gonality and Hurwitz spaces

Recall that the *gonality* of a curve X is the minimal degree of a nonconstant map from the curve to $\mathbf{P}^1$. Equivalently, the gonality $\operatorname{gon}(X)$ is the minimal degree of a divisor of rank 1 on X. The Brill-Noether Theorem tells us that the gonality of a general curve of genus g is $\lfloor \frac{g+3}{2} \rfloor$. On the other hand, Hurwitz theory (the theory of branched covers of $\mathbf{P}^1$) tells us that the Hurwitz space

$$\mathscr{H}_{g,k} = \{X \in \mathscr{M}_g \mid \operatorname{gon}(X) \leq k\}$$

is irreducible of dimension $2g+2k-5$. Since gonality is the most basic invariant of a smooth projective curve, aside from the genus, it is then natural to ask questions about the geometry of $W^r_d(X)$ for X general in $\mathscr{H}_{g,k}$.

The breakthrough paper of Jensen and Ranganathan [JR17] brings to bear a broad package of relatively new techniques from tropical geometry, Berkovich theory, and logarithmic deformation theory, along with a healthy dose of classical algebraic geometry (in the form of Maroni invariants and more general scrollar invariants) to prove the analogue of the Brill-Noether Theorem for the general curve of fixed genus and gonality.

3. Prior results for general curves of fixed gonality

Let us begin by recalling previous work of Coppens and Martens [CM99], and of Pflueger [Pfl17a], giving lower and upper bounds, respectively, for the dimension of $W^r_d(X)$, when X is general in $\mathscr{H}_{g,k}$.

Lower Bound ([CM99])**.** *Suppose X is general in $\mathscr{H}_{g,k}$. If $r \geq 1$ then*

$$\dim W^r_d(X) \geq \max_{\ell \in \{0,1,r-1,r\}} \rho(g, r-\ell, d) - \ell k.$$

For $r \leq 3$, the lower bound of Coppens and Martens agrees with the following upper bound of Pflueger.

Upper Bound ([Pfl17a])**.** *Suppose X is general in $\mathscr{H}_{g,k}$. If $r \geq 1$ then*

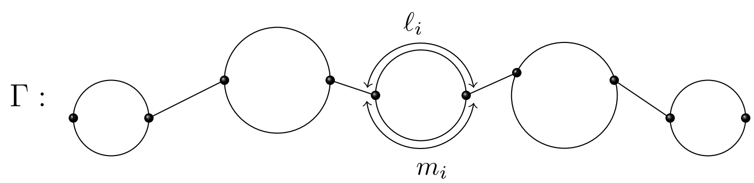
$$\dim W^r_d(X) \leq \max_{\ell \in \{0,\dots,r\}} \rho(g, r-\ell, d) - \ell k.$$

A word of caution is in order, when comparing with the classical Brill-Noether Theorem. For the general curve X in $\mathscr{H}_{g,k}$, the scheme $W^r_d(X)$ may have irreducible components of different dimensions. This contrasts with the case of curves general in $\mathscr{M}_g$, for which W^r_d has pure dimension.

The proof of the Coppens-Martens result is essentially classical and constructive. They start from a curve and a linear series of degree k and rank 1, and produce a locus of at least the specified dimension in W^r_d. Pflueger's proof is markedly different, using tropical and nonarchimedean geometry (specialization from curves to graphs, as in [Bak08]) together with an intricate combinatorial analysis of divisors of given degree and rank on a chain of loops in terms of certain ratios of edge lengths [Pfl17b], as we now discuss.

4. Chains of loops

Following earlier works studying the tropical geometry of divisors on chains of loops, such as [CDPR12], Pflueger considers a graph Γ consisting of g loops separated by bridges.



The top edge of the ith loop in Γ has length ℓ_i, and the bottom edge has length m_i. Pflueger defines the *torsion order* of the ith loop to be

$$\mu_i = \begin{cases} q & \text{if } \frac{\ell_i}{\ell_i + m_i} \text{ is rational and equal to the reduced fraction } \frac{p}{q}. \\ 0 & \text{otherwise.} \end{cases}$$

Note that the torsion orders of the first and last loops are irrelevant (depending only on the arbitrary choice of points on the first and last loops separating the top edges from the bottom edges). Pflueger therefore defines the *torsion profile* of Γ to be

$$\vec{\mu} = (\mu_2, \ldots, \mu_{g-1}),$$

and shows that the Brill-Noether theory of Γ (i.e., the geometry of the locus $W^r_d(\Gamma)$ parametrizing divisor classes of degree d and rank at least r on in the real torus $\mathrm{Pic}_d(\Gamma)$) depends only on $\vec{\mu}$. With this terminology, the main combinatorial result of [CDPR12] may be stated as follows.

Brill-Noether Theorem for a General Chain of Loops. *If* $\vec{\mu} = 0$ *then*

$$W^r_d(\Gamma) = \bigcup_t T(t)$$

where t ranges over standard tableaux on a $(r+1) \times (g-d+r)$ rectangle, with entries in $\{1, ..., g\}$ and $T(t)$ is a real torus of dimension $\rho(g, r, d)$.

The Brill-Noether number $\rho(g, r, d)$ is negative exactly when $g < (r+1)(g-d+r)$; in these cases, there are no such standard tableaux and $W^r_d(\Gamma)$ is empty. One can also improve this statement by giving an explicit description of the locus $T(t)$ in $\mathrm{Pic}_d(\Gamma)$. The reader interested in such details is encouraged to consult the original papers.

Given such fundamental foundational results from tropical and nonarchimedean geometry as the specialization lemma from curves to graphs [Bak08] and the identification of the skeleton of the Jacobian with the Jacobian of the skeleton [BR15], the classical Brill-Noether Theorem of Griffiths and Harris follows as an immediate corollary of this "tropical Brill-Noether theorem" for chains of loops. Indeed, for the classical Brill-Noether Theorem it is enough to produce a single curve for which the conclusion holds, and specialization shows that any curve X whose skeleton is a chain of loops Γ with torsion profile $\vec{\mu} = 0$ is sufficiently general.

We also know that every special divisor on a general chain of loops lifts to such an X.

Lifting Theorem ([CJP15])**.** *Let X be a curve whose skeleton is a chain of loops with torsion profile $\vec{\mu} = 0$. Then every divisor class in $W^r_d(\Gamma)$ is the specialization of a divisor class in $W^r_d(X)$.*

The original proof of this lifting result is a kludge, using intersections with translates of θ to reduce to a zero-dimensional lifting statement, a local computation for lifting intersections, as in [OP13, OR13], to show that in the zero-dimensional case each tropical divisor lifts to at most one divisor on X, and then an explicit combinatorial counting argument, showing that the number of tropical divisors is equal to the number of algebraic ones, to conclude that every tropical divisor lifts.

By contrast, the arguments of Jensen and Ranganathan give a clear and conceptual proof of this lifting result as an essentially trivial special case of their more general arguments, the case where the logarithmic deformation theory of the corresponding map to projective space is unobstructed. Note that Xiang He has also subsequently given an independent clear and conceptual proof of the lifting theorem [He17], using smoothing results in Brian Osserman's theory of limit linear series for curves not of compact type [Oss14, Oss17].

5. Special divisors on special chains of loops

Leaving the case of torsion profile $\vec{\mu} = 0$, which is an essentially trivial special case in the work of Jensen and Ranganathan, we return to Pflueger's classification of special divisors on special chains of loops, i.e., chains of loops for with the torsion profile $\vec{\mu}$ is not zero.

Pflueger defines a *$\vec{\mu}$-displacement tableau* to be a semi-standard tableaux on a $(r+1) \times (g-d+r)$ rectangle with entries in $\{1, \ldots, g\}$ such that any two occurrences of i have lattice distance (in the taxicab metric) divisible by μ_i. In other words, if i occurs at position (j, k) (i.e., as the jth entry in the kth row) and also in position (j', k'), then i divides $|j-j'|+|k-k'|$.

Classification of Special Divisors ([Pfl17b])**.** *Let Γ be a chain of loops with arbitrary torsion profile $\vec{\mu}$. Then $W^r_d(\Gamma)$ is a union of real tori*

$$W^r_d(\Gamma) = \bigcup_t T(t),$$

where the union is over $\vec{\mu}$-displacement tableaux t, and $T(t)$ is a real torus of dimension equal to the number of elements of $\{1, \ldots, g\}$ that do not appear in t.

Once again, the full result includes an explicit description of the locus $T(t)$, the details of which the reader may find in the original source. The varying dimensions of these tori reflect the fact that the general curve X in $\mathscr{H}_{g,k}$ can have Brill-Noether loci $W^r_d(X)$ with irreducible components of different dimensions.

Pflueger's upper bound on $\dim W^r_d(X)$ for X general in $\mathscr{H}_{g,k}$, stated in Section 3 above, is equal to $\dim W^r_d(\Gamma)$, when Γ is the k-gonal chain of loops whose torsion profile is given by

$$\mu_i = \begin{cases} k & \text{for } k \leq i \leq g-k+1. \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

6. The main theorem of Jensen and Ranganathan

The main result of Jensen and Ranganathan says that Pflueger's upper bound is sharp.

Brill-Noether Theorem for Curves of Fixed Gonality ([JR17])**.** *Suppose X is general in $\mathscr{H}_{g,k}$. If $r \geq 1$ then*

$$\dim W^r_d(X) = \max_{\ell \in \{0,\ldots,r\}} \rho(g, r-\ell, d) - \ell k.$$

As mentioned previously, $W^r_d(X)$ need not be pure dimensional, so $\dim W^r_d(X)$ is the maximum of the dimensions of its irreducible components. The proof given by Jensen and Ranganathan is a lifting argument, which we now sketch.

Sketch of proof. Fix a chain of loops Γ with torsion profile $\vec{\mu}$ as in (5.1), and fix a tableau t that uses the minimal possible number of entries. Then the corresponding torus $T(t)$ has the largest possible dimension, realizing Pflueger's upper bound.

Standard specialization arguments show that $\dim W^r_d(X) \leq \dim W^r_d(\Gamma)$, for any curve X of genus g with skeleton Γ. The goal is to show that that there is some X with skeleton Γ such that equality holds. Note that the dimension of the space of skeletons Γ with torsion profile μ is equal to the dimension of $\mathscr{H}_{g,k}$. Therefore, by applying standard specialization arguments to the space of pairs consisting of a k-gonal genus g curve X with a divisor of degree d and rank r, it is enough to show that, for *every* chain of loops with torsion profile Γ, there is an open dense subset of $T(t)$ consisting of points that lift to $W^r_d(X')$ for some k-gonal X' with skeleton Γ.

Fix Γ and a general divisor class $[D] \in T(t)$. We must show that there exists a curve X with skeleton Γ such that $[D]$ lifts to $W^r_d(X)$. The crux of the arguments will involve a mix of classical algebraic geometry, Berkovich theory, and logarithmic deformation theory. However, there is still some remaining input from tropical geometry involved in the setup of the logarithmic deformation argument, as follows.

A general tropical divisor class $[D]$ in $T(t)$ is vertex avoiding and hence determines a map to tropical projective space

$$\varphi_{[D]} : \Gamma \to \mathbf{P}^r_{\mathrm{trop}}.$$

Jensen and Ranganathan observe that, in the special case where $\vec{\mu} = 0$, this map is not superabundant. In other words, the image of each loop contains edges that span the full r-dimensional ambient space. One then describes, in a standard way, the special fiber (over the residue field) that would result from an algebraic morphism $X \to \mathbf{P}^r$ over a nonarchimedean field K tropicalizing to $\varphi_{[D]}$, and well-known arguments, as in [NS06, CFPU16] show that the logarithmic deformation theory of the corresponding special fiber is unobstructed. In this way, one recovers the lifting theorem from [CJP15] with a better, more conceptual proof.

However, for $\vec{\mu}$ given by (5.1), the tropical maps $\varphi_{[D]}$ are typically superabundant. Indeed, the image of each loop is contained in an affine subspace of dimension $k-1$, which may be much less than r. In such cases, the corresponding logarithmic deformation problem is badly obstructed, and apparently hopeless to solve directly.

Jensen and Ranganathan overcome the difficulties posed by this superabundance, and the resulting obstructions to the logarithmic deformation problem, in several steps. First, they observe that if $[D]$ lifts, then the corresponding map $X \to \mathbf{P}^r$ would factor through a rational scroll (i.e., the projectivization of a vector bundle on $\mathbf{P}^1$) of the special form

$$S(a,b) = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}^{\oplus a} \oplus \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus b}),$$

in such a way that composing with the projection to $\mathbf{P}^1$ is a given branched cover of degree k, with fiber D'. Indeed, a classical algebraic geometry argument shows that the map associated to a divisor D on X factors through $S(a,b)$ in this way if and only if

$$h^0(X, \mathcal{O}(D)) \geq b \text{ and } h^0(X, \mathcal{O}(D+D')) \geq h^0(X, \mathcal{O}(D)) + a + b.$$

The rational scroll $S(a,b)$ is a toric variety, and hence has a natural tropicalization $S(a,b)_{\mathrm{trop}}$. If the map $X \to \mathbf{P}^r$ factors through $S(a,b)$, then the map $\Gamma \to \mathbf{P}^r_{\mathrm{trop}}$ factors through $S(a,b)_{\mathrm{trop}}$. A direct tropical computation shows that the resulting map to $S(a,b)_{\mathrm{trop}}$ is still superabundant, but much more mildly so than the map to $\mathbf{P}^r_{\mathrm{trop}}$. The image of each loop spans at least an affine hyperplane, and each consecutive pair of loops spans the full space. Furthermore, Jensen and Ranganathan compute combinatorially that each loop satisfies the analogue of Speyer's well-spacedness condition for lifting maps of genus 1 curves [Spe14]. This turns out to be enough to set up and explicitly solve the logarithmic deformation problem for the map to $S(a,b)$ by using analytic computations in a neighborhood of each loop, together with a patching argument. $\square$

Please see the original paper [JR17] for further details!

References

[Bak08] M. Baker, *Specialization of linear systems from curves to graphs*, Algebra Number Theory **2** (2008), no. 6, 613–653.

[BR15] M. Baker and J. Rabinoff, *The skeleton of the Jacobian, the Jacobian of the skeleton, and lifting meromorphic functions from tropical to algebraic curves*, Int. Math. Res. Not. IMRN (2015), no. 16, 7436–7472.

[CDPR12] F. Cools, J. Draisma, S. Payne, and E. Robeva, *A tropical proof of the Brill-Noether theorem*, Adv. Math. **230** (2012), no. 2, 759–776.

[CFPU16] M.-W. Cheung, L. Fantini, J. Park, and M. Ulirsch, *Faithful realizability of tropical curves*, Int. Math. Res. Not. IMRN (2016), no. 15, 4706–4727.

[CJP15] D. Cartwright, D. Jensen, and S. Payne, *Lifting divisors on a generic chain of loops*, Canad. Math. Bull. **58** (2015), no. 2, 250–262.

[CM99] M. Coppens and G. Martens, *Linear series on a general k-gonal curve*, Abh. Math. Sem. Univ. Hamburg **69** (1999), 347–371.

[EH87] D. Eisenbud and J. Harris, *Irreducibility and monodromy of some families of linear series*, Ann. Sci. École Norm. Sup. (4) **20** (1987), no. 1, 65–87.

[FL81] W. Fulton and R. Lazarsfeld, *On the connectedness of degeneracy loci and special divisors*, Acta Math. **146** (1981), no. 3-4, 271–283.

[GH80] P. Griffiths and J. Harris, *On the variety of special linear systems on a general algebraic curve*, Duke Math. J. **47** (1980), no. 1, 233–272.

[Gie82] D. Gieseker, *Stable curves and special divisors: Petri's conjecture*, Invent. Math. **66** (1982), no. 2, 251–275.

[He17] X. He, *Smoothing of limit linear series on curves and metrized complexes of pseudocompact type*, preprint arXiv:1707.04624, 2017.

[JR17] D. Jensen and D. Ranganathan, *Brill-Noether theory for curves of a fixed gonality*, preprint arXiv:1701.06579, 2017.

[KL72] S. Kleiman and D. Laksov, *On the existence of special divisors*, Amer. J. Math. **94** (1972), 431–436.

[NS06] T. Nishinou and B. Siebert, *Toric degenerations of toric varieties and tropical curves*, Duke Math. J. **135** (2006), no. 1, 1–51.

[OP13] B. Osserman and S. Payne, *Lifting tropical intersections*, Doc. Math. **18** (2013), 121–175.

[OR13] B. Osserman and J. Rabinoff, *Lifting nonproper tropical intersections*, Tropical and non-Archimedean geometry, Contemp. Math., vol. 605, Amer. Math. Soc., Providence, RI, 2013, pp. 15–44.

[Oss14] B. Osserman, *Limit linear series for curves not of compact type*, arXiv:1406.6699v3, 2014.

[Oss17] ———, *Limit linear series and the Amini-Baker construction*, preprint arXiv:1707.03845, 2017.

[Pfl17a] N. Pflueger, *Brill-Noether varieties of k-gonal curves*, Adv. Math. **312** (2017), 46–63.

[Pfl17b] ———, *Special divisors on marked chains of cycles*, J. Combin. Theory Ser. A **150** (2017), 182–207.

[Spe14] D. Speyer, *Parameterizing tropical curves I: Curves of genus zero and one*, Algebra Number Theory **8** (2014), no. 4, 963–998.

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