SIMONS SYMPOSIUM 2013: OPEN PROBLEM SESSIONS

The 2013 Simons Symposium on Tropical and Nonarchimedean Geometry included three open problem sessions, in which leading experts shared questions and conjectures that they find interesting and important. One purpose of these sessions is to help set a direction for the future development of tropical and nonarchimedean geometry, and to give young researchers clear targets to aim for in their own research programs. As might be expected, some of these problems were resolved at the Symposium or shortly after. Most remain completely open. What follows is an attempt to record the as much of the essential content as possible from these open problem sessions, along with a brief record of subsequent developments.

1. Poincaré Lelong as a computational tool, and higher dimensional potential theory. (Matt Baker)

The recent paper by "Les Antoines" concerning forms and currents on Berkovich spaces contains, among many other results, a higher-dimensional version of Thuillier's non-Archimedean Poincaré-Lelong formula for analytic curves. The latter result has proven to be very useful for tropical geometry, including the construction of "faithful tropicalizations" and the method of tropical implicitization [Baker-Payne-Rabinoff 2011], as well as the theory of linear series on metrized complexes of curves [Amini-Baker 2012]. In principle, there should be similar applications of the higher-dimensional Poincaré-Lelong formula. However, discovering such applications and exploiting Poincaré-Lelong in this way will probably require a nontrivial unpackaging of the machinery of forms and currents. The following is the sort of test question one might ask.

Problem: Given the retraction of the divisor of a meromorphic function f on an analytic variety X to a skeleton Σ, can one recover the restriction of $\log |f|$ to Σ in a combinatorial way as in the case of curves (where this boils down to using Kirchhoff's Laws to solve a discrete Dirichlet problem)?

[Editor's note: Some progress has recently been made on this problem by Walter Gubler, Joe Rabinoff, and Annette Werner.]

2. Moduli space for tropical curves plus a line bundle. (Lucia Caporaso)

Problem: Construct a moduli space for tropical curves together with a line bundle (i.e. divisor class). Even simple examples would be interesting!

Date: June 1, 2013.

3. Bogomolov conjecture for abelian varieties with everywhere good reduction. (Antoine Chambert-Loir)

Let F be a number field or a function field in one variable k. Let A be an abelian variety over F and let L be an ample line bundle on A. Associated to this data one has a Néron-Tate canonical height function $\hat{h} : A(\bar{F}) \to \mathbb{R}_{\geq 0}$ with the property that for any $x \in A(\bar{F})^{\mathrm{tor}}$, then $\hat{h}(x) = 0$. If F is a number field, or if the ground field k is *finite*, then the converse holds: any point $x \in A(\bar{F})$ such that $\hat{h}(x) = 0$ is a torsion point. If the ground field k is infinite, then the converse assertion is more delicate, due to the possible presence of abelian subvarieties of non-zero $\bar{F}/k$-trace ("constant" subvarieties).

So let us say that a subvariety V of $A_{\bar{F}}$ is *special* if, F is a number field and V is the translate of an abelian subvariety by a torsion point (aka torsion subvariety), or F is a function field and, denoting by $G_V \subset A$ the stabilizer of V, the image of V/G_V in A/G_V is a translate of a subvariety of the k-trace of A/G_V by a torsion point.

Let X be a subvariety of A. The Manin-Mumford problem, solved by Raynaud (1983, number field case) and Hrushovski (2001, function field case), is to show that $X(\bar{F}) \cap A^{\mathrm{tor}}$ is not Zariski dense in X unless $X_{\bar{F}}$ is a special subvariety. This is equivalent to showing that the intersection is finite when X does not contain any special subvariety.

A refined question posed by Bogomolov (1980) in the number field case is whether (assuming that X does not contain a special subvariety) there exists $\varepsilon > 0$ such that $\hat{h}(x) \geq \varepsilon$ for an non-torsion $x \in X(\bar{F})$.

This was solved over number fields by Zhang (1998) using equidistribution techniques due to Szpiro-Ullmo-Zhang (1997) and a geometric idea due to Ullmo (1998). It is also known when F is a function field of characteristic zero and X is a curve by work of Zhang and Cinkir (2011). Gubler (2007) has solved the problem (using tropical and non-Archimedean geometry!) when F is an arbitrary function field but assuming that A is totally degenerate (then, the only special subvarieties are torsion subvarieties). Recent work of Gubler and Yamaki reduces the general case to the special case where A has everywhere good reduction. In this special case, tropical methods seem to be of no help since the canonical measures are one point masses.

Problem: What to do when A has everywhere good reduction?

4. Tropically unirational varieties. (Jan Draisma)

The original question. Let X be a unirational subvariety of a torus T^n over some algebraically closed field with a (possibly trivial) valuation. The question raised in (Draisma–Frenk 2012) is whether there always exists a rational map $\phi : T^m \dashrightarrow T^n$ such that $\mathrm{Trop}(\phi) : \mathbb{R}^m \to \mathrm{Trop}(X) \subseteq \mathbb{R}^n$ is surjective onto $\mathrm{Trop}(X)$. Here $\mathrm{Trop}(\phi)$ is the composition of the inclusion $\mathbb{R}^m \cong S(T^m) \to (T^m)^{\mathrm{an}}$, where $S(T^m)$ is the usual skeleton of T^m, and $\phi^{\mathrm{an}} : (T^m)^{\mathrm{an}} \dashrightarrow (T^n)^{\mathrm{an}}$, whose domain of definition is nonempty and Zariski open and hence contains $S(T^m)$, and the retraction

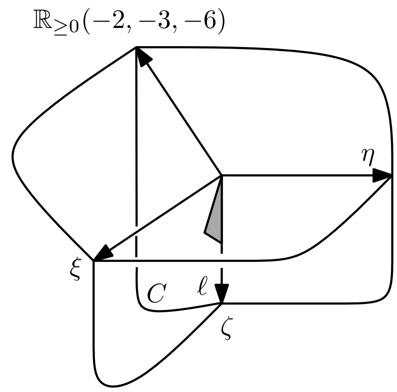

FIGURE 1. The tropicalization of the cubic surface X.

$(T^n)^{\mathrm{an}} \to S(T^n) \cong \mathbb{R}^n$. The term *tropically unirational varieties* for such (embedded) varieties X was coined in (Draisma–Frenk 2012). Several interesting examples were constructed: all rational curves, all intersections with T^n of affine linear spaces, the cones in $T^{\binom{n}{2}}$ over Grassmannians of two-spaces, the hypersurface in $T^{n\times n}$ defined by a determinant, etc.

Tevelev's counterexample. At the 2013 Simons symposium on non-Archimedean and tropical geometry, Jenia Tevelev constructed the following example of a unirational (indeed, rational) but not tropically unirational variety. All that follows is over $\mathbb{C}$ with the trivial valuation. Set

$$X = \{(x, y, z) \in T^3 \mid z = h(x, y)\}$$

where h is a polynomial with Newton polygon spanned by $1, x^3, y^2$ whose zero set in $\mathbb{C}^2$ is an elliptic curve. We use greek letters ξ, η, ζ for the tropical variables corresponding to x, y, z. The ray $\ell = \{(0, 0)\} \times \mathbb{R}_{\geq 0}$ lies in $\mathrm{Trop}(X)$, and around it are three 2-dimensional cones, one of which is the cone C given by the conditions $3\xi = 2\eta \leq \min\{0, \zeta\}$; see also Figure 1.

Assume that there exists a rational map $\phi = (\phi_1, \phi_2, \phi_3) : T^m \dashrightarrow X$ such that $\operatorname{im} \mathrm{Trop}(\phi) = \mathrm{Trop}(X)$. Write $\phi_i = f_i/g_i$ with f_i, g_i polynomials. Then there exists a 2-dimensional cone, spanned by $\alpha, \beta \in \mathbb{Z}^m$, on which all the $\mathrm{Trop}(f_i)$ and all the $\mathrm{Trop}(g_i)$ are linear, and such that $\mathrm{Trop}(\phi)(\alpha)$ equals some integral vector $(0, 0, a) \in \ell$ with $a > 0$ and $\mathrm{Trop}(\phi)(\beta)$ equals some integral vector $(2b, 3b, c) \in C$ with $c > 0, b < 0$. The sector spanned by the images of α and β is shaded in Figure 1.

After precomposing ϕ with a surjective monomial map $T^m \to T^m$, which does not alter the surjectivity of its tropicalisation, we may assume that $\alpha = (1, 0, \ldots)$ and $\beta = (0, 1, 0, \ldots, 0)$. Let $t_1, \ldots, t_m$ be the standard coordinates on T^m. Think of the f_i and g_i as polynomials in t_1, t_2 with coefficients in $\mathbb{C}[t_3, \ldots, t_m]$. Choose non-zero values for $t_3, \ldots, t_m$ such that none of those coefficients vanishes. Substituting these in ϕ gives a rational map $\phi' : T^2 \dashrightarrow X$ whose numerators f_i' and denominators g_i' have tropicalisations that are linear on $\mathbb{R}^2_{\geq 0}$. Change notation and call this map $\phi = (f_1/g_1, f_2/g_2, f_3/g_3)$.

Having a tropicalisation that is linear on $\mathbb{R}^2_{\geq 0}$ means that each f_i and each g_i has a monomial that divides all other monomials. Moreover, we have

$$\mathrm{Trop}(\phi)(\tau_1, \tau_2) = \tau_1(0, 0, a) + \tau_2(2b, 3b, c) = (\tau_2 2b, \tau_2 3b, \tau_1 a + \tau_2 c),$$

and this determines those smallest monomials. For instance, we infer that the quotient of the smallest monomial in f_1 by that in g_1 equals t_2^{2b}. Assuming, as we may, that we take f_1 and g_1 to have no common factors, we find that f_1 has a non-zero constant term and g_1 equals $t_2^{-2b} g_1'$ where g_1' has a non-zero constant term. Continuing in this fashion, we find that

$$\phi = \left(\frac{f_1}{t_2^{-2b} g_1'}, \frac{f_2}{t_2^{-3b} g_2'}, \frac{t_1^a t_2^c f_3'}{g_3}\right)$$

where $f_1, g_1', f_2, g_2', f_3', g_3$ have non-zero constant terms. Now extend ϕ to a rational map $\mathbb{C} \times T \dashrightarrow \overline{X} \subseteq T^2 \times \mathbb{C}$. As such, it is defined on (most of) $\{0\} \times T$, which it maps into the elliptic curve $E \subseteq T^2 \times \{0\}$ given by the equation $h(x, y) = 0$. Hence, by non-rationality of E, ϕ maps $\{0\} \times \mathbb{C}^*$ onto a single point $P \in T^2 \times \{0\}$. On the other hand, we observe from the formulas that $\phi(0, t_2)$ tends to $(\infty, \infty, 0)$ as t_2 tends to zero. This contradicts the fact that ϕ is constant on $T^2 \times \{0\}$. This concludes Tevelev's argument for why X is not tropically unirational.

The revised question. Tevelev's example gives a clear idea for obstructions to tropical unirationality: a non-unirational variety on the boundery of X in $\mathbb{C}^3$. A natural revised formulation of the original question is therefore:

Problem: Are such non-unirational boundaries of (partial) compactifications of X in toric varieties always obstructions to $X \subseteq T^n$ being unirational? If so, are they the only obstructions?

5. What is the right setting in tropical geometry in order to have a Hodge index theorem? (Eric Katz)

For simplicity, let's think about the "trivially valued" case. Suppose X is a surface over $\mathbb{C}$ embedded in some $(\mathbb{C}^\times)^n$ and consider the fan $\mathrm{Trop}(X)$. Assume (for simplicity) that X is *schön*. There is a natural intersection pairing on the rays of $\mathrm{Trop}(X)$ coming from intersection theory on the toric variety associated to Δ. This intersection pairing has exactly one positive eigenvalue.

Problem (asked by June Huh): For an abstract 2-dimensional fan, what do you need for this to be true?

Note that June Huh has examples of abstract tropical fans for which the Hodge index theorem is violated.

6. Weak factorization. (Kiran Kedlaya)

Suppose K has characteristic zero. Let $X \dashrightarrow Y$ be a birational morphisms of smooth proper varieties over K. It is known (Włodarczyk) that $X \dashrightarrow Y$ factors as a composition of blowups and blowdowns with smooth centers.

Problem: Does this work for non-Archimedean analytic spaces?

Michael Temkin made a remark that this question will in fact be resolved soon in joint work with Dan Abramovich!

7. Oort conjecture on lifting covers of curves. (Kiran Kedlaya)

Suppose k is an algebraically closed field of characteristic $p > 0$. Let $K = \mathrm{Frac} W(k)$ and let $K_n = K(\zeta_{p^n})$. Let $C' \to C$ be a $\mathbb{Z}/p^n\mathbb{Z}$-cover of smooth projective k-curves.

Problem: Can this be lifted to a $\mathbb{Z}/p^n\mathbb{Z}$-cover of smooth projective K_n°-curves?

It is known by recent work of Obus-Wewers (see `http://arxiv.org/abs/1203.5057`) and Pop (see `http://arxiv.org/abs/1203.1867`) that this is true over some algebraic extension of K_n. It could be very useful to understand their arguments in a more tropical / non-Archimedean way. Their proof involves a detailed study of ramification filtrations and some model theory, among other things. More specifically, the proof of Obus-Wewers involves so-called "Hurwitz trees" which ought to have a concrete interpretation in terms of Berkovich spaces.

Some possibly related material concerns ramification of maps between p-adic curves, e.g. the work of Faber and Baldassarri on the "Berkovich ramification locus". Baldassarri's point of view on this involves studying the convergence of local solutions of p-adic differential equations.

8. Tropical vector bundles. (Kiran Kedlaya)

Problem: What is a tropical vector bundle?

9. The openness conjecture of Demailly. (Kiran Kedlaya)

There are interesting and deep relationships between geometry of Berkovich spaces and analytic questions about complex singularities and plurisubharmonic (psh) functions. For example, the openness conjecture of Demailly and Kollar, which is a "purely analytic" statement about psh functions on complex manifolds, was reduced to a "purely algebraic" statement about Berkovich spaces by Jonsson and Mustaţa, see `http://arxiv.org/abs/1205.4273`. [*Update:* It seems that the conjecture has been proved by Berndtsson shortly after the Simons Symposium, see `http://arxiv.org/abs/1305.5781`. This proof remains in the world of complex analysis.]

Problem: It would be interesting to see whether it also transposes into the context of nonarchimedean spaces.

10. Algorithmic stable reduction. (Kiran Kedlaya)

Let X be an algebraic curve over a finite extension K of $\mathbb{Q}_p$. One would like to have an algorithm (and better yet an implementation) that divides X^{an} into "standard" pieces (i.e., each piece, when the base field is suitably extended, becomes a disjoint union of copies of a wide open subspace of some curve of good reduction).

The existence of such a covering is immediate from stable reduction; the main point is that one wants to compute the covering *over* K, without explicitly making the base extension.

Problem: If one can do this, it probably gives yet another proof of stable reduction, so it might be worth checking whether the work of Arzdorf-Wewers or Temkin can be used for this?

11. The direct summand conjecture. (Kiran Kedlaya)

The "direct summand conjecture" in commutative algebra states that for R a regular local ring and S the integral closure of R in a finite extension of $\mathrm{Frac}(R)$, the map $R \to S$ is split in the category of R-modules.

Problem: Is there an approach to this problem using analytic geometry?

Motivation for this question: the only open case is when R is of mixed characteristic, and one can even assume that $R = \mathbb{Z}_p[[x_1, ..., x_n]]$. If $S[1/p]$ is log-étale over $R[1/p]$, then one can prove this using Faltings' almost purity theorem (as observed by Bhatt). But almost purity now has analytic geometry versions coming from work of Scholze and Kedlaya-Liu, which would help *if* one could find a version of the direct summand conjecture which was somehow local on the generic fibre of R.

12. Cohomological understanding of p-adic volumes. (François Loeser)

By Grothendieck's function-sheaf correspondence, functions over finite fields may be expressed as the trace of the Frobenius operator on constructible sheaves. This has shown to be an extremely powerful device since it allows to use geometrical tools to compute or estimate some very complicated sums.

Problem: Figure out the right cohomogical / categorical framework governing the geometry of p-adic volumes and integrals.

13. Monodromy conjecture. (François Loeser)

The monodromy conjecture for p-adic and motivic Igusa zeta functions relates poles of non-archimedean integrals to eigenvalues of the local monodromy. Though some progress has been made in the last years in understanding some connections between non-archimedean integration and the monodromy action on the cohomology of the Milnor fibre, the conjecture is still very much an open problem. For further details:

[**1**] J. Denef and F. Loeser, "Geometry on arc spaces of algebraic varieties," *Proceedings of 3rd European Congress of Mathematics*, Barcelona, Progress in Mathematics **201**, 327–348 (2001), Birkhäuser.

[**2**] E. Hrushovski and F. Loeser, "Monodromy and the Lefschetz fixed point formula," preprint `http://arxiv.org/abs/1111.1954`.

[**3**] J. Nicaise, "An introduction to p-adic and motivic zeta functions and the monodromy conjecture," *Algebraic and analytic aspects of zeta functions and L-functions*, 141–166, MSJ Mem., **21**, Math. Soc. Japan, Tokyo, 2010.

[**4**] J. Nicaise and J. Sebag, "Motivic Serre invariants, ramification, and the analytic Milnor fiber," *Invent. Math.* **168** (2007), no. 1, 133–173.

14. Lefschetz theorems for theta divisors. (Sam Payne)

Let X be a smooth projective curve of genus g over a nonarchimedean field. Assume X is totally degenerate, so $h^1(X^{\mathrm{an}}) = g$. By work of Baker and Rabinoff (2013), the skeleton of $\mathrm{Jac}(X^{\mathrm{an}})$ is canonically identified with the tropical Jacobian of the skeleton of X^{an}, and retraction to the skeleton maps θ^{an} onto the tropical theta divisor θ^{trop}.

Problem: Is the projection $\pi : \theta^{\mathrm{an}} \to \theta^{\mathrm{trop}}$ a homotopy equivalence?

If so, this would be a case where Lefschetz theorems hold for the inclusion of the ample divisor θ^{an} in $\mathrm{Jac}(X^{\mathrm{an}})$ for cohomology groups and homotopy groups with integer coefficients, since $\mathrm{Jac}(X)^{\mathrm{trop}} \smallsetminus \theta^{\mathrm{trop}}$ consists of a single open Voronoi cell of dimension g.

15. Tropical cycles of multiplicity 2. (Bernd Sturmfels)

The problem of studying tropical cycles of multiplicity (degree) 1 is solved via the theory of *matroids*.

Problem: What about multiplicity 2? Can we find a description of these mildly non-linear local structures in some matroid-like way?

16. Integer affine structures on tropical charts. (Michael Temkin)

Suppose X is a Berkovich analytic space of dimension n, $U \subset X$ is an open subset, and $f_1, \dots, f_n \in \mathcal{O}^\times(U)$ are invertible analytic functions giving a morphism $f : U \to T^n$, where T^n is an n-dimensional analytic torus.

Chambert-Loir and Ducros have proved that the inverse image under f of the skeleton Σ of T^n has a natural *rational* piecewise linear structure.

Problem: Do we have a natural *integer* affine structure on $f^{-1}(\Sigma)$?

One can also ask which invariants of X can be read off the *direct limit* of $f^{-1}(\Sigma)$ over all such tropical charts f. Note that the support of the direct limit of all $f^{-1}(\Sigma)$ is the set of Abhyankar points of X. [Editors note: significant progress on this problem has now been made in recent work of Ducros.]

17. Skeletons without models. (Michael Temkin)

Problem: Find a construction of skeletons not depending on formal models. For example, can one extend the method of Hrushovski-Loeser to affinoid spaces?

18. Existence of log-smooth formal models. (Michael Temkin)

Problem: Assuming that K has residue characteristic zero and X is smooth, show that there exists a log-smooth formal model for X.

This is related to the weak semistable reduction theorem of Abramovich and Karu.

19. Complexity of moduli spaces and skeletons. (Jenia Tevelev)

The problem, stated informally, is to try to bound the complexity of compact moduli spaces of canonically polarized varieties as being at least the complexity of the parameter spaces of their canonical skeletons. 1 If X is a smooth proper variety over $\mathbb{C}((t))$ with K_X ample, then after a finite base change one can find a model whose central fiber X_0 has semi-log-canonical singularities and such that ω_{X_0} is ample. By work of Kollár, Shephard-Barron, and others there exists a proper moduli space for such objects.

More generally, one can work with pairs (X, D) with $K_X + D$ ample. Some interesting examples are:

- $X = \mathbf{P}^2$ and $D = L_1 + \cdots + L_n$ where the L_i are general lines.
- X is a del Pezzo surface and D is a line.

In these cases the skeleton of X can be computed via tropicalization and the moduli space admits a stratification essentially by combinatorial types of skeleta (Hacking–Keel–Tevelev). This generalizes the well-known fact that $\bar{M}_{g,n}$ has a stratification by combinatorial types of dual graphs of genus g with n infinite legs (skeleta of punctured curves).

Problem: One would like to understand the case where there is no boundary divisor D. In particular, one would like to have a "canonical skeleton".

There is a "naive canonical skeleton" which is the union of the Kontsevich–Soibelman skeleta $\mathrm{Sk}(\omega)$ as ω varies over all holomorphic pluri-canonical forms (this construction was worked out by Nicaise–Xu after the workshop). However, this does not suffice, because the pluri-canonical skeleton does not seem to detect degenerations with normal special fiber X_0 (which then have isolated log terminal singularities, so called T-singularities). Presence of these degenerations is the main difference between dimensions 1 and 2. It would be nice to upgrade the definition of the pluri-canonical skeleton to detect these degenerations.

20. Explore relationships between tropicalizations and extended skeleta in higher dimensions. (Annette Werner)

For example, try to generalize Theorem 6.22 of the paper *Nonarchimedean geometry, tropicalization, and metrics on curves* by Baker, Payne, and Rabinoff (`http://arxiv.org/abs/1104.0320`, version 2). The first step in this program is to define an *extended skeleton* associated to a pair (X, D) where X is a d-dimensional smooth variety over K and D is a simple normal crossings divisor. (More generally, Antoine Ducros suggested that there should be a natural skeleton associated to any log-smooth formal K°-scheme with log structure.)

Here is an outline of how this might go. The following is based on discussions with Walter Gubler. Suppose K is a discretely valued complete non-Archimedean field and that $\mathfrak{X}$ is a strictly semistable model over K°. Let D be a divisor on $\mathfrak{X}$ containing the special fiber such that locally (on the formal scheme) the pair $(X^{\mathrm{an}}, D^{\mathrm{an}})$ looks like

$$\mathrm{Spf} K^\circ\langle x_0, \ldots, x_d\rangle/(x_0 \cdots x_r - \pi)$$

for some $r \leq d$ with D defined by $\prod_{i=0}^{r+s} x_i = 0$. To such a pair, we associate the extended skeleton

$$S(\mathfrak{X}, D) := \{(u_0, \ldots, u_{r+s+1}) \in \mathbb{R}_+^{r+s+1} \ : \ u_0 + \cdots + u_r = v(\pi)\}.$$

This skeleton embeds in the analytic space and the construction glues. Probably there will be a deformation retraction onto the extended skeleton.

Now assume that $\mathcal{U} := \mathfrak{X} \backslash D$ is very affine, so we get an embedding $\varphi : \mathcal{U} \hookrightarrow T$ into a torus. There is a corresponding map $S(\mathfrak{X}, D) \to \mathrm{trop}_{\varphi}(\mathcal{U})$.

Problem: Under what hypotheses does one get a homeomorphism, and even better an isomorphism of integer-affine structures?