SUMMARY OF THE PROBLEM SESSIONS

MATT BAKER AND SAM PAYNE

At the 2017 Simons Symposium on Tropical and Non-Archimedean Geometry, there were two “Open Problem Sessions” in which participants were encouraged to come to the blackboard and describe a problem that they would like to see solved or better understood. The purpose of these sessions was to help shape the future development of tropical and non-Archimedean geometry by collecting suggestions from top experts in these fields. What follows is an attempt to capture some of the essential content of those sessions.

Session 1

Mattias Jonsson

Prove the Kontsevich-Soibelman conjecture.

Suppose we're given a family $X \to \mathbb{D}^*$ of Calabi-Yau varieties (meaning that the relative canonical bundle of the family is trivial), together with a relatively ample line bundle L on X. For each $t \in \mathbb{D}^*$, Yau's theorem furnishes us with a Kähler metric $\omega_t \in c_1(L_t)$ which turns X_t into a metric space (X_t, d_t), where d_t is normalized so that X_t has diameter 1. On the other hand, the family also induces a Kontsevich-Soibelman skeleton $\mathrm{Skel}(X)$, a simplicical complex that is naturally a subset of the Berkovich space over $\mathbb{C}((t))$ induced by X. The conjecture is that if the family is maximally degenerate (which means that the skeleton has dimension equal to the fibers of X), then (X_t, d_t) converges in the Gromov–Hausdorff metric to $(\mathrm{Skel}(X), d_{\mathrm{Skel}(X)})$, where $d_{\mathrm{Skel}(X)}$ is a real Calabi-Yau type metric on the skeleton: it comes from the Hessian of a convex function on the skeleton whose Monge-Ampère measure is equal to Lebesgue measure normalized by the integral affine structure.

Yuri Tschinkel

K3 surfaces and Kulikov models

Develop a theory of Kulikov models in residue characteristic p (especially mixed characteristic).

More precisely: given a family X of K3 surfaces over a discretely valued field K, find a semistable model over the valuation ring K° of K whose dual complex is a point, interval, or sphere.

Johannes Nicaise commented that this would follow from semistable reduction for K3 surfaces, which Michael Temkin says may be doable.

Yifeng Liu

Tropical cohomology of differential forms

Let K be an algebraically closed complete non-trivially valued non-archimedean field whose residue field $\widetilde{K}$ is algebraic over a finite field. Let X/K be a smooth

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proper scheme of dimension n. We have natural maps

$$\mathrm{H}^{p,q}_{\mathrm{trop}}(X) \times \mathrm{H}^{n-p,n-q}_{\mathrm{trop}}(X) \to \mathrm{H}^{n,n}_{\mathrm{trop}}(X) \to \mathbb{R},$$

where the last map is given by the integration. (Here, the tropical cohomology groups have real coefficients.)

Question 1: Is this a perfect pairing of finite-dimensional $\mathbb{R}$-vector spaces?

Alternately, instead of restricting $\widetilde{K}$, we could try to make further assumptions about X. For example, what if we assume that every point of X^{an} has an open neighborhood isomorphic to an open subset of $(\mathbb{P}^n)^{\mathrm{an}}$?

Or what if X has a semistable integral model where all strata are rational?

Question 2: Is there a good notion of harmonic representatives for elements of $\mathrm{H}^{p,q}_{\mathrm{trop}}(X)$?

When $q = 0$, one can define harmonic to mean that $\mathrm{d}'\omega = \mathrm{d}''\omega = 0$, in which case existence of harmonic representatives is equivalent to the natural monodromy map $\mathrm{H}^{p,0}_{\mathrm{trop}}(X) \to \mathrm{H}^{0,p}_{\mathrm{trop}}(X)$ induced by "flipping forms" being an isomorphism.

Annette Werner asked if there is a Laplacian operator in this context.

Kristin Shaw mentioned that in the tropical setting, there are challenges to constructing a Hodge-$\star$ operator.

Kristin Shaw

Tropical versus Dolbeault cohomology

In what situations is the tropical cohomology of a tropicalization isomorphic to the Dolbeault cohomology of superforms on an algebraic variety over K? Is it enough that a tropicalization is smooth?

Antoine Chambert-Loir

Tropical cohomology of currents

What can we say about the analogous cohomology of currents instead of forms?

There is a natural map $H^{p,q}_{\mathrm{trop}}(X) \to H^{q,p}_{\mathrm{curr}}(X)$; what can one say about it?

Another possible cohomology theory worth investigating would be "Bott–Chern cohomology", where we replace d' by $d'd''$, or more precisely:

$$\frac{\ker d'd''}{\mathrm{im} d' + \mathrm{im} d''}$$

How does this relate to the "usual" tropical cohomology? (Apparently Philipp Jell is working on this question...)

Annette Werner

Non-archimedean Kähler manifolds

What is the right analogue of the Kähler condition in non-Archimedean analytic geometry?

Antoine Ducros

Non-archimedean $\omega \wedge \bar{\omega}$

Given a global section $\omega \in H^0(X, \Omega^n_X)$ of the canonical sheaf on an n-dimensional complex manifold, we can define a natural (n,n)-form $\omega \wedge \bar{\omega}$. What is the analogue of this (n,n)-form if X is replaced by a smooth K-analytic space of dimension n, where K is a complete non-archimedean field?

Tony Yue Yu

Gromov compactness of spaces of stable maps

Is the space of stable maps with bounded area into a K-analytic space X compact if X is Kähler in some appropriate sense? This would be a non-archimedean analogue of Gromov compactness, and pondering this question might be useful for figuring out what the non-Archimedean definition of "Kähler" ought to be.

Ilia Itenberg

Refined tropical curve counting

It would be nice to better understand refined curve counting (in the sense of Block–Göttsche) for plane tropical curves. What do the coefficients of the resulting Laurent polynomials **mean** in terms of enumerating curves in the sense of Gromov–Witten theory?

Johannes Nicaise

Generic schönness

Let K be a non-archimedean field of residue characteristic 0.

Question: Is every algebraic variety X/K generically schön?

In other words: does there exist a non-empty open set $U \subseteq X$ and an embedding $U \hookrightarrow \mathbb{G}_m^n$ such that $\mathrm{Trop}(U)$ is schön? (Schön means that $\mathrm{in}_w(X)$ is smooth for all $w \in \mathbb{R}^n$.)

This is true for K discretely valued by the work of Luxton–Qu, using semistable reduction. What about the general case?

Michael Temkin

Polystable toroidal modifications

Given an algebraically closed complete non-archimedean field K, a scheme X/K, and a log-smooth formal model $\mathfrak{X}/K^\circ$, is there a toroidal modification of $\mathfrak{X}$ which is polystable (locally the product of semistable models)?

This is equivalent to the following combinatorial problem: given a rational polyhedral complex, does there exist a subdivision into rational polyhedra which are products of simplices?

If not, what is a reasonable kind of subdivision which we can hope to achieve?

Vladimir Berkovich

Exactness of the p-adic de Rham complex

On a smooth p-adic analytic space (i.e., a smooth K-analytic space with $K \subseteq \mathbb{C}_p$), Berkovich constructed a sheaf of algebras $\mathcal{S}_X \supset \mathcal{O}_X$ which gives rise to a de Rham complex

$$\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{S}_X.$$

Berkovich showed that this complex is exact in degrees 0 and 1.

Question: Is it exact in all degrees?

Session 2

Melody Chan

Homotopy type of $\Delta_{g,n}$

Question 1: Is the boundary complex $\Delta_{g,n}$ of $\mathcal{M}_{g,n}$ simply connected, for $g > 1$? It is simply connected (and homotopic to a wedge of spheres) for $g = 0$ and 1.

Question 2: Is the reduced homology with integer coefficients $\widetilde{H}_*(\Delta_{g,n};\mathbb{Z})$ supported in the top g degrees, for $g > 2$? It is supported in the top g degrees for $g = 0$, 1, and 2. Also, by Harer's computation of the virtual cohomological dimension of the mapping class group, the reduced homology with rational coefficients $\widetilde{H}_*(\Delta_{g,n};\mathbb{Q})$ is supported in the top g degrees, for all g.

Andrew Obus

Differential Hurwitz trees

Question 1: Can one give an explicit definition of the differential Hurwitz tree obstruction for as many groups as possible?

Question 2: Does every differential Hurwitz tree lift to characteristic zero? For cyclic groups, every differential Hurwitz tree does lift to characteristic zero.

Dustin Cartwright

Fundamental groups of nonarchimedean analytic spaces

Is every finitely presented group the topological fundamental group of some smooth and proper nonarchimedean analytic space? Can one find an example of a smooth and projective algebraic variety X such that $\pi_1^{\mathrm{top}}(X^{\mathrm{an}})$ does not contain a finite index subgroup which is a product of free groups?

Jérôme Poineau

Tameness of hybrid analytic spaces

In the hybrid setting, as presented in Mattias Jonsson's talk, is the Berkovich analytification X^{An} locally contractible? Does it have the homotopy type of a finite simplicial complex? Is compactness equivalent to sequential compactness for X^{An}? Is it angelic?

The same questions are also open for Berkovich analytifications of schemes of finite type over $\mathbb{Z}$.

Michael Temkin

Moduli for morphisms of stable curves

Is there a reasonable theory of moduli spaces for morphisms between stable curves? Assume that the ramification locus of such a morphism (between two families of stable curves over a fixed base scheme S) is étale over the base. If so, what geometric properties does this moduli space have, e.g. does it have a nice smooth open set with a modular toroidal compactification? The answer is yes when all of the ramification is tame, by work of Romagny on stacks of admissible covers. One should also look for relations between differential data and coordinates on this space. For instance, is the different function the norm of some coordinate?

Stefan Wewers

Extending finite covers

Given a finite cover of an open disc, can we find necessary and sufficient conditions to extend this to a finite cover of smooth algebraic curves? Finiteness of the number of branch points is necessary. There are examples over non-discretely valued fields where the domain has infinitely many type 2 points of positive genus (see lecture notes of Ducros on étale cohomology), so this should be excluded as well. It is unclear whether this latter obstruction can occur over discretely valued fields. Are these finiteness conditions sufficient? Does it help if you can extend the ground field?

<u>Matt Baker</u>

Tropical geometric class field theory

In algebraic geometry, geometric class field theory gives a natural bijection between finite abelian covers of curves and isogenies of Jacobians. In tropical geometry, the same is true for degree 2 covers, by recent work of David Jensen and Yoav Len. Can their results be extended to higher degree covers? Can you make sense of the fiber product of the Abel-Jacobi map with an isogeny, in tropical geometry? Topological fiber products are not enough; the relevant maps of tropical curves are not just topological covers, we must also allow metric expansion.

<u>Eric Katz</u>

Dual graphs for morphisms of curves with étale branch locus

Let $f : X \to Y$ be a morphism of curves over a DVR with horizontal branch locus B that is étale over the base. Which combinatorial types of marked dual graphs are possible? What if Y is smooth over the base and X is smooth over the generic point?

Andrew Obus remarks that the case of Galois covers is most important, and the general case is probably intractable. Also, the tame case is undersood.

<u>Antoine Ducros</u>

Nonarchimedean Kodaira embedding

Is there a good notion of (strict) positivity for $(1, 1)$-currents (in the sense of Chambert-Loir and Ducros) such that a line bundle L on a proper analytic space X is ample if (and only if) it can be endowed with a metric whose curvature is positive?

<u>Tony Yue Yu</u>

Balancing conditions

Let $f : X \to Y$ be a morphism of analytic spaces coming from a morphism of SNC models or pairs. What combinatorial constraints apply to the induced map on skeletons? The special case where X is a curve and Y is semistable was resolved "a long time ago" (in Tony Yue Yu's thesis), but the general case is open and interesting.

<u>Annette Werner</u>

Slope formulas in higher dimensions

What is the relation between slope formulas for $\log|f|$, as in the work of Gubler, Rabinoff, and Werner, and the Poincaré-Lelong formula of Chambert-Loir and Ducros? These agree for curves but look completely different in higher dimensions.