THE NON-ABELIAN BRILL-NOETHER DIVISOR ON $\overline{\mathcal{M}}_{13}$ AND THE KODAIRA DIMENSION OF $\overline{\mathcal{R}}_{13}$

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ABSTRACT. The paper is devoted to highlighting several novel aspects of the moduli space of curves of genus 13, the first genus g where phenomena related to $K3$ surfaces no longer govern the birational geometry of $\overline{\mathcal{M}}_g$. We compute the class of the non-abelian Brill-Noether divisor on $\overline{\mathcal{M}}_{13}$ of curves that have a stable rank 2 vector bundle with many sections. This provides the first example of an effective divisor on $\overline{\mathcal{M}}_g$ with slope less than $6+\frac{10}{g}$. Earlier work on the Slope Conjecture suggested that such divisors may not exist. The main geometric application of our result is a proof that the Prym moduli space $\overline{\mathcal{R}}_{13}$ is of general type. Among other things, we also prove the Bertram-Feinberg-Mukai and the Strong Maximal Rank Conjectures on $\overline{\mathcal{M}}_{13}$.

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1. INTRODUCTION

One of the defining achievements of modern moduli theory is the result of Harris, Mumford and Eisenbud that $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$ [HM, EH2]. An essential step in their proof is the calculation of the class of the *Brill-Noether divisor* $\overline{\mathcal{M}}_{g,r}^d$ consisting of those curves X of genus g such that $G^r_d(X) \neq \emptyset$ in the case $\rho(g,r,d) := g-(r+1)(g-d+r) = -1$. Recall that the *slope* of an effective divisor D on $\overline{\mathcal{M}}_g$ not containing any of the boundary divisors Δ_i in its support is defined as the quantity $s(D) := a/\min_i b_i$, where $[D] = a\lambda - b_0\delta_0 - \cdots - b_{\lfloor \frac{g}{2} \rfloor}\delta_{\lfloor \frac{g}{2} \rfloor} \in CH^1(\overline{\mathcal{M}}_g)$. Eisenbud and Harris showed that the slope of $\overline{\mathcal{M}}_{g,r}^d$ is $\frac{a}{b_0} = 6+\frac{12}{g+1}$ [EH2]. After these seminal results from the 1980s, the fundamental question arose whether one can construct effective divisors D on $\overline{\mathcal{M}}_g$ of slope $s(D) < 6+\frac{12}{g+1}$ by using conditions defined in terms of *higher rank* vector bundles on curves.

Each effective divisor D on $\overline{\mathcal{M}}_g$ of slope $s(D) < 6+\frac{12}{g+1}$ must contain the locus $\mathcal{K}_g \subseteq \mathcal{M}_g$ of curves lying on a $K3$ surface [FP]. Since curves on $K3$ surfaces possess stable rank two vector bundles with canonical determinant and unexpectedly many sections [Laz, Mu2, Vo], it is then natural to focus on conditions defined in terms of rank two vector bundles with canonical determinant.

For a smooth curve X of genus g, let $SU_X(2,\omega)$ be the moduli space of semistable rank 2 vector bundles E on X with $\det(E) \cong \omega_X$. For $k \geq 0$, Bertram-Feinberg [BF1, Conjecture, p.2] and Mukai

[Mu2, Problem 4.8] conjectured that for a general curve X the rank 2 Brill-Noether locus

$$SU_X(2,\omega,k) := \{E \in SU_X(2,\omega_X) : h^0(X,E) \geq k\}$$

has dimension $\beta(2,g,k) := 3g-3-\binom{k+1}{2}$. For a general curve X the *Mukai-Petri map*

$$\mu_E\colon \mathrm{Sym}^2 H^0(X,E) \to H^0\big(X, \mathrm{Sym}^2(E)\big) \tag{1}$$

is injective for each $E \in SU_X(2,\omega)$ [T2]. As a consequence, $SU_X(2,\omega,k)$ has the expected dimension $\beta(2,g,k)$, if it is nonempty. There are numerous partial results on the non-emptiness of $SU_X(2,\omega,k)$ [LNP, T1, Zh], although still no proof in full generality.

Assume now $3g-3 = \binom{k+1}{2}$. Then generically $SU_X(2,\omega,k)$ consists of finitely many vector bundles, if it is nonempty. We consider the *non-abelian* Brill-Noether divisor $\mathcal{MP}_g$ on $\mathcal{M}_g$ consisting of curves $[X]$ for which there exists $E \in SU_X(2,\omega_X,k)$ such that the Mukai-Petri map μ_E is not an isomorphism. In this paper, we focus on the first genuinely interesting case[1]

$$g = 13 \quad \text{and} \quad k = 8.$$

Our first main result proves this case of the Bertram-Feinberg-Mukai Conjecture and computes the class of the closure of the non-abelian Brill-Noether divisor.

Theorem 1.1. *A general curve X of genus 13 carries precisely three stable vector bundles $E \in SU_X(2,\omega,8)$. The closure in $\overline{\mathcal{M}}_{13}$ of the non-abelian Brill-Noether divisor on $\mathcal{M}_{13}$*

$$\mathcal{MP}_{13} := \Big\{[X] \in \mathcal{M}_{13} : \exists E \in \mathcal{SU}_X(2,\omega,8) \textit{ with } \mu_E\colon \mathrm{Sym}^2 H^0(E) \stackrel{\ncong}{\to} H^0\big(\mathrm{Sym}^2(E)\big)\Big\}$$

has slope equal to

$$s\big([\overline{\mathcal{MP}}_{13}]\big) = \frac{4109}{610} = 6.735... < 6 + \frac{10}{13} = 6.769\ldots$$

To explain the significance of this result, we recall that several infinite series of examples of divisors on $\overline{\mathcal{M}}_g$ for $g \geq 10$ with slope less than $6+\frac{12}{g+1}$ have been constructed in [FP, F, Kh, FJP], using syzygies on curves. Quite remarkably, the slopes $s(D)$ of all these divisors D on $\overline{\mathcal{M}}_g$ satisfy

$$6 + \frac{10}{g} \leq s(D) < 6 + \frac{12}{g+1}.$$

The slope $6+\frac{12}{g+1}$ appears as both the slope of the Brill-Noether divisors $\overline{\mathcal{M}}^d_{g,r}$, and as the slope of a Lefschetz pencil of curves of genus g on a $K3$ surface. Similarly, $6+\frac{10}{g}$ is the slope of the family of curves $\{X'_t\}_{t\in\mathbb{P}^1}$ in $\Delta_0 \subseteq \overline{\mathcal{M}}_g$ obtained from a Lefschetz pencil $\{X_t\}_{t\in\mathbb{P}^1}$ of curves of genus $g-1$ on a $K3$ surface S by identifying two sections corresponding to base points of the pencil. The natural question has been therefore raised in [CFM, p.2], whether a slight weakening of the Harris-Morrison Slope Conjecture [HMo] remains true and the inequality

$$s(D) \geq 6 + \frac{10}{g} \tag{2}$$

holds for every effective divisor D on $\overline{\mathcal{M}}_g$. Results from [FP, Ta] imply that inequality (2) holds for all $g \leq 12$. In particular, the divisor $\overline{\mathcal{K}}_{10}$ on $\overline{\mathcal{M}}_{10}$ consisting of curves lying on $K3$ surfaces, which was shown in [FP] to be the original counterexample to the Slope Conjecture, satisfies $s(\overline{\mathcal{K}}_{10}) = 7 = 6+\frac{10}{g}$. On $\overline{\mathcal{M}}_{12}$, since a general curve of genus 11 lies on a $K3$ surface, it follows that the pencils $\{X'_t\}_{t\in\mathbb{P}^1}$ cover the boundary divisor $\Delta_0 \subseteq \overline{\mathcal{M}}_{12}$, and consequently the inequality (2) holds. Therefore 13 is the smallest genus where inequality (2) can be tested, and Theorem 1.1 provides a negative answer to the question posed in [CFM].

[1] It is left to the reader to show that in the previous cases $k = 5, 6$, the corresponding divisors $\mathcal{MP}_6$ and $\mathcal{MP}_8$ are supported on the loci, in $\mathcal{M}_6$ and $\mathcal{M}_8$ respectively, of curves failing the Petri Theorem.

1.1. **The Kodaira dimension of the Prym moduli space $\overline{\mathcal{R}}_{13}$.** One application of Theorem 1.1 concerns the birational geometry of the moduli space $\overline{\mathcal{R}}_g$ of Prym curves of genus g. The Prym moduli space $\mathcal{R}_g$ classifying pairs $[X,\eta]$, where X is a smooth curve of genus g and η is a 2-torsion point in $\mathrm{Pic}^0(X)$, has been classically used to parametrize moduli of abelian varieties via the Prym map $\mathcal{R}_g \to \mathcal{A}_{g-1}$ [B]. The Deligne-Mumford compactification $\overline{\mathcal{R}}_g$ is uniruled for $g \leq 8$ (see [FV] and references therein), and was previously known to be of general type for $g \geq 14$ and $g \neq 16$ [FL, Br].

Theorem 1.2. *The Prym moduli space $\overline{\mathcal{R}}_{13}$ is of general type.*

In particular, 13 is the smallest genus g for which it is known that $\overline{\mathcal{R}}_g$ is of general type. The proof of Theorem 1.2 takes full advantage of Theorem 1.1. It also uses the *universal theta divisor* Θ_{13}, defined as the locus of Prym curves $[X,\eta] \in \mathcal{R}_{13}$ for which there exists a vector bundle $E \in SU_X(2,\omega,8)$ such that $H^0(X, E\otimes\eta) \neq 0$. In an indirect way (to be explained later), we calculate the class $[\overline{\Theta}_{13}]$ of the closure of Θ_{13} inside $\overline{\mathcal{R}}_{13}$ and show that

$$K_{\overline{\mathcal{R}}_{13}} \in \mathbb{Q}_{>0}\Big\langle \lambda, [\overline{\Theta}_{13}], [\overline{D}_{13:2}], \text{ boundary divisors}\Big\rangle, \tag{3}$$

where $D_{13:2}$ is the effective divisor on $\mathcal{R}_{13}$ introduced in [FL] consisting of Prym curves $[X,\eta]$ for which η can be written as the difference of two effective divisors of degree 6 on X. Since λ is big, it follows that $K_{\overline{\mathcal{R}}_{13}}$ is also big. Theorem 1.2 follows, since the singularities of $\overline{\mathcal{R}}_g$ do not impose adjunction conditions [FL].

1.2. **The Strong Maximal Rank Conjecture on $\overline{\mathcal{M}}_{13}$.** The proofs of both Theorems 1.1 and 1.2 are indirect and proceed through a study of the failure locus of the Strong Maximal Rank Conjecture [AF] on $\overline{\mathcal{M}}_{13}$. For a general curve X of genus 13 the Brill-Noether locus $W^5_{16}(X)$ is 1-dimensional, and $W^6_{16}(X)=\emptyset$. Counting dimensions shows that the multiplication map

$$\phi_L\colon \mathrm{Sym}^2 H^0(X,L) \to H^0(X, L^{\otimes 2})$$

has at least a one-dimensional kernel, since $h^0\big(X, L^{\otimes 2}\big) = 2\deg(L)+1-g = 20$. The space of pairs $[X,L]$ such that $\mathrm{Ker}(\phi_L)$ is at least 2-dimensional therefore has expected codimension 2 in the parameter space $\mathfrak{G}^5_{16}$ of all such pairs $[X,L]$. Since the fibres of the map $\sigma\colon \mathfrak{G}^5_{16} \to \mathcal{M}_{13}$ are in general 1-dimensional, the push-forward of this locus is expected to be a divisor on $\mathcal{M}_{13}$.

Our next result verifies this case of the Strong Maximal Rank Conjecture and computes the class of the closure of the divisorial part of the failure locus. This is essential input for the calculation of the non-abelian Brill-Noether divisor class in Theorem 1.1 and hence for the proof of Theorem 1.2.

Theorem 1.3. *The locus of curves $[X] \in \mathcal{M}_{13}$ carrying a line bundle $L \in W^5_{16}(X)$ such that the multiplication map $\phi_L\colon \mathrm{Sym}^2H^0(X,L) \to H^0(X,L^{\otimes 2})$ is not surjective is a proper subvariety of $\mathcal{M}_{13}$, having a divisorial part $\mathfrak{D}_{13}$, whose closure closure in $\overline{\mathcal{M}}_{13}$ has slope*

$$s(\overline{\mathfrak{D}}_{13}) = \frac{5059}{749} = 6.754\ldots < 6 + \frac{10}{13}.$$

The proof of Theorem 1.3 takes full advantage of the techniques we developed in [FJP] in the course of our work on $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$. To that end, we split Theorem 1.3 in two parts.

We consider the proper moduli stack $\sigma\colon \widetilde{\mathfrak{G}}^5_{16} \to \widetilde{\mathfrak{M}}_{13}$, where $\widetilde{\mathfrak{M}}_{13}$ is a suitable moduli stack of tree-like curves of genus 13 equal to $\mathfrak{M}_{13} \cup \Delta_0 \cup \Delta_1$ in codimension one (see §2 for a precise definition). We then construct a morphism of two vector bundles over $\widetilde{\mathfrak{G}}^5_{16}$ globalizing the multiplication maps ϕ_L considered before. The degeneracy locus $\mathfrak{U}$ of this morphism, due to its determinantal nature, carries a virtual class $[\mathfrak{U}]^{\mathrm{virt}}$ of codimension 2 inside $\widetilde{\mathfrak{G}}^5_{16}$. Set

$$[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}} := \sigma_*\big([\mathfrak{U}]^{\mathrm{virt}}\big) \in CH^1(\widetilde{\mathcal{M}}_{13}).$$

Theorem 1.4. *The following relation for the virtual class $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$ holds:*

$$[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}} = 3\big(5059\,\lambda - 749\,\delta_0 - 3929\,\delta_1\big) \in CH^1(\widetilde{\mathcal{M}}_{13}).$$

That the degeneracy locus $\mathfrak{U}$ does not map onto $\mathcal{M}_{13}$ is a particular case of the *Strong Maximal Rank Conjecture* of [AF]. We prove this case, along with a stronger result that guarantees that the virtual class $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$ is effective, using tropical geometry. In particular, we use the method of tropical independence on chains of loops, as introduced in [JP1, JP2]. Our construction of the required tropical independences is similar to the one used in our proof that $\overline{\mathcal{M}}_{22}$ and $\overline{\mathcal{M}}_{23}$ are of general type, with one important innovation. In [FJP], we were able to ignore certain loops called lingering loops. Here, this seems impossible; there are too few non-lingering loops. This difficulty shows up already in the simplest combinatorial case, which we call the vertex avoiding case; for a discussion of how we resolve this difficulty, see Remarks 4.3 and 4.9.

Theorem 1.5. *For a general curve $[X] \in \mathcal{M}_{13}$ the map $\phi_L \colon \mathrm{Sym}^2 H^0(X,L) \to H^0(X, L^{\otimes 2})$ is surjective for all $L \in W^5_{16}(X)$. Furthermore, there is no component of the degeneracy locus $\mathfrak{U}$ mapping with positive dimensional fibres onto a divisor in $\widetilde{\mathcal{M}}_{13}$.*

Theorem 1.5 implies that $\widetilde{\mathfrak{D}}_{13}$, defined as the divisorial part of $\sigma(\mathfrak{U})$, represents the class $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$. Together with Theorem 1.4, this completes the proof of Theorem 1.3.

The existence of effective divisors of exceptionally small slope on $\overline{\mathcal{M}}_{13}$ has direct applications to the birational geometry of the moduli space $\overline{\mathcal{M}}_{g,n}$ of n-pointed stable curves of genus g.

Theorem 1.6. *The moduli space $\overline{\mathcal{M}}_{13,n}$ is of general type for $n \geq 9$.*

This improves on Logan's result that $\overline{\mathcal{M}}_{13,n}$ is of general type for $n \geq 11$ [Log]. It is known that $\overline{\mathcal{M}}_{13,n}$ is uniruled for $n \leq 4$; see [AB] and references therein.

1.3. **The divisor $\mathfrak{D}_{13}$ and rank two Brill-Noether loci.** The link between Theorems 1.1 and 1.3 involves a reinterpretation of the divisor $\mathfrak{D}_{13}$ in terms of rank 2 Brill-Noether theory. Let $\mathcal{SU}_{13}(2,\omega,8)$ denote the moduli space of pairs $[X,E]$, where $[X] \in \mathcal{M}_{13}$ and $E \in SU_X(2,\omega,8)$. Consider the forgetful map

$$\vartheta \colon \mathcal{SU}_{13}(2,\omega,8) \to \mathcal{M}_{13}, \quad [X,E] \mapsto [X].$$

We will show that ϑ is a generically finite map of degree 3 (Theorem 6.5) and that $\mathcal{SU}_{13}(2,\omega,8)$ is unirational (Corollary 6.3). The fact that $\overline{\mathcal{M}}_{13}$ possesses a modular cover ϑ of such small degree is surprising; we do not know of parallels for other moduli spaces $\overline{\mathcal{M}}_g$.

We now fix a pair $[X,E] \in \mathcal{SU}_{13}(2,\omega,8)$ and consider the determinant map

$$d \colon \bigwedge^2 H^0(X,E) \to H^0(X,\omega_X).$$

It turns out that for a general $[X,E]$ as above, E is globally generated and the map d is surjective. In particular, $\mathbb{P}\big(\mathrm{Ker}(d)\big) \subseteq \mathbb{P}\big(\bigwedge^2 H^0(X,E)\big) \cong \mathbb{P}^{27}$ is a 14-dimensional linear space. Since $h^0(X,\omega_X) = 2h^0(X,E) - 3$, it follows that the set of pairs $[X,E]$ satisfying the condition

$$\mathbb{P}\big(\mathrm{Ker}(d)\big) \cap G\big(2, H^0(X,E)\big) \neq \emptyset \tag{4}$$

(the intersection being taken inside $\mathbb{P}\big(\bigwedge^2 H^0(X,E)\big)$) is expected to be a divisor on $\mathcal{SU}_{13}(2,\omega,8)$, and its image under projection by the generically finite map ϑ is expected to be also a divisor on $\mathcal{M}_{13}$. We refer to this locus as the *resonance divisor* $\mathfrak{Res}_{13}$, inspired by the algebraic definition of the *resonance variety*, see [AFPRW, Definition 2.4] and references therein.

Theorem 1.7. *The closure of the resonance divisor in $\mathcal{M}_{13}$*

$$\mathfrak{Res}_{13} := \Big\{[X] \in \mathcal{M}_{13} : \exists E \in \mathcal{SU}_X(2,\omega,8) \text{ with } \mathbb{P}\big(\mathrm{Ker}(d)\big) \cap G\big(2, H^0(X,E)\big) \neq \emptyset\Big\}$$

is an effective divisor in $\overline{\mathcal{M}}_{13}$. One has the following equality of divisors on $\overline{\mathcal{M}}_{13}$

$$\overline{\mathfrak{Res}}_{13} = \overline{\mathfrak{D}}_{13} + 3 \cdot \overline{\mathcal{M}}^1_{13,7}.$$

Here, we recall that $\overline{\mathcal{M}}^1_{13,7}$ is the Hurwitz divisor of heptagonal curves on $\overline{\mathcal{M}}_{13}$ whose class is computed in [HM]. The set-theoretic inclusion $\mathcal{M}^1_{13,7} \subseteq \mathfrak{Res}_{13}$ is relatively straightforward. The multiplicity 3 with which $\overline{\mathcal{M}}^1_{13,7}$ appears in $\mathfrak{Res}_{13}$ is explained by an excess intersection calculation carried out in §7 and confirms once more that the degree of the map $\vartheta\colon \mathcal{SU}_{13}(2,\omega,8) \to \mathcal{M}_{13}$ is 3.

We conclude this introduction by explaining the connection between the resonance divisor $\mathfrak{Res}_{13}$ and Theorems 1.1 and 1.3. On the one hand, using [FR] the class $[\widetilde{\mathfrak{Res}}_{13}]$ of the closure of $\mathfrak{Res}_{13}$ in $\widetilde{\mathcal{M}}_{13}$ can be computed in terms of the generators of $CH^1(\widetilde{\mathcal{M}}_{13})$ and a tautological class $\vartheta_*(\gamma)$, where γ is the push-forward of the second Chern class of the (normalized) universal rank 2 vector bundle on the universal curve over a suitable compactification of $\mathcal{SU}_{13}(2,\omega,8)$; see Definition 7.3 for details. On the other hand, Theorem 1.7 yields an explicit description of $\widetilde{\mathfrak{Res}}_{13}$. By combining this description with Theorem 1.3, we obtain a *second* calculation for the class $[\widetilde{\mathfrak{Res}}_{13}]$. In this way, we indirectly determine the tautological class $\vartheta_*(\gamma)$, see Proposition 7.7. Once the class of $[\widetilde{\mathfrak{Res}}_{13}]$ is known, the calculation of the class of the non-abelian Brill-Noether divisor $[\widetilde{\mathcal{MP}}_{13}]$ (Theorem 1.1) and that of the universal Theta divisor $[\widetilde{\Theta}_{13}]$ on $\overline{\mathcal{R}}_{13}$ (Theorems 1.2 and 8.3) follow from Grothendieck-Riemann-Roch calculations, after checking suitable transversality assumptions.

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2. The failure locus of the Strong Maximal Rank Conjecture on $\widetilde{\mathcal{M}}_{13}$

We denote by $\overline{\mathfrak{M}}_g$ the moduli stack of stable curves of genus $g \geq 2$ and by $\overline{\mathcal{M}}_g$ the associated coarse moduli space. We work throughout over an algebraically closed field K of characteristic 0 and the Chow groups that we consider are with rational coefficients. Via the isomorphism $CH^*(\overline{\mathfrak{M}}_g) \cong CH^*(\overline{\mathcal{M}}_g)$, we routinely identify cycle classes on $\overline{\mathfrak{M}}_g$ with their push forward to $\overline{\mathcal{M}}_g$. Recall that for $g \geq 3$ the group $CH^1(\overline{\mathcal{M}}_g)$ is freely generated by the Hodge class λ and by the classes of the boundary divisors $\delta_i = [\Delta_i]$, for $i = 0, \ldots, \lfloor \frac{g}{2} \rfloor$.

In this section, we realize the virtual divisor class $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$ as the push forward of the virtual class of a codimension 2 determinantal locus inside the moduli space $\widetilde{\mathfrak{G}}^5_{16}$ of limit linear series of type $\mathfrak{g}^5_{16}$ over an open substack $\widetilde{\mathfrak{M}}_{13}$ of $\overline{\mathfrak{M}}_{13}$, that differs from $\mathfrak{M}_{13} \cup \Delta_0 \cup \Delta_1$ outside a subset of codimension 2. We will use standard terminology from the theory of limit linear series [EH1], and begin by recalling a few of the basics.

Definition 2.1. Let X be a smooth curve of genus g with $\ell = (L, V) \in G^r_d(X)$ a linear series. The *ramification sequence* of ℓ at a point $q \in X$ is denoted

$$\alpha^\ell(q) : \alpha^\ell_0(q) \leq \cdots \leq \alpha^\ell_r(q).$$

This is obtained from the *vanishing sequence* $a^\ell(q) : a^\ell_0(q) < \cdots < a^\ell_r(q) \leq d$ of ℓ at q, by setting $\alpha^\ell_i(q) := a^\ell_i(q) - i$, for $i = 0, \ldots, r$. The *ramification weight* of q with respect to ℓ is $\mathrm{wt}^\ell(q) := \sum_{i=0}^r \alpha^\ell_i(q)$. We define $\rho(\ell, q) := \rho(g, r, d) - \mathrm{wt}^\ell(q)$.

A *generalized limit linear series* on a tree-like curve X of genus g consists of a collection

$$\ell = \{(L_C, V_C) : C \text{ is a component of } X\},$$

where L_C is a rank 1 torsion free sheaf of degree d on C and $V_C \subseteq H^0(C, L_C)$ is an $(r+1)$-dimensional space of sections satisfying compatibility conditions on the vanishing sequences at the nodes of X, see [EH2, p. 364]. Let $\overline{G}^r_d(X)$ be the variety of generalized limit linear series of type $\mathfrak{g}^r_d$ on X.

In this section we set

$$g = 13,\ r = 5,\ d = 16. \tag{5}$$

Although we are mainly interested in the case $g = 13$, some of the constructions are set up for an arbitrary genus g, making it easier to refer to results from [FJP].

We denote by $\mathcal{M}^5_{13,15}$ the subvariety of $\mathcal{M}_{13}$ parametrizing curves X such that $W^5_{15}(X) \neq \emptyset$. As explained in [FJP, §3], we have $\mathrm{codim}(\mathcal{M}^5_{13,5}, \mathcal{M}_{13}) \geq 2$.

Let $\Delta_1^\circ \subseteq \Delta_1 \subseteq \overline{\mathcal{M}}_g$ be the locus of curves $[X \cup_y E]$, where X is a smooth curve of genus $g-1$ and $[E, y] \in \overline{\mathcal{M}}_{1,1}$ is an arbitrary elliptic curve. The point of attachment $y \in X$ is chosen arbitrarily. Furthermore, let $\Delta_0^\circ \subseteq \Delta_0 \subseteq \overline{\mathcal{M}}_g$ be the locus of curves $[X_{yq} := X/y \sim q] \in \Delta_0$, where $[X, q]$ is a smooth curve of genus $g-1$ and $y \in X$ is an arbitrary point, together with their degenerations $[X \cup_q E_\infty]$, where E_∞ is a rational nodal curve (that is, E_∞ is a nodal elliptic curve and $j(E_\infty) = \infty$). Points of this form comprise the intersection $\Delta_0^\circ \cap \Delta_1^\circ$. We define the following open subset of $\overline{\mathcal{M}}_g$:

$$\overline{\mathcal{M}}_g^\circ := \mathcal{M}_g \cup \Delta_0^\circ \cup \Delta_1^\circ.$$

Along the lines of [FJP, §3], we introduce an even smaller open subspace of $\overline{\mathcal{M}}_g$ over which the calculation of $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$ can be completed. Let $\mathcal{T}_0 \subset \Delta_0^\circ$ be the locus of curves $[X_{yq} := X/y \sim q]$ where either $\overline{G}^{r+1}_d(X) \neq \emptyset$ or $\overline{G}^r_{d-2}(X) \neq \emptyset$. Similarly, let $\mathcal{T}_1 \subseteq \Delta_1^\circ$ be the locus of curves $[X \cup_y E]$, where X is a smooth curve of genus $g-1$ such that $G^{r+1}_d(X) \neq \emptyset$ or $G^r_{d-2}(X) \neq \emptyset$. We set

$$\widetilde{\mathcal{M}}_g := \overline{\mathcal{M}}_g^\circ \setminus \left(\overline{\mathcal{M}}^r_{g,d-1} \cup \mathcal{T}_0 \cup \mathcal{T}_1\right).$$

We define $\widetilde{\Delta}_0 := \widetilde{\mathcal{M}}_g \cap \Delta_0 \subseteq \Delta_0^\circ$ and $\widetilde{\Delta}_1 := \widetilde{\mathcal{M}}_g \cap \Delta_1 \subseteq \Delta_1^\circ$. Note that $\widetilde{\mathcal{M}}_g$ and $\mathcal{M}_g \cup \Delta_0 \cup \Delta_1$ agree away from a set of codimension 2 in each. We identify $CH^1(\widetilde{\mathcal{M}}_g) \cong \mathbb{Q}\langle \lambda, \delta_0, \delta_1 \rangle$, where λ is the Hodge class, $\delta_0 := [\widetilde{\Delta}_0]$ and $\delta_1 := [\widetilde{\Delta}_1]$.

2.1. **Stacks of limit linear series.** Let $\widetilde{\mathfrak{G}}^r_d$ be the stack of pairs $[X, \ell]$, where $[X] \in \widetilde{\mathcal{M}}_g$ and ℓ is a (generalized) limit linear series of type g^r_d on the tree-like curve X. We consider the proper projection

$$\sigma\colon \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathfrak{M}}_g.$$

Over a curve $[X \cup_y E] \in \widetilde{\Delta}_1$, we identify $\sigma^{-1}([X \cup_y E])$ with the variety of (generalized) limit linear series $\ell = (\ell_X, \ell_E) \in \overline{G}^r_d(X \cup_y E)$. The fibre $\sigma^{-1}([X_{yq}])$ over an irreducible curve $[X_{yq}] \in \widetilde{\Delta}_0 \setminus \widetilde{\Delta}_1$, is canonically identified with the variety $\overline{W}^r_d(X_{yq})$ of rank 1 torsion free sheaves L on X_{yq} having degree $d(L) = d$ and $h^0(X_{yq}, L) \geq r+1$.

Let $\widetilde{\mathfrak{C}}_g \to \widetilde{\mathfrak{M}}_g$ be the universal curve, and let $p_2\colon \widetilde{\mathfrak{C}}_g \times_{\widetilde{\mathfrak{M}}_g} \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathfrak{G}}^r_d$ be the projection map. We denote by $\mathfrak{Z} \subseteq \widetilde{\mathfrak{C}}_g \times_{\widetilde{\mathfrak{M}}_g} \widetilde{\mathfrak{G}}^r_d$ the codimension 2 substack consisting of pairs $[X_{yq}, L, z]$, where $[X_{yq}] \in \Delta_0^\circ$, the point z is the node of X_{yq} and $L \in \overline{W}^r_d(X_{yq}) \setminus W^r_d(X_{yq})$ is a *non-locally free* torsion free sheaf. Let

$$\epsilon\colon \widehat{\mathfrak{C}}_g := \mathrm{Bl}_{\mathfrak{Z}}\left(\widetilde{\mathfrak{C}}_g \times_{\widetilde{\mathfrak{M}}_g} \widetilde{\mathfrak{G}}^r_d\right) \to \widetilde{\mathfrak{C}}_g \times_{\widetilde{\mathfrak{M}}_g} \widetilde{\mathfrak{G}}^r_d$$

be the blow-up of this locus, and we denote the induced universal curve by

$$\wp := p_2 \circ \epsilon : \widehat{\mathfrak{C}}_g \to \widetilde{\mathfrak{G}}^r_d.$$

The fibre of $\wp$ over a point $[X_{yq}, L] \in \widetilde{\Delta}_0$, where $L \in \overline{W}^r_d(X_{yq}) \setminus W^r_d(X_{yq})$, is the semistable curve $X \cup_{\{y,q\}} R$ of genus g, where R is a smooth rational curve meeting X transversally at y and q.

2.2. **A degeneracy locus inside $\widetilde{\mathfrak{G}}^5_{16}$.** In order to define the degeneracy locus on $\widetilde{\mathfrak{G}}^5_{16}$ whose push-forward produces $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$, we first choose a Poincaré line bundle $\mathcal{L}$ over the universal curve $\widehat{\mathfrak{C}}_g$ with the following properties:

(i) If $[X\cup_y E]\in\widetilde{\Delta}_1$ and $\ell=(\ell_X,\ell_E)\in\overline{G}^r_d(X\cup E)$ is a limit linear series, then
$$\mathcal{L}_{|[X\cup_y E,\ell]}\in \mathrm{Pic}^d(X)\times\mathrm{Pic}^0(E).$$

(ii) For a point $t=[X_{yq},L]$, where $[X_{yq}]\in\widetilde{\Delta}_0$ and $L\in\overline{W}^r_d(X_{yq})\setminus W^r_d(X_{yq})$, thus $L=\nu_*(A)$ for some $A\in W^r_{d-1}(X)$, we have $\mathcal{L}_{|X}\cong A$ and $\mathcal{L}_{|R}\cong\mathcal{O}_R(1)$. Here, $\wp^{-1}(t)=X\cup R$, whereas $\nu\colon X\to X_{yq}$ is the normalization map.

We now introduce the following two sheaves over $\widetilde{\mathfrak{G}}^r_d$
$$\mathcal{E}:=\wp_*(\mathcal{L})\ \text{ and }\ \mathcal{F}:=\wp_*(\mathcal{L}^{\otimes 2}).$$
Both $\mathcal{E}$ and $\mathcal{F}$ are locally free; the proof by local analysis in [FJP, Proposition 3.6] goes through essentially without change.

There is a sheaf morphism over $\widetilde{\mathfrak{G}}^5_{16}$ globalizing the multiplication of sections
$$\phi\colon \mathrm{Sym}^2(\mathcal{E})\to\mathcal{F}. \tag{6}$$
We denote by $\mathfrak{U}\subseteq\widetilde{\mathfrak{G}}^5_{16}$ the locus where ϕ is not surjective (equivalently, where $\phi^\vee$ is not injective). Due to its determinantal nature, $\mathfrak{U}$ carries a virtual class in the expected codimension 2.

Definition 2.2. We define the virtual divisor class $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}:=\sigma_*([\mathfrak{U}]^{\mathrm{virt}})$ as
$$[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}:=\sigma_*\Big(c_2(\mathrm{Sym}^2(\mathcal{E})^\vee-\mathcal{F}^\vee)\Big)\in CH^1(\widetilde{\mathfrak{M}}_{13}).$$

If $\mathfrak{U}$ has pure codimension 2, then $\widetilde{\mathfrak{D}}_{13}$ is a divisor on $\widetilde{\mathcal{M}}_{13}$ and $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}=[\widetilde{\mathfrak{D}}_{13}]$. The following corollary provides a local description of the morphism ϕ.

Corollary 2.3. *The morphism $\phi\colon\mathrm{Sym}^2(\mathcal{E})\to\mathcal{F}$ has the following description on fibres:*
(i) For $[X,L]\in\widetilde{\mathfrak{G}}^r_d$, with $[X]\in\mathcal{M}_g\setminus\mathcal{M}^r_{g,d-1}$ smooth, the fibres are
$$\mathcal{E}_{(X,L)}=H^0(X,L)\ \textit{and}\ \mathcal{F}_{(X,L)}=H^0(X,L^{\otimes 2}),$$
and $\phi_{(X,L)}:\mathrm{Sym}^2H^0(X,L)\to H^0(X,L^{\otimes 2})$ is the usual multiplication map of global sections.

(ii) Suppose $t=(X\cup_y E,\ell_X,\ell_E)\in\sigma^{-1}(\widetilde{\Delta}_1)$, where X is a curve of genus $g-1$, E is an elliptic curve and $\ell_X=|L_X|$ is the X-aspect of the corresponding limit linear series with $L_X\in W^r_d(X)$ such that $h^0(X,L_X(-2y))\geq r$. If L_X has no base point at y, then
$$\mathcal{E}_t=H^0(X,L_X)\cong H^0\big(X,L_X(-2y)\big)\oplus K\cdot u\ \textit{and}\ \mathcal{F}_t=H^0\big(X,L_X^{\otimes 2}(-2y)\big)\oplus K\cdot u^2,$$
where $u\in H^0(X,L_X)$ is any section such that $\mathrm{ord}_y(u)=0$.

If L_X has a base point at y, then $\mathcal{E}_t=H^0(X,L_X)\cong H^0(X,L_X(-y))$ and the image of $\mathcal{F}_t\to H^0(X,L_X^{\otimes 2})$ is the subspace $H^0\big(X,L_X^{\otimes 2}(-2y)\big)\subseteq H^0(X,L_X^{\otimes 2})$.

(iii) Let $t=[X_{yq},L]\in\sigma^{-1}(\widetilde{\Delta}_0)$ be a point with $q,y\in X$ and let $L\in W^r_d(X_{yq})$ be a locally free sheaf of rank 1, such that $h^0(X,\nu^*L(-y-q))\geq r$, where $\nu\colon X\to X_{yq}$ is the normalization. Then one has the following description of the fibres:
$$\mathcal{E}_t=H^0(X,\nu^*L)\ \textit{and}\ \mathcal{F}_t=H^0\big(X,\nu^*L^{\otimes 2}(-y-q)\big)\oplus K\cdot u^2,$$
where $u\in H^0(X,\nu^*L)$ is any section not vanishing at both points y and q.

(iv) Let $t=[X_{yq},\nu_*(A)]$, where $A\in W^r_{d-1}(X)$ and set again $X\cup_{\{y,q\}}R$ to be the fibre $\wp^{-1}(t)$. Then $\mathcal{E}_t=H^0(X\cup R,\mathcal{L}_{X\cup R})\cong H^0(X,A)$ and $\mathcal{F}_t=H^0(X\cup R,\mathcal{L}^{\otimes 2}_{X\cup R})$. Furthermore, $\phi(t)$ is the multiplication map on $X\cup R$.

Proof. The proof is essentially identical to the proof of [FJP, Corollary 3.8]; we omit the details. □

2.3. **Test curves in $\widetilde{\mathcal{M}}_{13}$.** As in [FJP], the calculation of $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$ is carried out by understanding the restriction of the morphism ϕ along the pull backs of the three standard test curves F_0, F_{ell} and F_1 inside $\widetilde{\mathcal{M}}_{13}$. Let $[X,q]$ be a general pointed curve of genus $g-1$ and fix an elliptic curve $[E,y]$. We then define

$$F_0 := \Bigl\{X_{yq} := X/y \sim q : y \in X\Bigr\} \subseteq \Delta_0^\circ \subseteq \overline{\mathcal{M}}_g^\circ \text{ and } F_1 := \Bigl\{X \cup_y E : y \in X\Bigr\} \subseteq \Delta_1^\circ \subseteq \overline{\mathcal{M}}_g^\circ.$$

Furthermore, we define the curve

$$F_{\mathrm{ell}} := \Bigl\{[X \cup_q E_t] : t \in \mathbb{P}^1\Bigr\} \subseteq \Delta_1 \subseteq \overline{\mathcal{M}}_g, \tag{7}$$

where $\{[E_t,q]\}_{t\in\mathbb{P}^1}$ denotes a pencil of plane cubics and q is a fixed point of the pencil. We record the intersection of these test curves with the generators of $CH^1(\overline{\mathcal{M}}_g)$:

$$F_0\cdot\lambda = 0, \qquad F_0\cdot\delta_0 = 2-2g, \qquad F_0\cdot\delta_1 = 1 \text{ and } \qquad F_0\cdot\delta_j = 0 \text{ for } j = 2,\ldots,\Bigl\lfloor\frac{g}{2}\Bigr\rfloor,$$
$$F_{\mathrm{ell}}\cdot\lambda = 1, \qquad F_{\mathrm{ell}}\cdot\delta_0 = 12, \qquad F_{\mathrm{ell}}\cdot\delta_1 = -1 \text{ and } \qquad F_{\mathrm{ell}}\cdot\delta_j = 0 \text{ for } j = 2,\ldots,\Bigl\lfloor\frac{g}{2}\Bigr\rfloor.$$

Note also that $F_1\cdot\lambda = 0,\ F_1\cdot\delta_i = 4-2g,\ $ and $F_1\cdot\delta_j = 0$ for $j\neq 1$.

We now describe the pull back $\sigma^*(F_0)\subseteq\widetilde{\mathfrak{G}}^5_{16}$. Having fixed a general pointed curve $[X,q]\in\overline{\mathcal{M}}_{12,1}$, we introduce the variety

$$Y := \Bigl\{(y,L)\in X\times W^5_{16}(X) : h^0(X, L(-y-q)) \geq 5\Bigr\}, \tag{8}$$

together with the projection $\pi_1\colon Y\to X$. Arguing in a way similar to [FJP, Proposition 3.10], we conclude that Y has pure dimension 2, that is, its actual dimension equals its expected dimension as a degeneracy locus. We consider two curves inside Y, namely

$$\Gamma_1 := \Bigl\{(y, A(y)) : y\in X,\ A\in W^5_{15}(X)\Bigr\} \text{ and}$$
$$\Gamma_2 := \Bigl\{(y, A(q)) : y\in X,\ A\in W^5_{15}(X)\Bigr\},$$

intesecting transversely along finitely many points. We then introduce the variety $\widetilde{Y}$ obtained from Y by identifying for each $(y,A)\in X\times W^5_{15}(X)$, the points $(y,A(y))\in\Gamma_1$ and $(y,A(q))\in\Gamma_2$. Let $\vartheta\colon Y\to\widetilde{Y}$ the projection map.

Proposition 2.4. *With notation as above, there is a birational morphism*

$$f\colon \sigma^*(F_0)\to\widetilde{Y},$$

which is an isomorphism outside $\vartheta(\pi_1^{-1}(q))$. The restriction of f to $f^{-1}\bigl(\vartheta(\pi_1^{-1}(q))\bigr)$ forgets the aspect of each limit linear series on the elliptic curve E_∞. Furthermore, both $\mathcal{E}_{|\sigma^*(F_0)}$ and $\mathcal{F}_{|\sigma^*(F_0)}$ are pull backs under f of vector bundles on $\widetilde{Y}$.

Proof. The proof is identical to the proof of [FJP, Proposition 3.11]. □

We now describe the pull back $\sigma^*(F_1)\subseteq\widetilde{\mathfrak{G}}^5_{16}$ and we define the locus

$$Z := \Bigl\{(y,L)\in X\times W^5_{16}(X) : h^0(X, L(-2y)) \geq 5\Bigr\}. \tag{9}$$

Because X is general, we find that Z is pure of dimension 2. Next we observe that in order to estimate the intersection of $[\widetilde{\mathfrak{D}}_{13}]$ with the surface $\sigma^*(F_1)$ it suffices to restrict ourselves to Z:

Proposition 2.5. *The variety Z is an irreducible component of $\sigma^*(F_1)$ and*

$$c_2\bigl(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\bigr)_{|\sigma^*(F_1)} = c_2\bigl(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\bigr)_{|Z}.$$

Proof. Let $(\ell_X, \ell_E) \in \sigma^{-1}([X \cup_y E])$ be a limit linear series. Observe that that $\rho(13,5,16) = 1 \geq \rho(\ell_X, y) + \rho(\ell_E, y)$. Since $\rho(\ell_E, y) \geq 0$, it follows thats $\rho(\ell_X, y) \in \{0,1\}$. If $\rho(\ell_E, y) = 0$, then $\ell_E = 10y + |\mathcal{O}_E(6y)|$ and the aspect $\ell_X \in G^5_{16}(X)$ is a complete linear series with a cusp at the point $y \in X$. Therefore $(y, \ell_X) \in Z$, and in particular $Z \times \{\ell_E\} \cong Z$ is a union of irreducible components of $\sigma^*(F_1)$.

The remaining components of $\sigma^*(F_1)$ are indexed by Schubert indices

$$\alpha := \bigl(0 \leq \alpha_0 \leq \ldots \leq \alpha_5 \leq 11 = 16 - 5\bigr),$$

such that $\alpha \geq (0,1,1,1,1,1)$ holds lexicographically, and $\alpha_0 + \cdots + \alpha_5 \in \{6,7\}$, for $\rho(\ell_X, y) \geq -1$, for any point $y \in X$, see also [F, Theorem 0.1]. For a Schubert index α satisfying these conditions, we let $\alpha^c := (11 - \alpha_5, \ldots, 11 - \alpha_0)$ be the complementary Schubert index, and define

$$Z_\alpha := \bigl\{(y, \ell_X) \in X \times G^5_{16}(X) : \alpha^{\ell_X}(y) \geq \alpha\bigr\} \text{ and } W_\alpha := \bigl\{\ell_E \in G^5_{16}(E) : \alpha^{\ell_E}(y) \geq \alpha^c\bigr\}.$$

Then the following relation holds for certain natural coefficients m_α:

$$\sigma^*(F_1) = Z + \sum_{\alpha \geq (0,1,1,1,1,1)} m_\alpha \bigl(Z_\alpha \times W_\alpha\bigr).$$

We now finish the proof by invoking the pointed Brill-Noether Theorem [EH2, Theorem 1.1], which gives $\dim Z_\alpha = 1 + \rho(12,5,16) - (\alpha_0 + \cdots + \alpha_5) \leq 1$. In the definition of the test curve F_1, the point of attachment $y \in E$ is fixed, therefore the restrictions of both $\mathcal{E}$ and $\mathcal{F}$ are pulled-back from Z_α and one obtains $c_2\bigl(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\bigr)_{|Z_\alpha \times W_\alpha} = 0$ for dimension reasons. □

2.4. **Top Chern numbers on Jacobians.** We use various facts about intersection theory on Jacobians, for which we refer to [ACGH, Chapters VII–VIII]. We start with a general curve X of genus g, fix a Poincaré line bundle $\mathcal{P}$ on $X \times \mathrm{Pic}^d(X)$ and denote by

$$\pi_1 \colon X \times \mathrm{Pic}^d(X) \to X \text{ and } \pi_2 \colon X \times \mathrm{Pic}^d(X) \to \mathrm{Pic}^d(X)$$

the two projections. Let $\eta = \pi_1^*([x_0]) \in H^2(X \times \mathrm{Pic}^d(X), \mathbb{Z})$, where $x_0 \in X$ is a fixed point. We choose a symplectic basis $\delta_1, \ldots, \delta_{2g} \in H^1(X, \mathbb{Z}) \cong H^1(\mathrm{Pic}^d(X), \mathbb{Z})$, and then consider the class

$$\gamma := -\sum_{\alpha=1}^{g} \Bigl(\pi_1^*(\delta_\alpha)\pi_2^*(\delta_{g+\alpha}) - \pi_1^*(\delta_{g+\alpha})\pi_2^*(\delta_\alpha)\Bigr) \in H^2(X \times \mathrm{Pic}^d(X), \mathbb{Z}).$$

One has $c_1(\mathcal{P}) = d \cdot \eta + \gamma$, and the relations $\gamma^3 = 0$, $\gamma\eta = 0$, $\eta^2 = 0$, and $\gamma^2 = -2\eta\pi_2^*(\theta)$, for which we refer to [ACGH, page 335]. Assuming $W^{r+1}_d(X) = \emptyset$ (which is what happens in the case $g = 12, r = 5, d = 16$ relevant to us), the smooth variety $W^r_d(X)$ admits a rank $r+1$ vector bundle

$$\mathcal{M} := (\pi_2)_*\Bigl(\mathcal{P}_{|X \times W^r_d(X)}\Bigr)$$

with fibres $\mathcal{M}_L \cong H^0(X, L)$, for $L \in W^r_d(X)$. The Chern numbers of $\mathcal{M}$ are computed via the Harris-Tu formula [HT]. We write formally

$$\sum_{i=0}^{r} c_i(\mathcal{M}^\vee) = (1 + x_1) \cdots (1 + x_{r+1}).$$

For a class $\zeta \in H^*\bigl(\mathrm{Pic}^d(X), \mathbb{Z}\bigr)$, the Chern number $c_{j_1}(\mathcal{M}) \cdots c_{j_s}(\mathcal{M}) \cdot \zeta \in H^{\mathrm{top}}(W^r_d(X), \mathbb{Z})$ can be computed by using repeatedly the following formal identities[2]:

$$x_1^{i_1} \cdots x_{r+1}^{i_{r+1}} \cdot \theta^{\rho(g,r,d) - i_1 - \cdots - i_{r+1}} = g!\, \frac{\prod_{j>k}(i_k - i_j + j - k)}{\prod_{k=1}^{r+1}(g - d + 2r + i_k - k)!}. \tag{10}$$

[2]See [FJP, § 4.1] for a detailed discussion of how to read and apply the Harris-Tu formula in this context.

We now specialize to the case when X is a general curve of genus 12, thus $W^5_{16}(X)$ is a smooth 6-fold. By Grauert's Theorem, $\mathcal{N} := (R^1\pi_2)_*\Big(\mathcal{P}_{|X\times W^5_{16}(X)}\Big)$ is locally free of rank one. Set $y_1 := c_1(\mathcal{N})$. We now explain how y_1 determine the Chern numbers of $\mathcal{M}$.

Proposition 2.6. *For a general curve X of genus 12 set $c_i := c_i(\mathcal{M}^\vee)$, for $i = 1, \ldots, 6$, and $y_1 := c_1(\mathcal{N})$. Then the following relations hold in $H^*(W^5_{16}(X), \mathbb{Z})$:*

$$c_i = \frac{\theta^i}{i!} - \frac{\theta^{i-1}}{(i-1)!}y_1, \ \text{ for } i = 1, \ldots, 6.$$

Proof. For an effective divisor D of sufficiently large degree on X, there is an exact sequence

$$0 \to \mathcal{M} \to (\pi_2)_*\Big(\mathcal{P}\otimes\mathcal{O}(\pi^*D)\Big) \to (\pi_2)_*\Big(\mathcal{P}\otimes\mathcal{O}(\pi_1^*D)_{|\pi_1^*D}\Big) \to R^1\pi_{2*}\Big(\mathcal{P}_{|X\times W^5_{16}(X)}\Big) \to 0.$$

Recall that $\mathcal{N}$ is the vector bundle on the right in the exact sequence above. By [ACGH, Chapter VII], we have $c_{\mathrm{tot}}\Big((\pi_2)_*(\mathcal{P}\otimes\mathcal{O}(\pi_1^*D))\Big) = e^{-\theta}$, and the total Chern class of $(\pi_2)_*\Big(\mathcal{P}\otimes\mathcal{O}(\pi_1^*D)_{|\pi_1^*D}\Big)$ is trivial. We therefore obtain the formula

$$(1+y_1)\cdot e^{-\theta} = 1 - c_1 + c_2 - \cdots + c_6,$$

as claimed. $\square$

Using Proposition 2.6, any Chern number on $W^5_{16}(X)$ can be expressed in terms of monomials in y_1 and θ. The following identity on $H^{12}(W^5_{16}(X), \mathbb{Z})$ follows from (10) using the canonical isomorphism $H^1(X, L) \cong H^0\big(X, \omega_X\otimes L^\vee\big)^\vee$.

$$\Big(\theta^i\cdot y_1^{6-i}\Big)_{W^5_{16}(X)} = \frac{\theta^{12}}{(12-i)!} = i!\binom{12}{i}. \tag{11}$$

With this preparation in place, we now compute the classes of the loci Y and Z.

Proposition 2.7. *Let $[X, q]$ be a general pointed curve of genus 12, let $\mathcal{M}$ denote the tautological rank 6 vector bundle over $W^5_{16}(X)$, and set $c_i = c_i(\mathcal{M}^\vee) \in H^{2i}(W^5_{16}(X), \mathbb{Z})$ as before. The following formulas hold:*

(i) $[Z] = \pi_2^*(c_5) - 6\eta\theta\pi_2^*(c_3) + (54\eta + 2\gamma)\pi_2^*(c_4) \in H^{10}\big(X\times W^5_{16}(X), \mathbb{Z}\big)$.

(ii) $[Y] = \pi_2^*(c_5) - 2\eta\theta\pi_2^*(c_3) + (15\eta + \gamma)\pi_2^*(c_4) \in H^{10}\big(X\times W^5_{16}(X), \mathbb{Z}\big)$.

Proof. The locus Z has been defined by (9) as the degeneracy locus of a vector bundle morphism over the 7-dimensional smooth variety $X\times W^5_{16}(X)$ (observe again that $W^6_{16}(X) = \emptyset$). For each $(y, L) \in X\times W^5_{16}(X)$, there is a natural map

$$H^0(X, L\otimes\mathcal{O}_{2y})^\vee \to H^0(X, L)^\vee.$$

These maps viewed together induce a morphism $\zeta\colon J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})^\vee$ of vector bundles. Then Z is the first degeneracy locus of ζ and applying the Porteous formula,

$$[Z] = c_5\Big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\Big).$$

The Chern classes of the jet bundle $J_1(\mathcal{P})$ are computed using the standard exact sequence

$$0 \longrightarrow \pi_1^*(\omega_X)\otimes\mathcal{P} \longrightarrow J_1(\mathcal{P}) \longrightarrow \mathcal{P} \longrightarrow 0.$$

We compute the total Chern class of the formal inverse of the jet bundle as follows:

$$\begin{aligned} c_{\mathrm{tot}}\big(J_1(\mathcal{P})^\vee\big)^{-1} &= \Big(\sum_{j\geq 0}(d(L)\eta+\gamma)^j\Big)\cdot\Big(\sum_{j\geq 0}\big((2g(X)-2+d(L))\eta+\gamma\big)^j\Big), \\ &= \big(1+16\eta+\gamma+\gamma^2+\cdots\big)\cdot\big(1+38\eta+\gamma+\gamma^2+\cdots\big), \\ &= 1+54\eta+2\gamma-6\eta\theta. \end{aligned}$$

Multiplying this with the total class of $\pi_2^*(\mathcal{M})^\vee$, one finds the claimed formula for $[Z]$.

To compute the class of Y defined in (8), we consider the projections

$$\mu, \nu\colon X \times X \times \mathrm{Pic}^{16}(X) \to X \times \mathrm{Pic}^{16}(X),$$

and let $\Delta \subseteq X \times X \times \mathrm{Pic}^{16}(X)$ be the diagonal. Set $\Gamma_q := \{q\} \times \mathrm{Pic}^{16}(X)$ and consider the vector bundle $\mathcal{B} := \mu_*\big(\nu^*(\mathcal{P}) \otimes \mathcal{O}_{\Delta+\nu^*(\Gamma_q)}\big)$. There is a morphism $\chi\colon \mathcal{B}^\vee \to (\pi_2)^*(\mathcal{M})^\vee$ of vector bundles over $X \times W^5_{16}(X)$ obtained as the dual of the evaluation map and the surface Y is realized as its degeneracy locus. Since we also have that

$$c_{\mathrm{tot}}(\mathcal{B}^\vee)^{-1} = \Big(1 + (d(L)\eta + \gamma) + (d(L)\eta+\gamma)^2 + \cdots\Big) \cdot \big(1-\eta\big) = 1 + 15\eta + \gamma - 2\eta\theta,$$

we find the stated expression for $[Y]$ and finish the proof. □

We introduce two further vector bundles which appear in many of our calculation. Their Chern classes are computed via Grothendieck-Riemann-Roch.

Proposition 2.8. *Let $[X, q]$ be a general pointed curve of genus 12 and consider the vector bundles $\mathcal{A}_2$ and $\mathcal{B}_2$ on $X \times \mathrm{Pic}^{16}(X)$ having fibres*

$$\mathcal{A}_{2,(y,L)} = H^0\big(X, L^{\otimes 2}(-2y)\big) \text{ and } \mathcal{B}_{2,(y,L)} = H^0\big(X, L^{\otimes 2}(-y-q)\big),$$

respectively. One then has the following formulas for their Chern classes:

$$c_1(\mathcal{A}_2) = -4\theta - 4\gamma - 86\eta, \qquad c_1(\mathcal{B}_2) = -4\theta - 2\gamma - 31\eta,$$
$$c_2(\mathcal{A}_2) = 8\theta^2 + 320\eta\theta + 16\gamma\theta, \qquad c_2(\mathcal{B}_2) = 8\theta^2 + 116\eta\theta + 8\theta\gamma.$$

Proof. We apply Grothendieck-Riemann-Roch to the projection map

$$\nu\colon X \times X \times \mathrm{Pic}^{16}(X) \to X \times \mathrm{Pic}^{16}(X).$$

Via Grauert's Theorem, $\mathcal{A}_2$ can be realized as a push forward under the map ν, precisely

$$\mathcal{A}_2 = \nu_!\Big(\mu^*\big(\mathcal{P}^{\otimes 2} \otimes \mathcal{O}_{X\times X\times \mathrm{Pic}^{16}(X)}(-2\Delta)\big)\Big) = \nu_*\Big(\mu^*\big(\mathcal{P}^{\otimes 2} \otimes \mathcal{O}_{X\times X\times \mathrm{Pic}^{16}(X)}(-2\Delta)\big)\Big).$$

Applying Grothendieck-Riemann-Roch to ν, we find $\mathrm{ch}_2(\mathcal{A}_2) = 8\eta\theta$, and $\nu_*(c_1(\mathcal{P})^2) = -2\theta$. One then obtains $c_1(\mathcal{A}_2) = -4\theta - 4\gamma - (4d(L) + 2g(C) - 2)\eta$, which yields the formula for $c_2(\mathcal{A}_2)$. To determine the Chern classes of $\mathcal{B}_2$, we observe $c_1(\mathcal{B}_2) = -4\theta - 2\gamma - (2d-1)\eta$ and $\mathrm{ch}_2(\mathcal{B}_2) = 4\eta\theta$. □

3. The class of the virtual divisor $\widetilde{\mathfrak{D}}_{13}$

In this section we determine the virtual class $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}} := \sigma_*\Big(c_2(\mathrm{Sym}^2(\mathcal{E}))^\vee - \mathcal{F}^\vee\Big)$ on $\overline{\mathcal{M}}_{13}$. We begin by recording the following formulas for a vector bundle $\mathcal{V}$ of rank $r+1$ on a stack $\mathcal{X}$:

$$c_1\big(\mathrm{Sym}^2(\mathcal{V})\big) = (r+2)c_1(\mathcal{V}), \quad c_2\big(\mathrm{Sym}^2(\mathcal{V})\big) = \frac{r(r+3)}{2}c_1^2(\mathcal{V}) + (r+3)c_2(\mathcal{V}).$$

We apply these formulas for the first degeneracy locus of $\phi^\vee\colon \mathcal{F}^\vee \to \mathrm{Sym}^2(\mathcal{E})^\vee$. By Definition 2.2, its class $[\mathfrak{U}]^{\mathrm{virt}}$ is given by

$$\begin{aligned} c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big) &= c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee\big) - c_1\big(\mathrm{Sym}^2(\mathcal{E})^\vee\big)\cdot c_1(\mathcal{F}^\vee) + c_1^2(\mathcal{F}^\vee) - c_2(\mathcal{F}^\vee), \\ &= 20c_1^2(\mathcal{E}) + 8c_2(\mathcal{E}) - 7c_1(\mathcal{E})\cdot c_1(\mathcal{F}) + c_1^2(\mathcal{F}) - c_2(\mathcal{F}). \end{aligned} \tag{12}$$

In what follows we expand the virtual class in $CH^1(\widetilde{\mathcal{M}}_{13})$ as

$$[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}} = a\lambda - b_0\delta_0 - b_1\delta_1. \tag{13}$$

We compute the coefficients a, b_0 and b_1, by intersecting both sides of this expression with the test curves F_0, F_1 and F_{ell}. We start with the coefficient b_1.

Theorem 3.1. *Let X be a general curve of genus 12. The coefficient b_1 in (13) is:*

$$b_1 = \frac{1}{2g(X)-2}\sigma^*(F_1)\cdot c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big) = 11787.$$

Proof. We intersect the degeneracy locus of the map $\phi\colon \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ with $\sigma^*(F_1)$. By Proposition 2.5, it suffices to estimate the contribution coming from Z. We write

$$\sigma^*(F_1)\cdot c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big) = c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big)_{|Z}.$$

In Proposition 2.7, we constructed a morphism $\zeta\colon J_1(\mathcal{P})^\vee \to \pi_2^*(\mathcal{M})^\vee$ of vector bundles on Z, whose fibres are the maps $H^0(\mathcal{O}_{2y})^\vee \to H^0(X,L)^\vee$. The kernel sheaf $\mathrm{Ker}(\zeta)$ is locally free of rank 1. If U is the line bundle on Z with fibre

$$U(y,L) = \frac{H^0(X,L)}{H^0(X,L(-2y))} \hookrightarrow H^0(X, L\otimes \mathcal{O}_{2y})$$

over a point $(y,L)\in Z$, then one has the following exact sequence over Z:

$$0 \longrightarrow U \longrightarrow J_1(\mathcal{P}) \longrightarrow \big(\mathrm{Ker}(\zeta)\big)^\vee \longrightarrow 0.$$

In particular, by Proposition 2.7, we find that

$$c_1(U) = 2\gamma + 54\eta + c_1(\mathrm{Ker}(\zeta)). \tag{14}$$

The product of the Chern class of $\mathrm{Ker}(\zeta)$ with any class $\xi \in H^2(X\times W^5_{16}(X),\mathbb{Z})$ is given by the Harris-Tu formula [HT]:

$$\begin{aligned} c_1\big(\mathrm{Ker}(\zeta)\big)\cdot \xi_{|Z} &= -c_6\Big(\pi_2^*(\mathcal{M})^\vee - J_1(\mathcal{P})^\vee\Big)\cdot \xi_{|Z}, \\ &= -\Big(\pi_2^*(c_6) - 6\eta\theta\pi_2^*(c_4) + (54\eta+2\gamma)\pi_2^*(c_5)\Big)\cdot \xi_{|Z}. \end{aligned} \tag{15}$$

Similarly, one has the formula [HT] for the self-intersection on the surface Z:

$$c_1^2\big(\mathrm{Ker}(\zeta)\big) = \Big(\pi_2^*(c_7) - 6\eta\theta\pi_2^*(c_5) + (54\eta+2\gamma)\pi_2^*(c_6)\Big) \in H^{14}\big(X\times W^5_{16}(X),\mathbb{Z}\big)\cong \mathbb{Z}. \tag{16}$$

We also observe that $c_7 = 0$, since the bundle $\mathcal{M}$ has rank 6.

Let $\mathcal{A}_3$ denote the vector bundle on Z having fibres

$$\mathcal{A}_{3,(y,L)} = H^0(X, L^{\otimes 2})$$

constructed as a push forward of a line bundle on $X\times X\times \mathrm{Pic}^{16}(X)$. Then the line bundle $U^{\otimes 2}$ can be embedded in $\mathcal{A}_3/\mathcal{A}_2$. We consider the quotient

$$\mathcal{G} := \frac{\mathcal{A}_3/\mathcal{A}_2}{U^{\otimes 2}}.$$

The morphism $U^{\otimes 2} \to \mathcal{A}_3/\mathcal{A}_2$ vanishes along the locus of pairs (y,L) where L has a base point. It follows that the sheaf $\mathcal{G}$ has torsion along the locus $\Gamma \subseteq Z$ consisting of pairs $(q, A(q))$, where $A\in W^5_{16}(X)$. Furthermore, $\mathcal{F}_{|Z}$, as a subsheaf of $\mathcal{A}_3$, can be identified with the kernel of the map $\mathcal{A}_3 \to \mathcal{G}$. Summarizing, there is an exact sequence of vector bundles on Z

$$0 \longrightarrow \mathcal{A}_{2|Z} \longrightarrow \mathcal{F}_{|Z} \longrightarrow U^{\otimes 2} \longrightarrow 0. \tag{17}$$

Over a general point $(y,L)\in Z$, this sequence reflects the decomposition

$$\mathcal{F}(y,L) = H^0(X, L^{\otimes 2}(-2y)) \oplus K\cdot u^2,$$

where $u \in H^0(X,L)$ is a section such that $\mathrm{ord}_y(u) = 1$.

Via the exact sequence (17), one computes the Chern classes of $\mathcal{F}_{|Z}$:

$$c_1(\mathcal{F}_{|Z}) = c_1(\mathcal{A}_{2|Z}) + 2c_1(U), \qquad c_2(\mathcal{F}_{|Z}) = c_2(\mathcal{A}_{2|Z}) + 2c_1(\mathcal{A}_{2|Z})c_1(U).$$

Recalling that $\mathcal{E}_{|Z} = \pi_2^*(\mathcal{M})_{|Z}$ and using (12), we find that $\sigma^*(F_1)\cdot c_2\big((\mathrm{Sym}^2\mathcal{E})^\vee - \mathcal{F}^\vee\big)$ is equal to:

$$20c_1^2\big(\pi_2^*\mathcal{M}_{|Z}^\vee\big) + 8c_2\big(\pi_2^*\mathcal{M}_{|Z}^\vee\big) + 7c_1\big(\pi_1^*\mathcal{M}_{|Z}^\vee\big)\cdot c_1\big(\mathcal{A}_{2|Z}\big) + 4c_1^2(U) - \\ - c_2(\mathcal{A}_{2|Z}) + 14c_1\big(\pi_2^*\mathcal{M}_{|Z}^\vee\big)\cdot c_1(U) + c_1^2(\mathcal{A}_{2|Z}) + 2c_1^2(\mathcal{A}_{2|Z})\cdot c_1(U).$$

Here, $c_i(\pi_2^*\mathcal{M}_{|Z}^\vee) = \pi_2^*(c_i) \in H^{2i}(Z,\mathbb{Z})$. The Chern classes of $\mathcal{A}_{2|Z}$ have been computed in Proposition 2.8. Formula (14) expresses $c_1(U)$ in terms of $c_1((\mathrm{Ker}(\zeta))$ and the classes η and γ. When expanding $\sigma^*(F_1)\cdot c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big)$, one distinguishes between terms that *do* and those that *do not* contain the first Chern class of $\mathrm{Ker}(\zeta)$. The coefficient of $c_1\big(\mathrm{Ker}(\zeta)\big)$, as well as the contribution coming from $c_1^2\big(\mathrm{Ker}(\zeta)\big)$ in the expression of $\sigma^*(F_1)\cdot c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big)$ is evaluated using the formulas (15) and (16) respectively. To carry this out, we first consider the part of this product that *does not* contain $c_1\big(\mathrm{Ker}(\zeta)\big)$, and we obtain

$$8\pi_2^*(c_2) + 20\pi_2^*(c_1^2) + c_1^2(\mathcal{A}_{2|Z}) + 7\pi_2^*(c_1)\cdot c_1(\mathcal{A}_{2|Z}) - c_2(\mathcal{A}_{2|Z}) + 4(2\gamma+54\eta)^2 + \\ + 2(2\gamma+54\eta)\cdot c_1(\mathcal{A}_{2|Z}) + 14(2\gamma+54\eta)\cdot\pi_2^*(c_1) = \\ 20\pi_2^*(c_1^2) + 154\pi_2^*(c_1)\cdot\eta - 28\pi_2^*(c_1)\cdot\theta - 96\eta\theta + 8\theta^2 + 8\pi_2^*(c_2) \in H^4\big(X\times W^5_{16}(X),\mathbb{Z}\big).$$

This polynomial gets multiplied by the class $[Z]$, which is expressed in Proposition 2.7 as a degree 5 polynomial in θ, η and $\pi_2^*(c_i)$. We obtain a homogeneous polynomial of degree 7 viewed as an element of $H^{14}\big(X\times W^5_{16}(X),\mathbb{Z}\big)$.

Next we turn our attention to the contribution $\sigma^*(F_1)\cdot c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big)$ coming from terms that do contain $c_1\big(\mathrm{Ker}(\zeta)\big)$. This is given by the following formula:

$$4c_1^2(\mathrm{Ker}(\zeta) + c_1(\mathrm{Ker}(\zeta)\cdot\Big(8(2\gamma+54\eta) + 2c_1(\mathcal{A}_{2|Z}) + 14\pi_2^*(c_1)\Big).$$

Using (15) and (16), one ends up with the following homogeneous polynomial of degree 7 in η, θ, and $\pi_2^*(c_i)$ for $i=1,\ldots,6$:

$$84\pi_2^*(c_1c_4)\theta\eta - 48\pi_2^*(c_4)\theta^2\eta - 756\pi_2^*(c_1c_5)\eta + 440\pi_2^*(c_5)\theta\eta - 44\pi_2^*(c_6)\eta.$$

Adding together the parts that do and those that do not contain $c_1\big(\mathrm{Ker}(\zeta)\big)$, and using the fact that the only monomials that need to be retained are those containing η, after manipulations carried out using *Maple*, one finds

$$\sigma^*(F_1)\cdot c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big) = \eta\pi_2^*\Big(-602c_1c_5 + 432c_2c_4 - 120c_1^2c_3\theta + 168c_1c_3\theta^2 - \\ - 48c_3\theta^3 + 1080c_1^2c_4 - 1428c_1c_4\theta - 48c_2c_3\theta + 384c_4\theta^2 + 344c_5\theta - 44c_6\Big).$$

We suppress η and the remaining polynomial lives inside $H^{12}(W^5_{16}(X),\mathbb{Z})\cong\mathbb{Z}$. Using (2.6) this expression is equal to

$$\sigma^*(F_1)\cdot c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big) = \frac{193}{45}\theta^6 - \frac{1271}{30}\theta^5 y_1 + \frac{1607}{12}\theta^4 y_1^2 - 120\theta_3 y_1^3 = 259314,$$

where for the last step we used the formulas (11). We conclude

$$b_1 = \frac{1}{22}\sigma^*(F_1)\cdot c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big) = 11787,$$

as required. □

Theorem 3.2. *Let $[X,q]$ be a general pointed curve of genus 12 and let $F_0 \subseteq \widetilde{\Delta}_0 \subseteq \widetilde{\mathcal{M}}_{13}$ be the associated test curve. Then the coefficient of δ_0 in the expression (13) of $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$ is equal to*

$$b_0 = \frac{\sigma^*(F_0)\cdot c_2\Big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\Big) + b_1}{24} = 2247.$$

Proof. Using Proposition 2.4, we observe that

$$c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big)_{|\sigma^*(F_0)} = c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big)_{|Y}.$$

We shall evaluate the Chern classes of $\mathcal{F}_{|Y}$ via the line bundle V on Y with fibre

$$V(y,L) = \frac{H^0(X,L)}{H^0(X,L(-y-q))} \hookrightarrow H^0(X, L\otimes \mathcal{O}_{y+q})$$

over a point $(y,L)\in Y$. We write the following exact sequence over Y

$$0 \longrightarrow V \longrightarrow \mathcal{B} \longrightarrow \big(\mathrm{Ker}(\chi)\big)^\vee \longrightarrow 0,$$

where the morphism $\chi\colon \mathcal{B}^\vee \to \pi_2^*(\mathcal{M})^\vee$ was defined in the final part of the proof of Proposition 2.7. In particular, we have

$$c_1(V) = 15\eta + \gamma + c_1\big(\mathrm{Ker}(\chi)\big).$$

The effect on multiplying $c_1\big(\mathrm{Ker}(\chi)\big)$ against a class $\xi \in H^2\big(X\times W^5_{16}(X),\mathbb{Z}\big)$ is described by applying once more the Harris-Tu [HT] formula:

$$c_1\big(\mathrm{Ker}(\chi)\big)\cdot \xi_{|Y} = \Big(-\pi_2^*(c_6) - 2\eta\theta\pi_2^*(c_4) + (15\eta+\gamma)\pi_2^*(c_5)\Big)\cdot \xi_{|Y}, \tag{18}$$

where we recall that $\pi_2\colon X\times W^5_{16}(X) \to W^5_{16}(X)$ and $c_i \in H^{2i}(W^5_{16}(X),\mathbb{Z})$. Similarly, for the self-intersection on Y the following formula holds:

$$c_1^2\big(\mathrm{Ker}(\chi)\big) = -2\eta\theta\pi_2^*(c_5) + (15\eta+\gamma)\pi_2^*(c_6) \in H^{14}\big(X\times W^5_{16}(X),\mathbb{Z}\big). \tag{19}$$

We have also introduced in Proposition 2.8 the vector bundle $\mathcal{B}_2$ on $X\times \mathrm{Pic}^{16}(X)$ with fibres $\mathcal{B}_{2,(y,L)} = H^0(X, L^{\otimes 2}(-y-q))$ over a point (y,L). A local calculation along the lines of the one in the proof of Theorem 3.1 shows that one also has an exact sequence on Y, which can then be used to determine the Chern numbers of $\mathcal{F}_{|Y}$.

$$0 \longrightarrow \mathcal{B}_{2|Y} \longrightarrow \mathcal{F}_{|Y} \longrightarrow V^{\otimes 2} \longrightarrow 0.$$

This exact sequence reflects the fact for a general point $(y,L)\in Y$ one has a decomposition $\mathcal{F}(y,L) = H^0(X, L^{\otimes 2}(-y-q)) \oplus K\cdot u^2$, where $u \in H^0(X,L)$ is a section not vanishing at y and q. We thus obtain the formulas:

$$c_1(\mathcal{F}_{|Y}) = c_1(\mathcal{B}_{2|Z}) + 2c_1(V), \qquad\qquad c_2(\mathcal{F}_{|Y}) = c_2(\mathcal{B}_{2|Y}) + 2c_1(\mathcal{B}_{2|Y})c_1(V).$$

To estimate $c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big)_{|Y}$ we use (12) and write:

$$\begin{aligned}\sigma^*(F_0)\cdot c_2\big((\mathrm{Sym}^2\mathcal{E})^\vee - \mathcal{F}^\vee\big) &= 20c_1^2\big(\pi_1^*\mathcal{M}^\vee_{|Y}\big) + 8c_2\big(\pi_2^*\mathcal{M}^\vee_{|Y}\big) + 7c_1\big(\pi_1^*\mathcal{M}^\vee_{|Y}\big)\cdot c_1\big(\mathcal{B}_{2|Y}\big)\\ &+ 4c_1^2(V) - c_2(\mathcal{B}_{2|Y}) + 14c_1\big(\pi_2^*\mathcal{M}^\vee_{|Y}\big)\cdot c_1(V) + c_1^2(\mathcal{B}_{2|Y}) + 2c_1(\mathcal{B}_{2|Y})\cdot c_1(V).\end{aligned}$$

We expand this expression, collect the terms that do not contain $c_1\big(\mathrm{Ker}(\chi)\big)$, and obtain the following:

$$20\pi_2^*(c_1^2) - 7\eta\pi_2^*(c_1) - 28\theta\cdot\pi_2^*(c_1) + 4\theta\eta + 8\theta^2 + 8\pi_2^*(c_2).$$

This quadratic polynomial gets multiplied with the class $[Y]$ computed in Proposition 2.7. Next, we collect the terms in $\sigma^*(F_0)\cdot c_2\big(\mathrm{Sym}^2\mathcal{E}^\vee - \mathcal{F}^\vee\big)$ that do contain $c_1\big(\mathrm{Ker}(\chi)\big)$:

$$4c_1^2\big(\mathrm{Ker}(\chi)\big) + c_1\big(\mathrm{Ker}(\chi)\big)\Big(8(15\eta+\gamma) + 14\pi_2^*(c_1) + 2c_1(\mathcal{B}_{2|Y}\Big).$$

This part of the contribution is evaluated using formulas (18) and (19).

Putting everything together, we obtain a polynomial in $H^{14}\big(X\times W^5_{16}(X),\mathbb{Z}\big)\cong\mathbb{Z}$, as in the proof of Theorem 3.1:

$$\sigma^*(F_0)\cdot c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big) = \eta\pi_2^*\Big(-40c_1^2c_3\theta + 56c_1c_3\theta^2 - 16c_3\theta^3 + 300c_1^2c_4 - 392c_1c_4\theta -$$

$$-16c_2c_3\theta + 104c_4\theta^2 - 217c_1c_5 + 120c_2c_4 + 124c_5\theta + 2c_6\Big).$$

Applying (2.6) and then (11), after eliminating η, we obtain

$$\sigma^*(F_0)\cdot c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big) = \frac{161}{180}\theta^6 - \frac{28}{3}\theta^5 y_1 + \frac{755}{24}\theta^4 y_1^2 - 30\theta^3 y_1^3 = 42141.$$

□

We can now complete the calculation of $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$.

Proof of Theorem 1.4. We consider the curve $F_{\mathrm{ell}} \subseteq \widetilde{\mathcal{M}}_g$ defined in (7) obtained by attaching at the fixed point of a general curve X of genus 12 a pencil of plane cubics at one of the base points of the pencil. Then one has the relation

$$a - 12b_0 + b_1 = F_{\mathrm{ell}}\cdot \sigma_* c_2\big(\mathrm{Sym}^2(\mathcal{E})^\vee - \mathcal{F}^\vee\big) = 0.$$

Using Theorems 3.1 and 3.2, we thus find $a = 15177$, for the λ-coefficient in the expansion (13). This completes the calculation of the virtual class $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$. □

We finally explain how Theorem 1.4 and Theorem 1.5 (proved in §4) together imply Theorem 1.3.

Proof of Theorem 1.3. We write $[\overline{\mathfrak{D}}_{13}] = a\lambda - b_0\delta_0 - \cdots - b_6\delta_6$, where a, b_0 and b_1 are determined by Theorem 1.4. Applying [FP, Theorem 1.1] we have the inequalities $b_i \geq (6i+8)b_0 - (i+1)a \geq b_0$ for $i = 2, \ldots, 6$, which shows that $s(\overline{\mathfrak{D}}_{13}) = \frac{a}{b_0} = \frac{5059}{749}$. □

4. The Strong Maximal Rank Conjecture in genus 13

In this section and the next, we prove that $\widetilde{\mathfrak{D}}_{13}$ is not all of $\widetilde{\mathcal{M}}_{13}$ and that its condimension 1 part represents the virtual class $[\widetilde{\mathfrak{D}}_{13}]^{\mathrm{virt}}$.

To show that $\widetilde{\mathfrak{D}}_{13}$ is not all of $\widetilde{\mathcal{M}}_{13}$, it suffices to prove the existence of one Brill-Noether general smooth curve X of genus 13 such that, for every $L \in W^5_{16}(X)$, the multiplication map

$$\phi_L\colon \mathrm{Sym}^2 H^0(X, L) \to H^0(X, L^{\otimes 2})$$

is surjective. This is one case of the Strong Maximal Rank Conjecture [AF]. The locus of such curves is Zariski open; to prove that it is nonempty over every algebraically closed field of characteristic zero, it suffices to show this over one such field. Hence, we can and do assume that our ground field K is spherically complete with respect to a surjective valuation $\nu\colon K^\times \to \mathbb{R}$, and that K has residue characteristic zero. This allows us to discuss the nonarchimedean analytifications of curves, the skeletons of those analytifications, and the tropicalizations of rational functions, viewed as sections of L and $L^{\otimes 2}$. In this framework, we apply the method of tropical independence to give a lower bound for the rank of the multiplication map ϕ_L, for every $L \in W^5_{16}(X)$. The motivation and technical foundations for this approach are detailed in §§1.4-1.5, §§2.4-2.5, and §6 of [FJP], to which we refer the reader for details and further references.

After proving this case of the Strong Maximal Rank Conjecture, we will furthermore show that no component of the degeneracy locus $\mathfrak{U}$ in the parameter space $\widetilde{\mathfrak{G}}^5_{16}$ over $\widetilde{\mathcal{M}}_{13}$ maps with generically positive dimensional fibres onto a divisor in $\widetilde{\mathcal{M}}_{13}$. As in [FJP], this additional step is necessary to show that the push-forward of the virtual class is effective, and our proof involves analogous arguments on lower genus curves for linear series with ramification. In particular, we will consider linear series with ramification on curves of genus 11 and 12 in the next section, and so we set up our arguments here to work in this greater generality.

Let X be a smooth projective curve of genus $11 \leq g \leq 13$ over K whose Berkovich analytification X^{an} has a skeleton Γ which is a chain of g loops connected by bridges, as shown. In order to simplify notation later, the vertices of Γ are labeled $w_{13-g}, \ldots, w_{13}$, and $v_{14-g}, \ldots, v_{14}$, as shown in Figure 1.

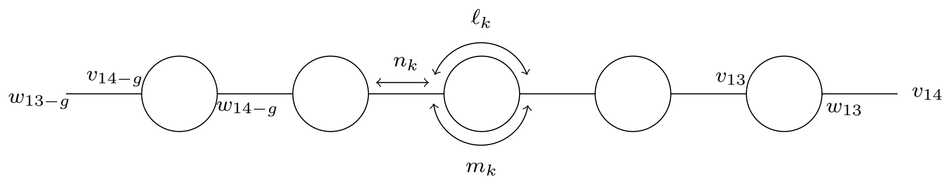

FIGURE 1. The chain of loops Γ.

We write γ_k for the kth loop, formed by the two edges of length ℓ_k and m_k between v_k and w_k, for $14 - g \leq k \leq 13$. We write β_k for the kth bridge, between w_{k-1} and v_k, for $0 \leq k \leq g$, which has length n_k. Except where explicitly noted otherwise, we assume that these edge lengths satisfy

$$\ell_{k+1} \ll m_k \ll \ell_k \ll n_{k+1} \ll n_k \quad \text{for all } k. \tag{20}$$

These conditions are precisely as in [FJP, §6.4]. Any curve X whose analytification has such a skeleton is Brill-Noether general [CDPR].

Given a line bundle L on X we choose an identification $L = \mathcal{O}_X(D_X)$ so that any linear series $V \subseteq H^0(X, L)$ is identified with a finite dimensional vector space of rational functions $V \subseteq K(X)$. The tropicalization of any nonzero rational function f on X is a piecewise linear function with integer slopes on Γ, and we write $\operatorname{trop} V$ for the set of all tropicalizations of nonzero functions in V.

Any sum of two functions in $\operatorname{trop} V$ is the tropicalization of a function in the image of the multiplication map $\phi_V \colon \operatorname{Sym}^2 V \to H^0(X, L^{\otimes 2})$. We say that a set of functions $\{\psi_0, \ldots, \psi_n\}$ on Γ is *tropically independent* if there are real numbers $b_0, \ldots, b_n$ such that

$$\min\{\psi_0 + b_0, \ldots, \psi_n + b_n\} \neq \min\{\psi_0 + b_0, \ldots, \widehat{\psi_j + b_j}, \ldots, \psi_n + b_n\} \quad \text{for } 0 \leq j \leq n.$$

In other words, $\{\psi_0, \ldots, \psi_n\}$ is tropically independent if there are real numbers $b_0, \ldots, b_n$ such that each $\psi_j + b_j$ achieves the minimum uniquely in $\min_i\{\psi_i + b_i\}$ at some point $v \in \Gamma$. The function $\theta = \min_i\{\psi_i + b_i\}$ is then called an *independence*, since it verifies that $\{\psi_0, \ldots, \psi_n\}$ is independent.

We recall that tropical independence is a sufficient condition for linear independence; if $f_0, \ldots, f_n$ are nonzero rational functions on X such that $\{\operatorname{trop}(f_0), \ldots, \operatorname{trop}(f_n)\}$ is tropically independent on Γ, then $\{f_0, \ldots, f_n\}$ is linearly independent in $K(X)$. Therefore, the relevant case of the Strong Maximal Rank Conjecture, and hence the fact that $\widetilde{\mathfrak{D}}_{13}$ is a divisor, follows immediately from the following.

Theorem 4.1. *Let X be a curve of genus 13 with skeleton Γ. Let V be a linear series of degree 16 and dimension 5 on X, and let $\Sigma = \operatorname{trop} V$. Then there is an independence θ among 20 pairwise sums of functions in Σ.*

We will use the following generalization of Theorem 4.1 in our proof that $\widetilde{\mathfrak{D}}_{13}$ represents the virtual class; the generalization involves analogous statements for linear series satisfying certain ramification conditions in genus 12 and 11. The situation is closely parallel to that in [FJP, §8.2]. Recall that $a_0^V(p) < \ldots < a_r^V(p)$ denotes the vanishing sequence of a linear series V of rank r at a point p.

Theorem 4.2. *Let X be a curve of genus $g \in \{11, 12, 13\}$ with skeleton Γ, and let $p \in X$ be a point specializing to w_{13-g}. Let V be a linear series of degree 16 and dimension 5 on X, and let $\Sigma = \operatorname{trop} V$. Assume that*

(i) if $g = 12$, then $a_1^V(p) \geq 2$, and
(ii) if $g = 11$, then either $a_1^V(p) \geq 3$, or $a_0^V(p) + a_2^V(p) \geq 5$.

Then there is an independence θ among 20 pairwise sums of functions in Σ.

The remainder of this section is devoted to the proof of Theorem 4.2. In all three cases, the adjusted Brill-Noether number $\rho(V, p)$ is equal to 1. In particular, this means that there is at most one lingering loop, and there is exactly one when Σ is unramified and vertex avoiding. Our approach to constructing the independence is similar to that of [FJP], with a few important differences which we highlight when they arise.

Remark 4.3. The differences are subtle but crucial. Even in the unramified and vertex avoiding case, if we apply the algorithm of [FJP, §7] naively, we obtain an independence among only 19 functions in Σ. To overcome this difficulty, we divide the graph into blocks in such a way that the lingering loop is the last loop in its block and has exactly two permissible functions. This allows us to assign a function to the lingering loop, raising the total number of functions in the independence to 20. See Remark 4.9.

Throughout, we let D_X be a divisor class on X with $V \subseteq H^0(X, \mathcal{O}(D_X))$. We write $D = \mathrm{Trop}(D_X)$, and we assume throughout that D is a break divisor.

4.1. **The unramified vertex avoiding case.** We first consider the case where $g = 13$ and $\Sigma = \mathrm{trop}\, V$ is vertex avoiding and unramified. Unramified means that the ramification weights of $\mathrm{trop}\, V$ at w_0 and v_{14}, in the sense of [FJP, Definition 6.17], are zero. Vertex avoiding means that, for $0 \leq i \leq 5$, there is a unique divisor $D_i \sim D$ such that $D_i - iw_0 - (r-i)v_{14}$ is effective. A vertex avoiding divisor is unramified if and only if the support of $D_i - iw_0 - (r-i)v_{14}$ contains neither w_0 nor v_{14}, for all i.

For $\psi \in \Sigma$, we write $s_k(\psi)$ and $s'_k(\psi)$ for the rightward slopes along the incoming and outgoing bridges of the kth loop γ_k, at v_k and w_k, respectively. Since $\dim V = 6$, the functions in Σ have exactly 6 distinct slopes along each tangent vector in Γ.

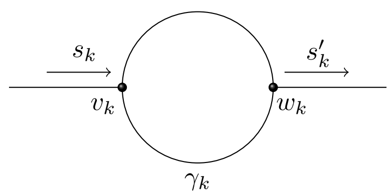

FIGURE 2. The slopes s_k and s'_k.

Definition 4.4. Let $s_k[0] < \cdots < s_k[5]$ and $s'_k[0] < \cdots < s'_k[5]$ denote the 6 distinct rightward slopes that occur as $s_k(\psi)$ and $s'_k(\psi)$, for $\psi \in \Sigma$.

Since D is vertex avoiding, there is a function $\varphi_i \in \Sigma$ such that

$$s_k(\varphi_i) = s_k[i] \text{ and } s'_k(\varphi_i) = s'_k[i] \text{ for all } k,$$

and it is unique up to additive constants. Since Σ is also unramified, there is a unique *lingering loop* γ_ℓ, i.e., a unique loop γ_ℓ such that $s'_\ell[i] = s_\ell[i]$ for all i. Moreover, there is no function $\varphi \in \Sigma$ with the property that $s_\ell(\varphi) \leq s_\ell[i]$ and $s'_\ell(\varphi) \geq s'_\ell[i+1]$. This last condition means that γ_ℓ is not a *switching loop*, in the sense of [FJP, Definition 6.19].

Our assumption that Σ is unramified implies that the break divisor D satisfies $\deg_{w_0} D = 3$, and the rightward slopes of the functions ψ_i at w_0 are

$$(s'_0[0], \ldots, s'_0[5]) = (-2, -1, 0, 1, 2, 3).$$

Let us consider how the slope vector $(s'_k[0], \ldots, s'_k[5])$ changes as we go from left to right across the graph. When crossing a loop other than the lingering loop γ_ℓ, one of these slopes increases by 1, and the other 5 remain the same. So, after the first non-lingering loop, the slopes are $(-2, -1, 0, 1, 2, 4)$, and after the second non-lingering loop, the slopes are either $(-2, -1, 0, 1, 2, 5)$ or $(-2, -1, 0, 1, 3, 4)$. The data of these slopes is recorded by a standard Young tableau on a rectangle with 2 rows and 6 columns, filled with the symbols 1 through 13, excluding ℓ. If the symbol k appears in column i, then it is the $(5-i)$th slope that increases on the loop γ_k, i.e., $s'_k[5-i] = s_k[5-i] + 1$. Note, in particular, that each slope increases exactly twice, so $s'_{13} = (0, 1, 2, 3, 4, 5)$ and no slope is ever greater than 5.

We first divide the graph into three *blocks*. Within each block, the slope of θ will be nearly constant on each bridge, equal to 4 on bridges in the first block, 3 on bridges in the second block, and 2 on bridges in the third block. Let

$$z_1 = \min\{6, \ell\}$$
$$z_2 = \max\{7, \ell\}.$$

We will construct our independence θ so that its incoming slope at the loop γ_k is:

$$s_k(\theta) = \begin{cases} 4 & \text{if } k \leq z_1, \\ 3 & \text{if } z_1 < k \leq z_2, \\ 2 & \text{if } z_2 < k \leq 13. \end{cases}$$

In other words, γ_{z_1} and γ_{z_2} are the last loops of the first and second blocks, respectively. Note that either z_1 or z_2 is equal to ℓ, so the lingering loop γ_ℓ is always the last loop in its block.

The following definition gives a natural necessary condition for a function in Σ to achieve the minimum at some point of a given loop.

Definition 4.5. Let $\psi \in \mathrm{PL}(\Gamma)$ be a function. We say that ψ is *permissible* on γ_k if

$$s_k(\psi) \leq s_k(\theta) \qquad \text{and} \qquad s'_k(\psi) \geq s_k(\theta).$$

We say that ψ is *permissible* on a block if it is permissible on some loop in that block. A permissible function ψ is *new* if $s_k(\psi) \leq s_k(\theta) - 1$, and it is *departing* if $s'_k(\psi) \geq s_k(\theta) + 1$.

To understand this definition, recall that θ has nearly constant slopes along bridges, and the bridges adjacent to a loop are much longer than the bridge itself. If $s_k(\psi) \geq s_k(\theta) + 1$, then the value of ψ at v_k exceeds the value of θ at v_k by at least the length of the bridge β_k (or half this length, if β_k is the bridge between two blocks). Since this bridge is much longer than the loop γ_k, it follows that ψ cannot achieve the minimum at any point of γ_k. A similar argument shows that if $s'_k(\psi) \leq s_k(\theta) - 1$, then ψ cannot achieve the minimum at any point of γ_k. Our choice of z_1 and z_2 completely determines which loops have new permissible functions.

Proposition 4.6. *There is no new permissible function on γ_k if and only if $k = \ell$ or*

(i) $\ell > 6$ *and* $k = 6$;
(ii) $\ell > 7$ *and* $k = 7$;
(iii) $\ell < 9$ *and* $k = 9$; *or*
(iv) $\ell \leq 7$, $s'_7[5] = 4$, *and* $k = 8$.

Proof. There is no new permissible function on the lingering loop γ_ℓ. Suppose $k \neq \ell$. Let j be the unique integer satisfying $s'_k[j] = s_k[j] + 1$. There is a new permissible function on γ_k if and only if either the function φ_{jj} is both new and departing, or there is an integer i such that $s'_k(\varphi_{ij}) = s_k(\theta)$. We now examine when such an i exists.

The values $s'_k[i]$ are 6 distinct integers between -2 and 5. Let a and b be the two integers in this range that are not equal to $s'_k[i]$ for any i. On the hth non-lingering loop, one has

$$h = \sum_{i=0}^{5} (s'_k[i] + 2 - i) = 9 - (a + b).$$

Since $s'_k[j] = s_k[j] + 1$, we must have that $s'_k[j]$ is equal to either $a + 1$ or $b + 1$. Without loss of generality, assume that it is equal to $a + 1$. There does not exist i such that $s'_k[i] + s'_k[j] = s'_k(\theta)$ if and only if $s'_k(\theta) - (a + 1)$ is greater than 5, smaller than -2, or equal to either a or b. If it is equal to a, then the function φ_{jj} is both new and departing. Since $s'_k(\theta) \leq 4$ and $a + 1 \geq -1$, we see that $s'_k(\theta) - (a+1)$ cannot be greater than 5, and $s'_k(\theta) - (a+1)$ is smaller than -2 if and only if $s'_k(\theta) = 2$ and $a = 4$. By the above calculation, $b = s'_k(\theta) - (a + 1)$ if and only if $h = 10 - s'_k(\theta)$.

The 6th non-lingering loop is contained in the first block if and only if $\ell > 6$. The 7th non-lingering loop is contained in the second block if and only if $\ell > 7$. The 8th non-lingering loop is contained in the third block if and only if $\ell < 9$. Finally, if $a = 4$, then γ_k is one of the first 7 non-lingering loops. If γ_k is in the third block, then since $z_2 \geq 7$, we have $\ell \leq 7$, and γ_k is the first loop in the third block. $\square$

The functions that appear in the tropical independence θ are as follows. Let

$$\mathcal{B} = \{\varphi_i + \varphi_j : 0 \leq i \leq j \leq 5\}.$$

Note that $\mathcal{B}$ has 21 elements. The tropical independence that we construct uses only 20 elements, which form a subset $\mathcal{B}' \subseteq \mathcal{B}$. We describe how to choose this subset $\mathcal{B}'$. We make our choices so that the number of permissible functions on each block is 1 more than the number of loops in that block.

If $\ell \leq 7$, then we let $\psi \in \mathcal{B}$ be a function that is permissible on the second block and $\mathcal{B}' = \mathcal{B} \smallsetminus \{\psi\}$. If $\ell > 7$, then we let $\psi \in \mathcal{B}$ be a function that is permissible on the third block and $\mathcal{B}' = \mathcal{B} \smallsetminus \{\psi\}$.

Remark 4.7. Note that there are several functions that are permissible on the second or third block; it does not matter which of these we choose to omit from the set $\mathcal{B}'$.

Lemma 4.8. *On each block, the number of permissible functions in $\mathcal{B}'$ is one more than the number of loops.*

Proof. This follows directly from Proposition 4.6. Specifically, since $z_1 = \min\{6, \ell\}$, there is a new permissible function in $\mathcal{B}$ on each loop of the first block, except for the last one. Since there are precisely two pairs (i, j) such that $s_1(\varphi_{ij}) = 4$, we see that the number of permissible functions on the first block is 1 more than the number of loops. By symmetry, if $z_2 \leq 7$, then the number of permissible functions in $\mathcal{B}$ on the third block is 1 more than the number of loops, and if $z_2 > 7$, it is 2 more. But when $z_2 > 7$, one of these functions is not in $\mathcal{B}'$.

Finally, we consider the middle block. We count the number of pairs (i, j) such that $s'_{z_1}(\varphi_{ij}) = 3$. Since 3 is odd, if (i, j) is such a pair, then $i \neq j$. It follows that there are 3 such pairs if and only if $s'_{z_1}[i] + s'_{z_1}[5 - i] = 3$ for all i, which implies that there are precisely 6 non-lingering loops in the first block. It follows that, if $\ell < 7$, then there are precisely two such pairs, and if $\ell \geq 7$, there are three such pairs. By Proposition 4.6, if $\ell < 7$, there is a new permissible function on every loop of the middle block. If $\ell = 7$, then the middle block contains only one loop, and since this loop is lingering, there are no new permissible functions on it. In both of these cases, the number of permissible functions in $\mathcal{B}$ on the middle block is therefore 2 more than the number of loops, but one of these functions is not in $\mathcal{B}'$. If $\ell > 7$, then there are no new permissible functions on γ_7 or γ_ℓ, so the number of permissible functions is 1 more than the number of loops. □

Remark 4.9. The algorithm for constructing the tropical independence is identical to that presented in [FJP, § 7], with one exception. Specifically, we do *not* skip the lingering loop γ_ℓ. Instead, since γ_ℓ is the last loop in its block, there are precisely two unassigned permissible functions on γ_ℓ. These two functions do not have identical restrictions to γ_ℓ. Thus, if we adjust their coefficients upward so that they agree at w_ℓ, one of them will obtain the minimum uniquely at some point of the loop γ_ℓ. We assign this function to γ_ℓ and adjust its coefficient upward by a small amount ϵ, small enough so that it still obtains the minimum uniquely at some point of γ_ℓ. The other will then obtain the minimum uniquely at w_ℓ, and we assign this function to the bridge $\beta_{\ell+1}$.

We now verify that this algorithm produces a tropical independence.

Lemma 4.10. *Suppose that φ_{ij} is assigned to the loop γ_k or the bridge β_k. Then φ_{ij} does not achieve the minimum at any point to the right of v_{k+1}.*

Proof. If γ_k is a non-lingering loop, then the proof is the same as [FJP, Lemma 7.21]. On the other hand, if γ_k is the lingering loop, then it is the last loop in its block. Since v_{k+1} is the start of the next block, φ_{ij} cannot achieve the minimum at any point to the right of v_{k+1}. □

This completes the proof of Theorem 4.2 in the vertex avoiding case.

Remark 4.11. For future reference, we note that the proof of Lemma 4.10 does not depend on the relative lengths of the bridges. It only uses that the bridges are much longer than the loops. The assumption that each bridge is much longer than the next is only used later, when there are decreasing bridges, decreasing loops, or switching loops.

Remark 4.12. If Γ' is the subgraph of Γ to the right of w_1, then Γ' is a chain of 12 loops whose edge lengths satisfy the required conditions, and if the first loop is non-lingering, then the restriction of Σ to Γ' satisfies the ramification condition of Theorem 4.2, with equality. Similarly, the subgraph to the right of w_2 is a chain of 11 loops whose edge lengths satisfy the required conditions, and the restriction of Σ to this subgraph satisfies the ramification condition of Theorem 4.2, with equality. To produce an independence in these cases, assign each function in $\mathcal{B}'$ with slope greater than 4 to the first bridge, and then proceed as above. There are precisely $15 - g$ such functions, and they have distinct slopes along the first bridge, as in [FJP, Lemma 8.25]. Because of this, we can choose coefficients so that each one obtains the minimum uniquely at some point of the first bridge. Thus the unramified vertex avoiding cases of Theorem 4.2 for $g = 11$ and 12 follow from essentially the same argument as for $g = 13$. Our choice to index the vertices starting at w_{13-g} reflects the idea that these linear series with ramification on a chain of $g = 11$ or 12 loops behave like linear series on a chain of 13 loops restricted to the subgraph to the right of w_{13-g}.

Example 4.13. We illustrate the construction with an example. Let $[D]$ be a vertex avoiding class of degree 16 and rank 5 associated to the tableau in Figure 3.

01	03	04	08	09	10
02	05	07	11	12	13

FIGURE 3. The tableau corresponding to the divisor D.

The independence $\theta = \min_{ij}\{\varphi_{ij} + c_{ij}\}$ that we construct is depicted schematically in Figure 4. The graph should be read from left to right and top to bottom, so the first 6 loops appear in the first row, with γ_1 on the left and γ_6 on the right, and γ_{13} is the last loop in the third row. The rows correspond to the three blocks. The 31 dots indicate the support of the divisor $D' = 2D + \mathrm{div}(\theta)$. Note that $\deg(D') = 32$; the point on the bridge β_4 appears with multiplicity 2, as marked. Because $\ell = 6$, there is a function that is permissible on the second block in $\mathcal{B}$ but not $\mathcal{B}'$. The functions in $\mathcal{B}$ that are permissible on the second block are precisely φ_{05}, φ_{14}, and φ_{23}; we have chosen (arbitrarily) to omit φ_{23} from $\mathcal{B}'$. Each of the 20 functions φ_{ij} in $\mathcal{B}'$ achieves the minimum uniquely on the connected component of the complement of $\mathrm{Supp}(D')$ labeled ij.

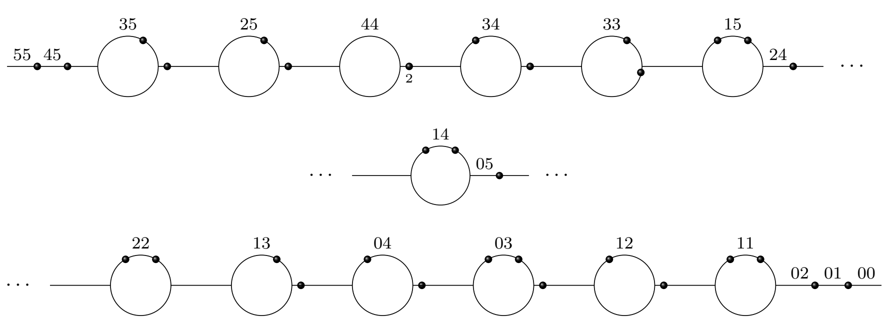

FIGURE 4. The divisor $D' = 2D + \mathrm{div}(\theta)$. The function φ_{ij} achieves the minimum uniquely on the region labeled ij in $\Gamma \smallsetminus \mathrm{Supp}(D')$.

4.2. **No switching loops.** Recall that a loop γ_ℓ is a *switching loop* for Σ if there is some $\varphi \in \Sigma$ and some h such that $s_\ell(\varphi) \leq s_\ell[h]$ and $s'_\ell(\varphi) \geq s'_\ell[h+1]$. It is a *lingering loop* if it is not a switching loop and $s_\ell[i] = s'_\ell[i]$ for all i. Recall also that γ_ℓ is a *decreasing loop* if $s_\ell[h] > s'_\ell[h]$. Similarly β_ℓ is a *decreasing bridge* if $s'_{\ell-1}[h] > s_\ell[h]$.

Because we are only considering cases where the adjusted Brill-Noether number is 1, by [FJP, Proposition 6.18], we know that there is either a lingering loop, a positive ramification weight, a decreasing loop, a decreasing bridge, or a switching loop, and these possibilities are mutually exclusive. (More precisely, the sum of all ramification weights plus all multiplicities of loops and bridges, as defined in [FJP, §6.7], is equal to 1; for decreasing loops and bridges, this means that the index h is unique and the decrease in slope is exactly 1.) In this subsection, we consider all cases where there is no switching loop. The cases with a switching loop are discussed in §4.3.

Assume Σ has no switching loops. Then, for all i there is a function $\varphi_i \in \Sigma$ such that

$$s_k(\varphi_i) = s_k[i] \quad \text{and} \quad s'_k(\varphi_i) = s'_k[i] \text{ for all } k.$$

We again let $\mathcal{B} = \{\varphi_i + \varphi_j \colon 0 \leq i \leq j \leq 5\}$. As in the unramified vertex avoiding case, we choose a subset $\mathcal{B}' \subseteq \mathcal{B}$ of 20 functions, and we choose integers z_1 and z_2 in order to divide the graph Γ into 3 blocks. We make our choices to satisfy the following conditions:

(i) no two functions in $\mathcal{B}'$ that are permissible on γ_k differ by a constant on γ_k,
(ii) the number of functions in $\mathcal{B}'$ that are permissible on each block is at most one more than the number of loops in that block,
(iii) no function in $\mathcal{B}'$ is permissible on more than one block,
(iv) if γ_k is a lingering loop, then it is the last loop in its block,
(v) if γ_k is a decreasing loop and j is the unique value such that $s'_k[j] < s_k[j]$, then no function of the form $\varphi_{ij} \in \mathcal{B}'$ is permissible on γ_k, and
(vi) if β_k is a decreasing bridge and j is the unique value such that $s_k[j] < s'_{k-1}[j]$, then either β_k is a bridge between blocks, or no function of the form $\varphi_{ij} \in \mathcal{B}'$ is permissible on γ_{k-1}.

Proposition 4.14. *If $\mathcal{B}'$ satisfies conditions (i)-(vi), then the functions in $\mathcal{B}'$ are independent.*

Proof. The algorithm for constructing the tropical independence is identical to that presented in [FJP, §7], with the following exceptions. First, as in Remark 4.12, we assign every function with slope greater than 4 to the first bridge. Second, we do not skip lingering loops and instead treat them as in the vertex avoiding case above. Finally, the procedure for Proceeding to the Next Block must be altered slightly when the bridge between the blocks is a decreasing bridge. In this case, there is a unique point v on the bridge where one of the functions φ_i is locally nonlinear. We initialize the coefficients of the new permissible functions on the next block so that they are equal to θ at a point to the right of v. In the case where one of the blocks contains zero loops, we set the coefficient of the unique function with slope equal to that of θ so that it is equal to θ at a point to the right of v, and initialize the coefficients of the new permissible functions on the next block so that they are equal to θ at a point to the right of this.

In [FJP, § 7], Step 4 of the Loop Subroutine requires that there are at most 3 non-departing permissible functions on each loop. In this case, this follows directly from the fact that the rank is 5.

To see that this algorithm produces an independence, suppose that φ_{ij} is assigned to the loop γ_k or the bridge β_k. We show that φ_{ij} does not achieve the minimum at any point to the right of v_{k+1}. If γ_k and β_k both have multiplicity zero, then the argument is the same as in [FJP, Lemma 7.21]. On the other hand, if γ_k has positive multiplicity, then either γ_k is a decreasing loop, or by (iv) it is the last loop in its block. If γ_k is a decreasing loop, then by (v) there is no function in $\mathcal{B}'$ that is permissible on γ_k and contains the decreasing function as a summand, so the result holds again by [FJP, Lemma 7.21]. We may therefore assume that γ_k is the last loop in its block, in which case the argument is identical to the vertex avoiding case above.

Similarly, if β_k has positive multiplicity, then by (vi) there are two possibilities. If φ_{ij} does not contain the decreasing function as a summand, then there is nothing to show. Otherwise, β_k is a bridge between blocks. By (iii) the function φ_{ij} is only permissible on one block. Since v_{k+1} is the start of the next block, φ_{ij} cannot achieve the minimum at any point to the right of v_{k+1}. $\square$

For the rest of this section, we explain how to choose z_1, z_2, and the set $\mathcal{B}'$ in order to satisfy conditions (i)-(vi). This is done by a careful case analysis, depending on combinatorial properties of the tropical linear series Σ.

Case 1: There are no loops or bridges of positive multiplicity. This guarantees that the linear series is ramified either at w_{13-g} or v_{14} (but not both, since $\rho = 1$). In this case, we choose z_1 and z_2 so that γ_{z_1} is the first loop in the first block with no new function, and γ_{z_2+1} is the last loop in the last block with no departing function. These loops are guaranteed to exist by a counting argument, but we can in fact be more explicit.

If Σ is ramified at v_{14}, let k be the smallest positive integer such that $s'_k[5] = 6$, and define

$$z_1 = \begin{cases} 6 & \text{if } k \geq 7; \\ 7 & \text{if } k \leq 6; \end{cases} \qquad \text{and} \qquad z_2 = \max\{k-1, 7\}. \tag{21}$$

If Σ is ramified at w_{13-g}, let k be the largest positive integer such that $s_k[0] = -3$, and define

$$z_1 = \min\{k, 6\}, \qquad \text{and} \qquad z_2 = \begin{cases} 6 & \text{if } k \geq 8; \\ 7 & \text{if } k \leq 7. \end{cases} \tag{22}$$

Let $\psi \in \mathcal{B}$ be a function that is permissible on the second block, and let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\psi\}$. (In the case where $z_1 = z_2$, let $\psi \in \mathcal{B}$ be a function with $s_{z_1+1}(\psi) = 3$.)

If there is a loop or bridge of positive multiplicity, then since $\rho = 1$, there is only one such loop or bridge, and it has multiplicity 1.

Case 2: There is a bridge β_ℓ of multiplicity 1. If $\ell \geq 8$ and $s'_{\ell-1}[5] = 6$, then define z_1 and z_2 as in (21). If $\ell \leq 7$ and $s_\ell[0] = -3$, then define z_1 and z_2 as in (22). Otherwise, define

$$z_1 = \min\{\ell - 1, 6\}, \qquad \text{and} \qquad z_2 = \ell - 1.$$

If $\ell \geq 8$ and $s_{\ell-1}[5] = 6$ or $\ell \leq 7$ and $s_\ell[0] = -3$, then as above, we let $\psi \in \mathcal{B}$ be a function that is permissible on the second block, and let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\psi\}$. Otherwise, let h be the unique integer such that $s_\ell[h] < s'_{\ell-1}[h]$. If $\ell \neq 5, 6$, then we will see in Lemma 4.15 that either there is a unique i such that $s'_{\ell-1}[h] + s'_{\ell-1}[i] = s_{\ell-1}(\theta)$, or $2s'_{\ell-1}[h] = s_{\ell-1}(\theta) + 1$, but not both. In the first case, we let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\varphi_{hi}\}$, and in the second case, we let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\varphi_{hh}\}$. (The function in $\mathcal{B} \smallsetminus \mathcal{B}'$ is permissible on both blocks to either side of the bridge β_ℓ.) If $\ell = 5$ or 6, then we will see in Lemma 4.15 that there is a unique i such that $s'_{\ell-1}[h] + s'_{\ell-1}[i] = s_{\ell-1}(\theta) - 1$, and we again let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\varphi_{hi}\}$.

It remains to consider the cases where there is a loop of multiplicity one. The case of a switching loop is left to the next subsection. In the case of a lingering loop, we construct an independence exactly as in the unramified vertex avoiding case. We now discuss the only other case.

Case 3: There is a decreasing loop γ_ℓ. If $\ell \geq 8$ and $s_\ell[5] = 6$, then define z_1 and z_2 as in (21). If $\ell \leq 7$ and $s'_\ell[0] = -3$, then define z_1 and z_2 as in (22). Otherwise, define

$$z_1 = \begin{cases} \ell & \text{if } \ell < 6, \\ 5 & \text{if } \ell = 6, \\ 6 & \text{if } \ell > 6, \end{cases} \qquad \text{and} \qquad z_2 = \begin{cases} \ell - 1 & \text{if } \ell > 8, \\ 8 & \text{if } \ell = 8, \\ 7 & \text{if } \ell < 8. \end{cases}$$

If $\ell \geq 8$ and $s_\ell[5] = 6$ or $\ell \leq 7$ and $s_\ell[0] = -3$, then as above, we let $\psi \in \mathcal{B}$ be a function that is permissible on the second block, and let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\psi\}$. Otherwise, let h be the unique integer such that $s'_\ell[h] < s_\ell[h]$. If $\ell < 6$ or $\ell = 7, 8$ then γ_ℓ is the last loop in its block, and we will see in Lemma 4.15 that either there is a unique i such that $s_\ell[h] + s_\ell[i] = s_\ell(\theta)$, or $2s_\ell[h] = s_\ell(\theta) + 1$, but not both. In the first case, we let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\varphi_{hi}\}$, and in the second case, we let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\varphi_{hh}\}$. If $\ell > 8$ or $\ell = 6$, then we will see that either there is a unique i such that $s_\ell[h] + s_\ell[i] = s_{\ell-1}(\theta)$, or $2s_\ell[h] = s_{\ell-1}(\theta) + 1$. Again, in the first case, we let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\varphi_{hi}\}$, and in the second case, we let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\varphi_{hh}\}$.

In the cases above, we asserted several times that certain functions exist with specified slopes. To prove this, we need to generalize Proposition 4.6. We first define the following function.

$$\tau(k) = \sum_{i=0}^{5}(s'_k[i] + 2 - i).$$

Note that, if there is a loop of positive multiplicity and γ_ℓ is the kth loop of multiplicity zero, then $k = \tau(\ell)$. The following observation serves as the basis for our counting arguments.

Lemma 4.15. *For a fixed k, suppose that $-2 \le s'_k[i] \le 5$ for all i. Let j be an integer such that $s'_k[j] - 1$ is not equal to -3 or $s'_k[i]$ for any i. For s in the range $2 \le s \le 5$, there does not exist i such that $s'_k[i] + s'_k[j] = s$ if and only if one of the following holds:*

(i) $\tau(k) = 10 - s$;
(ii) $s = 5$, $j = 0$, *and* $s'_k[0] = -1$;
(iii) $s = 2$, $j = 5$, *and* $s'_k[5] = 5$;
(iv) $2s'_k[j] = s + 1$.

Proof. The argument is identical to that of Proposition 4.6. □

There are additional relevant cases, when $s'_k[5] = 6$ or $s'_k[0] = -3$.

Lemma 4.16. *If $s'_k[5] = 6$, then there does not exist i such that $s'_k[i] + 6 \le 3$. Similarly, if $s_k[0] = -3$, then there does not exist i such that $s'_k[i] - 2 \ge 4$.*

Proof. Since $\rho = 1$, if $s'_k[5] = 6$, then $s'_k[0] \ge -2$. It follows that $s'_k[i] + 6 \ge 4$ for all i. Similarly, if $s_k[0] = -3$, then $s'_k[i] \le 5$ for all i. It follows that $s'_k[i] - 2 \le 3$ for all i. □

Lemma 4.17. *The set $\mathcal{B}'$ satisfies conditions (i)-(vi).*

Proof. Condition (i): If $s'_k[i] \ge s_k[i]$ for all i, then the result is immediate, so we may assume that γ_k is a decreasing loop. Let h be the unique integer such that $s'_k[h] = s_k[h] + 1$, and let h' be the unique integer such that $s'_k[h'] = s_k[h'] - 1$. If $\varphi_{hh'}$ is not permissible on γ_k, then again there is nothing to show. If $\varphi_{hh'}$ is permissible, then by Lemma 4.15, we must have $s_k(\theta) = 10 - k$. By construction, this occurs if and only if $k = 7$, in which case $\varphi_{hh'} \notin \mathcal{B}'$.

Condition (ii): Consider the first block first. There are two functions $\psi \in \mathcal{B}$ with the property that $s'_{13-g}(\psi) = 4$. The result will therefore hold for the first block if and only if the first block contains a loop with no new permissible functions. Let γ_k be a loop of multiplicity zero that is contained in the first block. By Lemmas 4.15 and 4.16, there is no new permissible function on γ_k if and only if $\tau(k) = 6$ or $s'_k[0] = s_k[0]+1 = -2$. Thus, the number of permissible functions in $\mathcal{B}$ on the first block is at most 2 more than the number of loops in Cases 2 or 3 when $\ell \le 6$ and $s_\ell[0] \ge -2$, and 1 more than the number of loops in the remaining cases. In Cases 2 and 3 when $\ell \le 6$ and $s_\ell[0] \ge -2$, the function in $\mathcal{B} \smallsetminus \mathcal{B}'$ is permissible on the first block. Since this function is not in $\mathcal{B}'$, the number of functions in $\mathcal{B}'$ that are permissible on the first block is one less than the number in $\mathcal{B}$. The third block follows from a completely symmetric argument.

For the second block, note that if $\tau(z_1) = 6$, then there are 3 functions $\psi \in \mathcal{B}$ with the property that $s'_{z_1}(\psi) = 3$, and otherwise there are only two such functions. In every case, either $\tau(z_1) < 6$ or by Lemma 4.15, the second block contains a loop with no new permissible functions. Since the function in $\mathcal{B} \smallsetminus \mathcal{B}'$ is permissible on the second block, we see that the number of permissible functions on the second block is one more than the number of loops. (Note that this holds even in the case where the second block contains zero loops, in which case there is exactly one permissible function on the second block.)

Condition (iii): Suppose that $\varphi_{ij} \in \mathcal{B}$ is permissible on more than one block. First, consider the case where β_ℓ is a bridge of multiplicity one, and let h be the unique integer such that $s_\ell[h] = s_{\ell-1}[h] - 1$. If φ_{ij} is permissible on more than one block, then $j = h$ and either $s'_{\ell-1}[h] + s'_{\ell-1}[i] = s_{\ell-1}(\theta)$, or $i = h$ and $2s'_{\ell-1}[h] = s_{\ell-1}(\theta) + 1$. If $-2 \le s_\ell[h] \le 5$, then by Lemma 4.15, such an i exists if and only if $\ell \ne 5, 6$, and by construction, we have $\varphi_{hi} \notin \mathcal{B}'$. Similarly, if $s_\ell[h] = -3$, then by Lemma 4.16, such an i exists if and only if $\ell \geq 8$, and if $s_\ell[h] = 5$, then such an i exists if and only if $\ell \leq 7$. In both cases, we have $\varphi_{hi} \notin \mathcal{B}'$.

Next, consider the case where γ_ℓ is a decreasing loop. By construction, γ_ℓ is either the first or last loop in its block. Let h be the unique integer such that $s'_\ell[h] = s_\ell[h] - 1$. If γ_ℓ is the last loop in its block and φ_{ij} is permissible on both the block containing γ_ℓ and the next block, then $j = h$ and either $s_\ell[h] + s_\ell[i] = s_\ell(\theta)$, or $i = h$ and $2s_\ell[h] = s_\ell(\theta) + 1$. But then $\varphi_{ij} \notin \mathcal{B}'$. Similarly, if γ_ℓ is the first loop in its block, and φ_{ij} is permissible on both the block containing γ_ℓ and the preceding block, then $j = h$ and either $s_\ell[h] + s_\ell[i] = s_{\ell-1}(\theta)$, or $2s_\ell[h] = s_{\ell-1}(\theta) + 1$. If $\ell \neq 7$, then again $\varphi_{ij} \notin \mathcal{B}'$. Finally, note that if γ_ℓ is *both* the first and last loop in its block, then $\ell = 7$, and the only functions φ_{ij} that are permissible on γ_7 satisfy $s'_\ell[i] + s'_\ell[j] = 3$. The result follows.

Condition (iv): If γ_ℓ is a lingering loop, then we follow the construction of the vertex avoiding case of the previous subsection, in which γ_ℓ is the last loop in its block.

Condition (v): Let γ_k be a decreasing loop, let h be the unique integer such that $s'_k[h] = s_k[h] + 1$, and let h' be the unique integer such that $s'_k[h'] = s_k[h'] - 1$. If $\varphi_{hh'}$ is permissible, then $\varphi_{hh'} \notin \mathcal{B}'$, as shown in the proof of condition (i).

Condition (vi): Let β_k be a decreasing bridge and let j is the unique value such that $s_k[j] < s'_{k-1}[j]$. If β_k is not a bridge between blocks, then by construction either $j = 0$, $k \leq 7$, and $s_k[0] = -3$, or $j = 5$, $k \geq 8$, and $s_k[5] = 5$. In both cases, by Lemma 4.16, we see that there is no i such that $\varphi_{ij} \in \mathcal{B}'$ is permissible on γ_{k-1}. □

This completes the proof of Theorem 4.2 in all cases where there is no switching loop for Σ.

4.3. **Switching loops.** We now consider the case where there is a switching loop γ_ℓ, which switches slope h. This means that $s_\ell[i] = s'_\ell[i]$ for all i, and there exists a function $\varphi \in R(D)$ satisfying

$$s_\ell(\varphi) = s_\ell[h],\ s'_\ell(\varphi) = s'_\ell[h] + 1 = s'_\ell[h+1].$$

In this case, we define z_1 and z_2 as follows:

$$z_1 = \begin{cases} \ell & \text{if } \ell < 6, \\ 5 & \text{if } \ell = 6, \\ 6 & \text{if } \ell > 6; \end{cases} \qquad \text{and} \qquad z_2 = \begin{cases} 7 & \text{if } \ell < 6, \\ \ell & \text{if } \ell \geq 6. \end{cases}$$

By [FJP, Proposition 9.18], there is a pencil $W \subseteq V$ with φ_A, φ_B, and φ_C in $\mathrm{trop}(W)$ such that:

(i) $s'_k(\varphi_A) = s'_k[h]$ for all $k < \ell$;
(ii) $s_k(\varphi_B) = s_k[h+1]$ for all $k > \ell$;
(iii) $s_k(\varphi_C) = s_k[h+1]$ for all $k \leq \ell$, and $s'_k(\varphi_C) = s_k[h]$ for all $k \geq \ell$;
(iv) $s_k(\varphi_\bullet) \in \{s_k[h], s_k[h+1]\}$ and $s'_k(\varphi_\bullet) \in \{s'_k[h], s'_k[h+1]\}$ for all k.

Moreover, by [FJP, Lemma 9.19], there are functions $\varphi^0_h, \varphi^0_{h+1}$, and φ^∞_h in $R(D)$ such that:

(1) $s_k(\varphi^0_h) = s_k[h]$ and $s'_k(\varphi^0_h) = s'_k[h]$ for all k;
(2) $s_k(\varphi^0_{h+1}) = s_k[h+1]$ and $s'_k(\varphi^0_{h+1}) = s'_k[h+1]$ for all k;
(3) $s_k(\varphi^\infty_h) = s_k[h]$, $s'_{k-1}(\varphi^\infty_h) = s'_{k-1}[h]$ for all $k \leq \ell$,
and $s_k(\varphi^\infty_h) = s_k[h+1]$, $s'_{k-1}(\varphi^\infty_h) = s'_{k-1}[h+1]$ for all $k > \ell$.
(4) The function φ_A is a tropical linear combination of the functions φ^0_h and φ^∞_h, where the two functions simultaneously achieve the minimum at a point to the right of γ_ℓ.
(5) The function φ_B is a tropical linear combination of the functions φ^0_{h+1} and φ^∞_h, where the two functions simultaneously achieve the minimum at a point to the left of γ_ℓ.
(6) The function φ_C is a tropical linear combination of the functions φ^0_h and φ^0_{h+1}, where the two functions simultaneously achieve the minimum on the loop γ_ℓ where they agree.

Note that $\varphi^0_h, \varphi^0_{h+1}$, and φ^∞_h are in $R(D)$ but not necessarily in Σ. Let

$$\mathcal{A} = \{\varphi_i : i \neq h, h+1\} \cup \{\varphi^0_h, \varphi^0_{h+1}, \varphi^\infty_h\},$$

and let $\mathcal{B}$ be the set of pairwise sums of elements of $\mathcal{A}$. We first choose a subset $\mathcal{B}'' \subseteq \mathcal{B}$, and then construct a tropical linear combination ϑ of the elements of $\mathcal{B}''$. Then, as in [FJP, § 9.3], we will obtain an independence as the best approximation of ϑ by certain pairwise sums of elements of Σ.

We now construct the set $\mathcal{B}''$. If there exists a j such that $s'_\ell[h+1] + s'_\ell[j] = s_\ell(\theta) + 1$, then we let $\mathcal{B}' = \mathcal{B} \smallsetminus \{\varphi_h^0 + \varphi_j\}$. If $\ell \leq 7$, $\ell \neq 6$, then we let $\psi \in \mathcal{B}'$ be a function that is permissible on the second block, and let $\mathcal{B}'' = \mathcal{B}' \smallsetminus \{\psi\}$. If $\ell = 6$, then there exists a unique j such that $s'_5[j] = s_5[j] + 1$, and by Lemma 4.15, there exists a unique i such that $s'_5[i] + s'_5[j] = 4$. We let $\mathcal{B}'' = \mathcal{B}' \smallsetminus \{\varphi_{ij}\}$. If $\ell > 7$ and there exists a j such that $s'_\ell[h+1] + s'_\ell[j] = s_\ell(\theta)$, then we let $\mathcal{B}'' = \mathcal{B}' \smallsetminus \{\varphi_h^0 + \varphi_j, \varphi_h^\infty + \varphi_j\}$. Otherwise, if $\ell > 7$ and no such j exists, then we let $\psi \in \mathcal{B}$ be a function that is permissible on the third block, and let $\mathcal{B}'' = \mathcal{B}' \smallsetminus \{\psi\}$.

To construct the tropical linear combination ϑ, we follow the algorithm of [FJP, § 7], with the following modification. This can be seen as a simplified version of the algorithm from [FJP, § 8]. For any function $\varphi \in \mathcal{A}$, if $\varphi + \varphi_h^0$ is assigned to a loop γ_k with $k < \ell$, and if both $\varphi + \varphi_h^0$ and $\varphi + \varphi_h^\infty$ are in $\mathcal{B}''$, then we also assign $\varphi + \varphi_h^\infty$ to γ_k. Similarly, if $\varphi + \varphi_{h+1}^0$ is assigned to a loop γ_k with $k \geq \ell$, and if both $\varphi + \varphi_{h+1}^0$ and $\varphi + \varphi_h^\infty$ are in $\mathcal{B}''$, then we also assign $\varphi + \varphi_h^\infty$ to γ_k. The output of this algorithm satisfies the following.

Lemma 4.18. *For each $\varphi \in \mathcal{A}$, one of the following holds:*

(i) $\varphi + \varphi_h^0 \notin \mathcal{B}''$,
(ii) $\varphi + \varphi_h^\infty \notin \mathcal{B}''$,
(iii) $\varphi + \varphi_h^\infty$ *is assigned to the same loop as* $\varphi + \varphi_h^0$, *or*
(iv) $\varphi + \varphi_h^\infty$ *is assigned to the same loop as* $\varphi + \varphi_{h+1}^0$.

Proof. By construction, it suffices to consider the case where $\varphi + \varphi_h^0$ is not assigned to a loop γ_k with $k < \ell$, but $\varphi + \varphi_{h+1}^0$ is.

If $\ell < 6$, then by Lemma 4.15 there is a departing function on γ_k for all $k < \ell$. If ℓ is equal to 6 or 7, then it is the first loop in its block. In each of these cases, it follows that

$$s_\ell(\varphi) + s_\ell[h+1] \geq s_\ell(\theta) + 1.$$

Since $\varphi + \varphi_h^0$ is not assigned to a loop γ_k with $k < \ell$, we must have equality in the expression above. By construction, in this case $\varphi + \varphi_h^0 \notin \mathcal{B}''$.

If $\ell > 7$, then we could instead have

$$s_\ell(\varphi) + s_\ell[h+1] = s_\ell(\theta).$$

But in this case, $\varphi + \varphi_h^\infty \notin \mathcal{B}''$. □

We have the following.

Lemma 4.19. *The best approximation of ϑ by $\varphi_C + \varphi_j$ achieves equality on the region where either $\varphi_h^0 + \varphi_j$ or $\varphi_{h+1}^0 + \varphi_j$ achieves the minimum.*

Proof. If $\mathcal{B}''$ contains both $\varphi_h^0 + \varphi_j$ and $\varphi_{h+1}^0 + \varphi_j$, then this is immediate. If not, it does not contain $\varphi_h^0 + \varphi_j$, and $s'_\ell[h+1] + s'_\ell[j] = s_\ell(\theta) + 1$. In this case, $\varphi_C + \varphi_j$ has slope greater than $s_\ell(\theta)$ on β_ℓ, so it must achieve equality to the left of $\gamma_{\ell-1}$, where it agrees with $\varphi_{h+1}^0 + \varphi_j$. □

As in [FJP, § 9.3], if the best approximation by $\varphi_C + \varphi_j$ achieves equality where $\varphi_h^0 + \varphi_j$ achieves the minimum, then we replace $\mathcal{B}_j = \mathcal{B}'' \cap \{\varphi_h^0 + \varphi_j, \varphi_{h+1}^0 + \varphi_j, \varphi_h^\infty + \varphi_j\}$ with $\{\varphi_B + \varphi_j, \varphi_C + \varphi_j\}$. Otherwise, if it achieves equality where $\varphi_{h+1}^0 + \varphi_j$ achieves the minimum, then we replace $\mathcal{B}_j$ with $\{\varphi_A + \varphi_j, \varphi_C + \varphi_j\}$. If $\varphi_h^\infty + \varphi_j \notin \mathcal{B}''$, then we replace only with $\{\varphi_C + \varphi_j\}$. Similarly, we replace the subset of $\mathcal{B}''$ consisting of pairwise sums of elements of $\{\varphi_h^0, \varphi_{h+1}^0, \varphi_h^\infty\}$ with three pairwise sums of elements of $\{\varphi_A, \varphi_B, \varphi_C\}$. The best approximation of ϑ by these replacements is then a tropical independence among pairwise sums of elements of Σ. This completes the proof of Theorem 4.2 in all cases where there is a switching loop for Σ.

5. Effectivity of the virtual class

Recall that $\widetilde{\mathfrak{M}}_{13}$ is an open substack of the moduli stack of stable curves, and $\widetilde{\mathfrak{G}}^r_d$ is a stack of generalized limit linear series of rank r and degree d over $\widetilde{\mathfrak{M}}_{13}$. There is a morphism of vector bundles $\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$ over $\widetilde{\mathfrak{G}}^r_d$, whose degeneracy locus is denoted by $\mathfrak{U}$.

The case of Theorem 4.2 where $g = 13$ shows that the push forward $\sigma_*[\mathfrak{U}]^{\mathrm{virt}}$ under the proper forgetful map $\sigma : \widetilde{\mathfrak{G}}^r_d \to \widetilde{\mathcal{M}}_g$ is a divisor, not just a divisor class. In our proof that $\sigma_*[\mathfrak{U}]^{\mathrm{virt}}$ is effective, we will use the additional cases where $g = 12$ or 11. Theorem 4.2 implies the following:

Theorem 5.1. *Let X be a general curve of genus $g \in \{11, 12, 13\}$, and let $p \in X$ be a general point. Let $V \subseteq H^0(X, L)$ be a linear series of degree 16 and rank 5. Assume that*

(i) if $g = 12$, then $a_1^V(p) \geq 2$, and
(ii) if $g = 11$, then either $a_1^V(p) \geq 3$, or $a_0^V(p) + a_2^V(p) \geq 5$.

Then the multiplication map $\phi_V : \mathrm{Sym}^2 V \to H^0(X, L^{\otimes 2})$ is surjective.

We now prove that $\mathfrak{U}$ is generically finite over each component of $\sigma_*[\mathfrak{U}]^{\mathrm{virt}}$, which implies that $\sigma_*[\mathfrak{U}]^{\mathrm{virt}}$ is effective. Our argument follows closely that of [FJP, § 10]. Indeed, in many cases the arguments are identical, and we omit the proof. As in [FJP, § 10], we suppose that $Z \subseteq \overline{\mathcal{M}}_{13}$ is an irreducible divisor and that $\sigma|_{\mathfrak{U}}$ has positive dimensional fibres over the generic point of Z. Let $\jmath_2 : \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_{13}$ be the map obtained by attaching an arbitrary pointed curve of genus 2 to a fixed general pointed curve (X, p) of genus 11. Since $g = 13$ is odd, by [FJP, Proposition 2.2], it suffices to show the following:

(1) Z is the closure of a divisor in $\mathcal{M}_{13}$,
(2) $\jmath_2^*(Z) = 0$, and
(3) Z does not contain any codimension 2 stratum $\Delta_{2,j}$.

The only irreducible boundary divisors in $\widetilde{\mathcal{M}}_{13}$ are Δ_0° and Δ_1°. Therefore, item (1), that Z is the closure of a divisor in $\mathcal{M}_{13}$, is a consequence of the following.

Proposition 5.2. *The image of the degeneracy locus $\mathfrak{U}$ does not contain Δ_0° or Δ_1°.*

Proof. The proof is identical to [FJP, Proposition 10.3]. □

The proofs of (2) and (3) use the following lemma.

Lemma 5.3. *If $[X] \in Z$ and $p \in X$, then there is a linear series $V \in G^5_{16}(X)$ that is ramified at p such that ϕ_V is not surjective.*

Proof. The proof is identical to [FJP, Lemma 10.4]. □

5.1. Pulling back to $\overline{\mathcal{M}}_{2,1}$.

In order to verify (2), we consider the preimage of Z under the map $\jmath_2$.

Lemma 5.4. *The preimage $\jmath_2^{-1}(Z)$ is contained in the Weierstrass divisor $\overline{\mathcal{W}}_2$ in $\overline{\mathcal{M}}_{2,1}$.*

Proof. The proof is identical to [FJP, Lemma 10.5]. □

To prove that $\jmath_2^*(Z) = 0$, we consider the following construction. Let Γ be a chain of 13 loops with the following restrictions on edge lengths:

(i) $m_2 = \ell_2$ (that is, the second loop has torsion index 2),
(ii) $n_3 \gg n_2$, and
(iii) $\ell_{k+1} \ll m_k \ll \ell_k \ll n_{k+1} \ll n_k$ for all $k \neq 2$.

The last condition says that, subject to the constraints of conditions (i) and (ii), the edge lengths otherwise satisfy (20). Let X be a smooth curve of genus 13 over K whose skeleton is Γ. We first note the following.

Lemma 5.5. *If $[X] \notin Z$, then $\jmath_2^*(Z) = 0$.*

Proof. This proof is identical to the first part of the proof of [FJP, Proposition 10.6]. □

Proposition 5.6. *We have $\jmath_2^*(Z) = 0$.*

Proof. By Lemma 5.5, it suffices to show that $[X] \notin Z$. We divide Γ into two subgraphs $\widetilde{\Gamma}'$ and $\widetilde{\Gamma}$, to the left and right, respectively, of the midpoint of the long bridge β_3. Let $q \in X$ be a point specializing to v_{14}. If $[X] \in Z$, by Lemma 5.3 there is a linear series in the degeneracy locus over X that is ramified at q. We now show that this impossible.

Let $\ell = \big(L, V\big) \in G^5_{16}(X)$ be a linear series ramified at q. We may assume that $L = \mathcal{O}(D_X)$, where $D = \mathrm{Trop}(D_X)$ is a break divisor, and consider $\Sigma = \mathrm{trop}(V)$. We will show that there are 20 tropically independent pairwise sums of functions in Σ using a variant of the arguments in § 4. It follows that the multiplication map ϕ_ℓ is surjective, and hence $[X]$ cannot be in Z.

To produce 20 tropically independent pairwise sums of functions in Σ, following the methods of § 4, we first consider the slope sequence along the long bridge β_3. First, suppose that either $s_3[4] \leq 2$ or $s_3[3] + s_3[5] \leq 5$. In this case, even though the restriction of Σ to $\widetilde{\Gamma}$ is not the tropicalization of a linear series on a pointed curve of genus 11 with prescribed ramification, it satisfies all of the combinatorial properties of the tropicalization of such a linear series. The proof of Theorem 4.2 then goes through verbatim, yielding a tropical linear combination of 20 functions in Σ such that each function achieves the minimum uniquely at some point of $\widetilde{\Gamma} \subseteq \Gamma$.

For the remainder of the proof, we therefore assume that $s_3[4] \geq 3$ and $s_3[3] + s_3[5] \geq 6$. Since $\deg D_{|\widetilde{\Gamma}'} = 5$, we see that $s_3[5] \leq 5$. Moreover, since the divisor $D_{|\widetilde{\Gamma}'} - s_3[4]w_2$ has positive rank on $\widetilde{\Gamma}'$, and no divisor of degree 1 on $\widetilde{\Gamma}'$ has positive rank, $s_3[4]$ must be exactly 3. Since the canonical class is the only divisor class of degree 2 and rank 1 on $\widetilde{\Gamma}'$, we see that $D_{|\widetilde{\Gamma}'} \sim K_{\widetilde{\Gamma}'} + 3w_2$. This yields an upper bound on each of the slopes $s_3[i]$, and these bounds determine the slopes for $i \geq 2$:

$$s_3[5] = 5,\ s_3[4] = 3,\ s_3[3] = 1,\ s_3[2] = 0.$$

Moreover, we must have $s_2'[i] = s_3[i]$ for $2 \leq i \leq 5$. Since ℓ is ramified at q, we also have $s_{14}[5] \geq 6$. These conditions together imply that the sum of the multiplicities of all loops and bridges on $\widetilde{\Gamma}$ is at most 1.

To construct an independence on Γ, we first construct an independence among 5 functions on $\widetilde{\Gamma}'$. This is done exactly as in [FJP, Figure 26], and we omit the details.

Next, we construct an independence among 15 pairwise sums of functions in Σ restricted to $\widetilde{\Gamma}$, with the property that any function ψ that obtains the minimum on $\widetilde{\Gamma}$ satisfies $s_2'(\psi) \leq 4$. Note that each of the functions ψ that obtains the minimum on $\widetilde{\Gamma}'$ satisfies $s_3(\psi) \geq 5$. Since the bridge β_3 is very long, it follows that no function that obtains the minimum on one of the two subgraphs can obtain the minimum on the other. Thus, we have constructed a tropical linear combination of 20 pairwise sums of functions in Σ in which 5 achieve the minimum uniquely at some point of $\widetilde{\Gamma}'$ and 15 achieve the minimum uniquely at some point of $\widetilde{\Gamma}$. In particular, this is an independence, as required.

It remains to construct an independence among 15 pairwise sums of functions in Σ restricted to $\widetilde{\Gamma}$. To do this, we run the algorithm from [FJP], with one change. (Indeed, one could imagine that Γ is simply the first 13 loops in a chain of 23 loops. We construct the independence from [FJP, § 10.3], and restrict it to Γ.) At the start, we skip the step named "Start at the First Bridge". Instead, we do not assign any function ψ with $s_3(\psi) \geq 5$, and we start with the Loop Subroutine applied to γ_3. Following this construction, there will only be two blocks, and there will be two functions with slope 2 along the last bridge β_{14}. We eliminate one of these functions from $\mathcal{B}$, and assign the other to β_{14}. The rest of the argument is exactly the same as that of [FJP]. □

5.2. **Higher codimension boundary strata.** It remains to verify (3), that Z does not contain any of the codimension 2 boundary strata $\Delta_{2,j} \subseteq \overline{\mathcal{M}}_{13}$.

Proposition 5.7. *The component Z does not contain any codimension 2 stratum $\Delta_{2,j}$.*

Proof. The proof is again a variation on the independence constructions from the proof of Theorem 4.2. We fix $\ell = 11 - j$. Let Y_1 be a smooth curve of genus 2 over K whose skeleton Γ_1 is a chain of

2 loops with bridges, and let $p \in Y_1$ be a point specializing to the right endpoint of Γ_1. Similarly, let Y_2 and Y_3 be smooth curves of genus ℓ and j, respectively, whose skeletons Γ_2 and Γ_3, are chains of ℓ loops and j loops with edge lengths satisfying (20). Suppose further that the edges in the final loop of Γ_2 are much longer than those in the first loop of Γ_3. Let $p, q \in Y_2$ be points specializing to the left and right endpoints of Γ_2, respectively, and let $q \in Y_3$ be a point specializing to the left endpoint of Γ_3. We show that $[Y'] = [Y_1 \cup_p Y_2 \cup_q Y_3] \in \Delta_{2,j}$ is not contained in Z.

As in the proof of [FJP, Proposition 10.6], if $[Y'] \in Z$, then Z contains points $[X]$ corresponding to smooth curves whose skeletons are arbitrarily close to the skeleton of Y' in the natural topology on $\overline{M}_{13}^{\mathrm{trop}}$. In particular, there is an $X \in Z$ with skeleton a chain of loops Γ_X whose edge lengths satisfy all the conditions of (20), except that the bridges β_3 and β_ℓ are exceedingly long in comparison to the other edges. Let Γ be the subgraph of Γ_X to the right of the midpoint of the bridge β_3. Note that Γ is a chain of 11 loops, labeled $\gamma_3, \ldots, \gamma_{13}$, with bridges labeled $\beta_3, \ldots, \beta_{14}$.

By Lemma 5.3, there is a linear series V of degree 16 and rank 5 on X that is ramified at a point x specializing to the righthand endpoint v_{14}, and such that ϕ_V is not surjective. We will show that this is not possible, using the tropical independence construction from § 4. Let $\Sigma = \mathrm{trop}(V)$. We have that either $s_2'[4] \leq 2$ or $s_2'[3] + s_2'[4] \leq 5$. Also, since V is ramified at x, we have $s_{14}[5] \geq 6$. These conditions imply that the multiplicity of every loop and bridge is zero. In particular, for each i there is a function φ_i satisfying

$$s_k(\varphi_i) = s'_{k-1}(\varphi_i) = s_k[i] = s'_{k-1}[i] \text{ for all } k.$$

These functions have constant slope along bridges, and the slopes $s_k(\varphi_i)$ are nondecreasing in k. These properties guarantee that, even though the bridge β_ℓ is very long, a function φ_{ij} can only obtain the minimum on a loop or bridge where it is permissible.

Even though the restriction of Σ to Γ is not the tropicalization of a linear series on a curve of genus 11 with prescribed ramification at two specified points specializing to the left and right endpoints of Γ, it satisfies all of the combinatorial properties of the tropicalization of such a linear series, and we may apply the algorithm from § 4. Because we are in a situation where the relative lengths of the bridges do not matter (Remark 4.11) the construction yields an independence among 20 pairwise sums of functions in Σ, and the proposition follows. □

6. The Bertram-Feinberg-Mukai Conjecture in genus 13

The aim of this section is to prove the existence part of the Bertram-Feinberg-Mukai conjecture on $\overline{\mathcal{M}}_{13}$. For a smooth curve X of genus g, we denote by $SU_X(2, \omega)$ the moduli space of S-equivalence classes of semistable rank 2 vector bundles E on X with $\det(E) \cong \omega_X$. For an integer $k \geq 0$, the Brill-Noether locus

$$SU_X(2, \omega, k) := \left\{E \in SU_X(2, \omega) : h^0(X, E) \geq k\right\}$$

has the structure of a Lagrangian degeneracy locus and each component of $SU_X(2, \omega, k)$ has dimension at least $\beta(2, g, k) = 3g - 3 - \binom{k+1}{2}$, see [Mu2]. Furthermore, $SU_X(2, \omega, k)$ is smooth of dimension $\beta(2, g, k)$ at a point $[E]$ corresponding to a stable vector bundle if and only if the Mukai-Petri map (1) is injective. Of particular interest to us is the case

$$g = 13 \text{ and } k = 8,$$

in which case $\beta(2, 13, 8) = 0$. First, using linkage methods, we show that a general curve of genus 13 carries a stable vector bundle $E \in SU_X(2, \omega, 8)$. Then using a Hecke correspondence, we compute the fundamental class of $SU_X(2, \omega, 8)$.

Theorem 6.1. *A general curve X of genus 13 caries a stable vector bundle E of rank 2 with $\det(E) \cong \omega_X$ and $h^0(X, E) = 8$.*

As a first step towards proving Theorem 6.1, we determine the extension type of the vector bundles in question.

Proposition 6.2. *For a general curve X of genus 13, every vector bundle $E \in SU_X(2,\omega,8)$ can be represented as an extension*

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow E \longrightarrow \omega_X(-D) \longrightarrow 0, \tag{23}$$

where D is an effective divisor of degree 6 on X, such that $L := \omega_X(-D) \in W^6_{18}(X)$ is very ample and the map $\phi_L\colon \mathrm{Sym}^2 H^0(X,L) \to H^0(X,L^{\otimes 2})$ is not surjective. Conversely, a very ample $L \in W^6_{18}(X)$ with ϕ_L not surjective induces a stable vector bundle $E \in SU_X(2,\omega,8)$.

Proof. Using a result of Segre [LN, Proposition 3.1], every semistable vector bundle E on X of rank 2 and canonical determinant carries a line subbundle $\mathcal{O}_X(D) \hookrightarrow E$ with $\deg(D) \geq \frac{g-2}{2}$. Therefore, in our case $\deg(D) \geq 6$.

If $h^0(X,\mathcal{O}_X(D)) \geq 2$, since $h^0(X,\mathcal{O}_X(D)) + h^0(X,\omega_X(-D)) \geq h^0(X,E) = 8$ it follows from the Brill-Noether Theorem and Riemann-Roch that $\deg(D) = 8$ and the multiplication map $\phi_{\omega_X(-D)}$ is not surjective, which contradicts Theorem 1.5. Therefore $h^0(X,\mathcal{O}_X(D)) = 1$, in which case necessarily $\deg(D) = 6$ and $h^0(X,E) = h^0(X,\mathcal{O}_X(D)) + h^0(X,\omega_X(-D))$. Setting $L := \omega_X(-D) \in W^6_{18}(X)$, an extension E satisfies $h^0(X,E) = 8$ if and only if the extension class of E in $\mathrm{Ext}^1(L,D)$ lies in the kernel of the linear map

$$\mathrm{Ext}^1(L,D) \to H^0(L)^\vee \otimes H^1(D).$$

Thus, an extension (23) exists if and only if the multiplication map

$$\phi_L\colon \mathrm{Sym}^2 H^0(L) \to H^0(X,L^{\otimes 2}) \cong \mathrm{Ext}^1(L,D)^\vee$$

is not surjective. We claim that L is very ample. Otherwise, there exist points $x,y \in X$ such that $L' := L(-x-y) \in W^5_{16}(X)$. Since X is general, by Theorem 1.5 the multiplication map $\phi_{L'}\colon \mathrm{Sym}^2 H^0(X,L') \to H^0\big(X,(L')^{\otimes 2}\big)$ is surjective, implying the inclusion $H^0\big(X,(L')^{\otimes 2}(x+y)\big) \subseteq \mathrm{Im}(\phi_L)$. We deduce that $[E]$ lies in the kernel of the map

$$\mathrm{Ext}^1(L,D) \to \mathrm{Ext}^1(L(-x-y),D).$$

That is, the vector bundle E can also be represented as an extension

$$0 \longrightarrow L(-x-y) \longrightarrow E \longrightarrow \mathcal{O}_X(D+x+y) \longrightarrow 0,$$

thus contradicting the semistability of E. We conclude that L has to be very ample.

Conversely, each *very ample* linear system $L \in W^6_{18}(X)$, for which the map ϕ_L is not surjective induces a stable vector bundle E; see also [CGZ, 7.2]. Indeed, let us assume E is not semistable. In view of the extension (23), a maximally destabilizing line subbundle of E is of the form $L(-M)$, where M is an effective divisor on X with $\deg(M) \leq 6$. Therefore, apart from (23), E can also be realized as an extension

$$0 \longrightarrow L(-M) \longrightarrow E \longrightarrow \mathcal{O}_X(D+M) \longrightarrow 0.$$

By applying Riemann-Roch, one can then write

$$h^0\big(X,L(-M)\big) + h^1\big(X,L(-M)\big) = h^0(X,L) + h^1(X,L) - 2\dim \frac{H^0(X,L)}{H^0(X,L(-M))} + \deg(M).$$

Since

$$h^0(X,L) + h^1(X,L) = h^0(X,E) \leq h^0\big(X,L(-M)\big) + h^1\big(X,L(-M)\big),$$

it follows that

$$\deg(M) \geq 2\dim \frac{H^0(L)}{H^0(L(-M))}.$$

Since L is very ample, we find $\deg(M) \in \{4,5,6\}$. In each case, the Brill-Noether number of $L(-M)$ is negative, contradicting the generality of X. Therefore E is stable. $\square$

Proof of Theorem 6.1. By Proposition 6.2, it suffices to show that for a general curve X of genus 13, there exists a very ample linear system $L \in W^6_{18}(X)$ such that ϕ_L is not surjective. We use a method inspired by Verra's proof of the unirationality of $\overline{\mathcal{M}}_{14}$ [Ve]. To illustrate the idea behind the proof, first suppose that there exists an embedding $\varphi_L \colon X \hookrightarrow \mathbb{P}^6$ given by $L \in W^6_{18}(X)$, such that the map ϕ_L is not surjective. In particular, $X \subseteq \mathbb{P}^6$ lies on at least $5 = \binom{8}{2} - h^0(X, L^{\otimes 2}) - 1$ quadrics. We expect the base locus of this system of quadrics to be a reducible curve (of degree 32), containing X as a component and write accordingly

$$X + C = \mathrm{Bs}\,\big|\mathcal{I}_{X/\mathbb{P}^6}(2)\big|.$$

Assuming that X and C intersect transversally, we obtain that C is a curve of degree 14 and $2g(X) - 2g(C) = (10-7)\big(\deg(X) - \deg(C)\big) = 12$, therefore $g(C) = 7$.

We now reverse this procedure and start with a general curve $C \subseteq \mathbb{P}^6$ of genus 7 embedded by a 7-dimensional linear system $V \subseteq H^0(C, L_C)$, where $L_C \in \mathrm{Pic}^{14}(C)$ is a general line bundle, therefore $h^0(C, L_C) = 8$. Consider the multiplication map

$$\phi_V \colon \mathrm{Sym}^2(V) \to H^0(C, L_C^{\otimes 2})$$

and observe that $\mathrm{Ker}(\phi_V)$ has dimension at least $6 = \dim \mathrm{Sym}^2(V) - h^0(L_C^{\otimes 2})$. Choose a general 5-dimensional system of quadrics $W \in G\big(5, H^0(\mathbb{P}^6, \mathcal{I}_{C/\mathbb{P}^6}(2))\big)$. We then expect

$$\mathrm{Bs}\,|W| = C + X \subseteq \mathbb{P}^6 \tag{24}$$

to be a nodal curve, and the curve X linked to C to be a smooth curve of degree 18 and genus 13. Setting $L := \mathcal{O}_X(1) \in W^6_{18}(X)$, by construction L is very ample and the embedded curve $X \subseteq \mathbb{P}^6$ lies on at least 5 quadrics, therefore ϕ_L is not surjective.

To carry this out, one needs to check some transversality statements. Let $\mathcal{P}ic^{14}_7$ be the universal Picard variety parametrizing pairs $[C, L_C]$, where C is a smooth curve of genus 7 and $L_C \in \mathrm{Pic}^{14}(C)$. As pointed out in [Ve, Theorem 1.2], it follows from Mukai's work [Mu1] that $\mathcal{P}ic^{14}_7$ is unirational. We introduce the variety:

$$\mathcal{Y} := \Big\{[C, L_C, V, W] : [C, L_C] \in \mathcal{P}ic^{14}_7,\ V \in G\big(6, H^0(C, L_C)\big), W \in G\big(5, \mathrm{Ker}(\phi_V)\big)\Big\}$$

The forgetful map $\mathcal{Y} \to \mathcal{P}ic^{14}_7$ has the structure of an iterated locally trivial projective bundle over $\mathcal{P}ic^{14}_7$, therefore $\mathcal{Y}$ is unirational as well. Moreover,

$$\dim(\mathcal{Y}) = \dim(\mathcal{P}ic^{14}_7) + \dim G(7,8) + \dim G(5,6) = 4 \cdot 7 - 3 + 7 + 5 = 37.$$

One has a rational *linkage map*

$$\chi \colon \mathcal{Y} \dashrightarrow \mathcal{SU}_{13}(2, \omega, 8), \quad [C, L_C, V, W] \mapsto [X, L, E],$$

where X is defined by (24), $L := \mathcal{O}_X(1) \in W^6_{18}(X)$ and $E \in SU_X(2, \omega, 8)$ is the rank 2 vector bundle defined uniquely by the extension $0 \to \omega_X \otimes L^\vee \to E \to L \to 0$.

To show that χ is well-defined it suffices to produce one example of a point in $\mathcal{Y}$ for which all these assumptions are realized. To that end, we consider 11 general points $p_1, \ldots, p_5$ and $q_1, \ldots, q_6$ respectively in $\mathbb{P}^2$ and the linear system

$$H \equiv 6h - 2(E_{p_1} + \cdots + E_{p_5}) - (E_{q_1} + \cdots + E_{q_6})$$

on the blow-up $S = \mathrm{Bl}_{11}(\mathbb{P}^2)$ at these points. Here h denotes the pullback of the line class from $\mathbb{P}^2$. Via *Macaulay2* one checks that $S \overset{|H|}{\hookrightarrow} \mathbb{P}^6$ is an embedding and the graded Betti diagram of S is the following:

$$\begin{array}{ccccc} 1 & - & - & - & - \\ - & 5 & - & - & - \\ - & - & 15 & 16 & 15 \end{array}$$

Next we consider a general curve $C \subseteq S$ in the linear system

$$C \equiv 10h - 4(E_{p_1} + E_{p_2} + E_{p_3} + E_{p_4}) - 3E_{p_5} - 2(E_{q_1} + E_{q_2}) - (E_{q_3} + E_{q_4} + E_{q_5} + E_{q_6}).$$

Via *Macaulay2*, we verify that C is smooth, $g(C) = 7$ and $\deg(C) = 14$. Furthermore, using that $H^1\big(\mathbb{P}^6, \mathcal{I}_{S/\mathbb{P}^6}(2)\big) = 0$, we have an exact sequence

$$0 \longrightarrow H^0\big(\mathbb{P}^6, \mathcal{I}_{S/\mathbb{P}^6}(2)\big) \longrightarrow H^0\big(\mathbb{P}^6, \mathcal{I}_{C/\mathbb{P}^6}(2)\big) \longrightarrow H^0\big(S, \mathcal{O}_S(2H - C)\big) \longrightarrow 0.$$

Since $\mathcal{O}_S(2H-C) = \mathcal{O}_S(2h - E_{p_5} - E_{q_3} - E_{q_4} - E_{q_5} - E_{q_6})$, clearly $h^0\big(S, \mathcal{O}_S(2H-C)\big) = 1$, therefore $h^0\big(\mathbb{P}^6, \mathcal{I}_{C/\mathbb{P}^6}(2)\big) = 6$. That is, $C \subseteq \mathbb{P}^6$ is a 2-normal curve.

One also verifies with *Macaulay2* that $C \subseteq \mathbb{P}^6$ is scheme-theoretically cut out by quadrics. Using [Ve, Proposition 2.2], C lies on a smooth surface $Y \subseteq \mathbb{P}^6$ which is a complete intersection of 4 quadrics containing C. Furthermore, the linear system $\big|\mathcal{O}_Y(2H - C)\big|$ is base point free, so a general element $X \in \big|\mathcal{O}_Y(2H - C)\big|$ is a smooth curve of genus 13 meeting C transversally. Finally, a standard argument using the exact sequence $0 \to \mathcal{O}_Y(H - X) \to \mathcal{O}_Y(H) \to \mathcal{O}_X(H) \to 0$ shows that since C is 2-normal, the residual curve X is 1-normal. That is, $h^1\big(X, \mathcal{O}_X(1)\big) = 1$. This implies that the map $\chi \colon \mathcal{Y} \dashrightarrow \mathcal{SU}_{13}(2, \omega, 8)$ is well-defined and dominant. □

Corollary 6.3. *The parameter space $\mathcal{SU}_{13}(2, \omega, 8)$ is unirational.*

Proof. This follows from the proof of Theorem 6.1 and from the unirationality of $\mathcal{Y}$. □

6.1. **The fundamental class of $SU_X(2, \omega, 8)$ for a general curve.** It is essential for our calculations to determine the degree of the map

$$\vartheta \colon \mathcal{SU}_{13}(2, \omega, 8) \to \mathcal{M}_{13}, \quad \vartheta\big([X, E]\big) = [X].$$

We fix a general curve X of genus g and a point $p \in X$. Since the moduli space $SU_X(2, \omega)$ is singular, in order to determine the fundamental class of the non-abelian Brill-Noether locus $SU_X(2, \omega, k)$, following [Ne], [LN], [Mu2] one uses instead the Hecke correspondence relating $SU_X(2, \omega)$ to the smooth moduli space $SU_X(2, \omega(p))$ of stable rank 2 vector bundles F on X with $\det(F) \cong \omega_X(p)$.

Recall that $SU_X(2, \omega(p))$ is a fine moduli space. Hence there is a universal rank 2 vector bundle $\mathcal{F}$ on $X \times SU_X(2, \omega(p))$ and we consider the *Hecke correspondence*

$$\mathbf{P} := \mathbb{P}\Big(\mathcal{F}_{|\{p\} \times SU_X(2, \omega(p))}\Big),$$

endowed with the projection $\pi_1 \colon \mathbf{P} \to SU_X(2, \omega(p))$. The points of $\mathbf{P}$ are exact sequences

$$0 \longrightarrow E \longrightarrow F \longrightarrow K(p) \longrightarrow 0, \tag{25}$$

where $F \in SU_X(2, \omega(p))$, and therefore $\det(E) \cong \omega_X$. One has a diagram

$$\begin{tikzcd} & \mathbf{P} \arrow[dl, "\pi_1"'] \arrow[dr, "\rho"] & \\ SU_X(2, \omega(p)) & & SU_X(2, \omega) \end{tikzcd}$$

where ρ assigns to a sequence (25) the semistable vector bundle E . Set

$$h := c_1\big(\mathcal{O}_{\mathbf{P}}(1)\big) = \rho^* c_1(\mathcal{L}_{\mathrm{ev}}),$$

where $\mathcal{L}_{\mathrm{ev}}$ is the determinant line bundle on $SU_X(2, \omega)$, associated to the effective divisor

$$\Theta := \big\{E \in SU_X(2, \omega) : H^0(X, E) \neq 0\big\}.$$

Set $\alpha := c_1\big(\mathcal{L}_{\mathrm{odd}}\big) \in H^2\big(SU_X(2, \omega(p)), \mathbb{Z}\big)$, where $\mathcal{L}_{\mathrm{odd}}$ is the ample generator of $\mathrm{Pic}\big(SU_X(2, \omega(p))\big)$. Note that $\mathrm{Pic}(\mathbf{P})$ is generated by h and by $\pi_1^*(\alpha)$.

For each $k \in \mathbb{N}$, the non-abelian Brill-Noether locus

$$B_{\mathbf{P}}(k) := \Big\{\big[0 \to E \to F \to K(p) \to 0\big] \in \mathbf{P} : h^0(X, E) \geq k\Big\}$$

has the structure of a Lagrangian degeneracy locus of expected codimension $\beta(2,g,k)+1 = 3g-2-\binom{k+1}{2}$, see [Mu2, Section 5] and [LN, Section 2]. As such, its virtual class $[B_{\mathbf{P}}(k)]^{\mathrm{virt}} \in H^*(\mathbf{P},\mathbb{Q})$ can be computed in terms of certain tautological classes, whose definition we recall now.

Following [Ne], we consider the Künneth decomposition of the Chern classes of $\mathcal{F}$, using that $\det(\mathcal{F}) \cong \omega_X(p) \boxtimes \mathcal{L}_{\mathrm{odd}}$ and write

$$c_1(\mathcal{F}) = \alpha + (2g-1)\varphi \ \text{ and } \ c_2(\mathcal{F}) = \chi + \psi + g\alpha \otimes \varphi,$$

where $\varphi \in H^2(X,\mathbb{Q})$ is the fundamental class of the curve, $\chi \in H^4\big(SU_X(2,\omega(p)),\mathbb{Q}\big)$ and ψ is in $H^3\big(SU_X(2,\omega(p)),\mathbb{Q}\big) \otimes H^1(X,\mathbb{Q})$. Finally, we define the class

$$\gamma \in H^6\big(SU_X(2,\omega(p)),\mathbb{Q}\big)$$

by the formula $\psi^2 = \gamma \otimes \varphi$. One has the relation

$$h^2 = \alpha h - \frac{\alpha^2-\beta}{4} \in H^4(\mathbf{P},\mathbb{Q}),$$

from which we can recursively determine all powers of h. We summarize as follows:

Proposition 6.4. *For each $n \geq 2$, the following relation holds in $H^*(\boldsymbol{P},\mathbb{Q})$:*

$$h^n = \frac{h(-2\alpha+2h)\sqrt{\beta}+\alpha^2-2\alpha h+\beta}{\sqrt{\beta}(\alpha^2-\beta)}\Big(\frac{\alpha+\sqrt{\beta}}{2}\Big)^n + \frac{h(2\alpha-2h)\sqrt{\beta}+\alpha^2-2\alpha h+\beta}{\sqrt{\beta}(\alpha^2-\beta)}\Big(\frac{\alpha-\sqrt{\beta}}{2}\Big)^n.$$

In this formula $\sqrt{\beta}$ is a formal root of the class β. Applying [LN, Section 3] or [Mu2] one can endow $B_{\mathbf{P}}(k)$ with the structure of a Lagrangian degeneracy locus as follows. Let $\mathcal{E}$ be the vector bundle on $X \times \mathbf{P}$ defined by the following exact sequence:

$$0 \longrightarrow \mathcal{E} \longrightarrow \big(\mathrm{id}\times\pi_1\big)^*(\mathcal{F}) \to (p_2)_*\big(\mathcal{O}_{\mathbf{P}}(1)\big) \longrightarrow 0,$$

where $p_2\colon X\times\mathbf{P} \to \mathbf{P}$ is the projection. Choose an effective divisor D of large degree on X and also denote by D its pull-back under $X\times\mathbf{P}\to X$. Then $(p_2)_*\big(\mathcal{E}/\mathcal{E}(-D)\big)$ and $(p_2)_*(\mathcal{E}(D))$ are Lagrangian subbundles of $(p_2)_*\big(\mathcal{E}(D)/\mathcal{E}(-D)\big)$. For each point $t := [0\to E\to F\to K(p)\to 0] \in \mathbf{P}$, one has

$$(p_2)_*\Big(\mathcal{E}(D)\Big)(t) \cap (p_2)_*\Big(\mathcal{E}/\mathcal{E}(-D)\Big)(t) \cong H^0(X,E).$$

Assume from now on $g = 13$ and $k = 8$, therefore we expect $B_{\mathbf{P}}(8)$ to be 1-dimensional. Applying the formalism for Lagrangian degeneracy loci [Mu2, Proposition 1.11], we find the following determinantal formula for its virtual fundamental class

$$[B_{\mathbf{P}}(8)]^{\mathrm{virt}} = \begin{vmatrix} c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} & c_{12} & c_{13} \\ c_4 & c_5 & c_6 & c_7 & c_8 & c_9 & c_{10} & c_{11} \\ c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 & c_9 \\ c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ 0 & 0 & c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ 0 & 0 & 0 & 0 & c_0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_0 & c_1 \end{vmatrix} \tag{26}$$

where $c_i \in H^{2i}(\mathbf{P},\mathbb{Q})$ are defined recursively by the following formulas, see [LN, Corollary 4.2]:

$$c_1 = h,\ c_2 = \frac{h^2}{2},\ c_3 = \frac{1}{3}\Big(\frac{h^3}{2}+\frac{\beta h}{4}-\frac{\gamma}{2}\Big),\ c_4 = \frac{1}{4}\Big(\frac{h^4}{6}+\frac{\beta h^2}{3}-\frac{2\gamma h}{3}\Big), \tag{27}$$

and for each $n \geq 1$,

$$(n+4)c_{n+4} - \frac{n+2}{2}\beta c_{n+2} + \Big(\frac{\beta}{4}\Big)^2 n c_n = hc_{n+3} - \Big(\frac{\beta h}{4}+\frac{\gamma}{2}\Big)c_{n+1}. \tag{28}$$

In order to evaluate the determinant giving $[B_{\mathbf{P}}(8)]^{\mathrm{virt}}$, we shall use Proposition 6.4 coupled with the formula of Thaddeus [Th] determining all top intersection numbers of tautological classes on $SU_X(2,\omega(p))$. Precisely, for $m+2n+3p=3g-3$, one has

$$\int_{SU_X(2,\omega(p))}\alpha^m\cdot\beta^n\cdot\gamma^p=(-1)^{g-p}\frac{g!m!}{(g-p)!q!}2^{2g-2-p}(2^q-2)B_q, \tag{29}$$

where $q=m+p+1-g$ and B_q denotes the Bernoulli number; those appearing in our calculation are:

$$B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ B_6=\frac{1}{42},\ B_8=-\frac{1}{30}, B_{10}=\frac{5}{66},\ B_{12}=-\frac{691}{2730},\ B_{14}=\frac{7}{6},$$
$$B_{16}=-\frac{3617}{510},\ B_{18}=\frac{43867}{798},\ B_{20}=-\frac{174611}{330},\ B_{22}=\frac{854513}{138},\ B_{24}=-\frac{236364091}{2730}.$$

Theorem 6.5. *For a general curve X of genus 13, the locus $SU_X(2,\omega,8)$ consists of three reduced points corresponding to stable vector bundles.*

Proof. As explained, the Lagrangian degeneracy locus $B_{\mathbf{P}}(8)$ is expected to be a curve and we write

$$[B_{\mathbf{P}}(8)]^{\mathrm{virt}}=f(\alpha,\beta,\gamma)+h\cdot u(\alpha,\beta,\gamma),$$

where $f(\alpha,\beta,\gamma)$ and $u(\alpha,\beta,\gamma)$ are homogeneous polynomials of degree $36=3g-3$ and $35=3g-4$, respectively.

Observe that if $E\in SU_X(2,\omega,8)$ then necessarily E is a stable bundle. Otherwise E is strictly semistable, in which case $E=B\oplus(\omega_X\otimes B^\vee)$, where $B\in W^3_{12}(X)$, which contradicts the Brill-Noether Theorem on X. Since ρ is a $\mathbb{P}^1$-fibration over the locus of stable vector bundles, it follows that $B_{\mathbf{P}}(8)$ is a $\mathbb{P}^1$-fibration over $SU_X(2,\omega,8)$. Furthermore, applying [T2], the Mukai-Petri map μ_E is an isomorphism for each vector bundle $E\in SU_X(2,\omega,8)$, therefore $SU_X(2,\omega,8)$ is a reduced zero-dimensional cycle. We denote by a its length, thus we can write

$$[B_{\mathbf{P}}(8)]=[B_{\mathbf{P}}(8)]^{\mathrm{virt}}=a\rho^*([E_0])=f(\alpha,\beta,\gamma)+h\cdot u(\alpha,\beta,\gamma), \tag{30}$$

where $[E_0]\in SU_X(2,\omega)$ is general. Intersecting both sides of (30) with h, we obtain

$$h\cdot f(\alpha,\beta,\gamma)=-h\cdot\alpha u(\alpha,\beta,\gamma).$$

Next observe that $\rho^*([E_0])\cdot\alpha=2$. Indeed, since ρ is a $\mathbb{P}^1$-fibration over the open locus of stable bundles and $\omega_{\mathbf{P}}=\rho^*(\mathcal{L}_{\mathrm{ev}})\otimes\pi^*(-\alpha)$, it follows that

$$-2=\deg\big(\omega_{\mathbf{P}|\rho^*([E_0])}\big)=\omega_{\mathbf{P}}\cdot\rho^*([E_0])=-\alpha\cdot\rho^*([E_0]).$$

Intersecting both sides of (30) with α, we find $2a=h\cdot\alpha u(\alpha,\beta,\gamma)=-h\cdot f(\alpha,\beta,\gamma)$, so

$$a=\big|SU_X(2,\omega,8)\big|=\frac{1}{2}\int_{\mathbf{P}}hf(\alpha,\beta,\gamma)=\frac{1}{2}\int_{SU_X(2,\omega(p))}f(\alpha,\beta,\gamma).$$

We are left with the task of computing the degree 36 polynomial $f(\alpha,\beta,\gamma)$, which is a long but elementary calculation. We consider the determinant (26) computing the class of $B_{\mathbf{P}}(8)$. First we substitute for each of the classes $c_1,\ldots,c_{15}$ the expression in terms of α,β,γ,h given by the recursion (28), starting with the initial conditions (27). Evaluating this determinant, we obtain a polynomial of degree 36 in the classes α,β,γ and h. We recursively express all the powers h^n with $n\geq 2$ and obtain a formula of the form $[B_{\mathbf{P}}(8)]=f(\alpha,\beta,\gamma)+h\cdot u(\alpha,\beta,\gamma)$. We set $h=0$ in this formula and then we evaluate each monomial of degree 36 in α,β,γ using Thaddeus' formulas (29). At the end, we obtain $f(\alpha,\beta,\gamma)=-6$, which completes the proof of Theorem 6.5[3]. $\square$

[3]The *Maple* file describing all the calculations explained here can be found at *https://www.mathematik.hu-berlin.de/farkas/gen13bn.mw*

7. THE NON-ABELIAN BRILL-NOETHER DIVISOR ON $\overline{\mathcal{M}}_{13}$

In this section we determine the class of the non-abelian Brill-Noether divisor $\overline{\mathcal{MP}}_{13}$ and prove Theorem 1.1. The results in this section also lay the groundwork for the proof that $\overline{\mathcal{R}}_{13}$ is of general type.

7.1. **Tautological classes on the universal non-abelian Brill-Noether locus.**

Definition 7.1. Let $\mathcal{M}_{13}^{\sharp}$ be the open subset of $\overline{\mathcal{M}}_{13}$ consisting of (i) smooth curves X of genus 13 with $SU_X(2,\omega,9)=\emptyset$, or of (ii) 1-nodal irreducible curves $[X/y\sim q]$, where X is a 7-gonal smooth genus 12 curve, $y,q\in X$, and such that the multiplication map $\phi_L\colon \mathrm{Sym}^2H^0(X,L)\to H^0(X,L^{\otimes 2})$ is surjective for each $L\in W^5_{15}(X)$.

Note that $\mathcal{M}_{13}^{\sharp}$ and $\mathcal{M}_{13}\cup\Delta_0$ agree in codimension one, in particular we identify $CH^1(\mathcal{M}_{13}^{\sharp})$ with $\mathbb{Q}\langle\lambda,\delta_0\rangle$. We let $\mathcal{SU}_{13}^{\sharp}(2,\omega,8)$ be the moduli space of pairs $[X,E]$, with $[X]\in\mathcal{M}_{13}^{\sharp}$ and E is a rank 2 vector bundle on X with $\det(E)\cong\omega_X$ and $h^0(X,E)\geq 8$. We still denote by $\vartheta\colon \mathcal{SU}_{13}^{\sharp}(2,\omega,8)\to\mathcal{M}_{13}^{\sharp}$ the forgetful map.

Proposition 7.2. *The map $\vartheta\colon \mathcal{SU}_{13}^{\sharp}(2,\omega,8)\to\mathcal{M}_{13}^{\sharp}$ is proper. Furthermore, for each $[X,E]\in\mathcal{SU}_{13}^{\sharp}(2,\omega,8)$ the corresponding vector bundle E is globally generated.*

Proof. Suppose $\mathcal{X}\to T$ is a flat family of stable curves of genus 13, such that its generic fibre X_η is smooth and the special fibre X_0 corresponds to a 1-nodal curve in $\mathcal{M}_{13}^{\sharp}$. The moduli space $SU_{X_\eta}(2,\omega)$ specializes to a moduli space $SU_{X_0}(2,\omega)$ that is a closed subvariety of the moduli space $U_{X_0}(2,24)$ of S-equivalence classes of torsion free sheaves of rank 2 and degree 24 on X_0. The points in $SU_{X_0}(2,\omega)$ are described in [Su].

We claim that if $E\in SU_{X_0}(2,\omega)$ satisfies $h^0(X_0,E)\geq 8$, then necessarily E is locally free, in which case $\bigwedge^2 E\cong\omega_{X_0}$. Suppose $\nu\colon X\to X_0$ is the normalization map, let $y,q\in X$ denote the inverse images of the node p of X_0 and assume E is not locally free at p. Denoting by $\mathfrak{m}_p\subseteq\mathcal{O}_{X_0,p}$ the maximal ideal, either (i) $E_p\cong\mathfrak{m}_p\oplus\mathfrak{m}_p$, or else (ii) $E_p\cong\mathcal{O}_{X_0,p}\oplus\mathfrak{m}_p$. In the first case $E=\nu_*(F)$, where F is a vector bundle of rank 2 on X with $\det(F)\cong\omega_X$, that is, $SU_X(2,\omega,8)\neq\emptyset$. Note that

$$h^0\big(X,\det(F)\big)=12\leq 2h^0(X,F)-4,$$

implying that F has a subpencil $A\hookrightarrow F$. Then $A\in W^1_7(X)$ and $L:=\omega_X\otimes A^\vee\in W^5_{15}(X)$ is such that $\phi_L\colon\mathrm{Sym}^2H^0(X,L)\to H^0(X,L^{\otimes 2})$ is not surjective. This is ruled out by the definition of $\mathcal{M}_{13}^{\sharp}$. In case (ii), when $E_p\cong\mathcal{O}_{X_0,p}\oplus\mathfrak{m}_p$, one has an exact sequence

$$0\longrightarrow E\longrightarrow\nu_*(\widetilde{F})\longrightarrow K(p)\longrightarrow 0,$$

where $\widetilde{F}=\nu^*(E)/\mathrm{Torsion}$ is a vector bundle on the smooth curve X and satisfies $\det(F)=\omega_X(y)$, or $\det(F)\cong\omega_X(q)$, see also [Su, 1.2]. Observe that also in this case F necessarily carries a subpencil, and we argue as before to rule out this possibility.

We now turn out to the last part of Proposition 7.2. Choose $[X,E]\in\mathcal{SU}_{13}^{\sharp}(2,\omega,8)$ and assume for simplicity X is smooth (the case when X is 1-nodal being similar). Assume E is not globally generated at a point $q\in X$. Then there exists a vector bundle $F\in SU_X(2,\omega(-q),8)$, obtained from F by an elementary transformation at q. Note that $h^0\big(X,\det(F)\big)\leq 2h^0(X,F)-4$, which forces F to have a subpencil $A\hookrightarrow F$. Necessarily, $\deg(A)=7$. Since $h^0(F)=h^0(A)+h^0\big(\omega_X\otimes A^\vee(-q)\big)$, setting $L:=\omega_X\otimes A^\vee\in W^6_{17}(X)$, it follows that the multiplication map

$$H^0(X,L)\otimes H^0(X,L(-q))\to H^0\big(X,L^{\otimes 2}(-q)\big)$$

is not surjective, and in particular the map $\mathrm{Sym}^2H^0(X,L)\to H^0(X,L^{\otimes 2})$ is not surjective either. Then X possesses a stable rank 2 vector bundle with canonical determinant and $9=h^0(X,A)+h^0(X,L)$ sections, which is not the case. $\square$

Let us consider the universal genus 13 curve

$$\wp\colon \mathcal{C}_{13}^{\sharp} \to \mathcal{SU}_{13}^{\sharp}(2,\omega,8),$$

then let $\mathfrak{E}$ be the universal rank two bundle over $\mathcal{SU}_{13}^{\sharp}(2,\omega,8)$. Note that we can normalize $\mathfrak{E}$ is such a way that $\det(\mathfrak{E}) \cong \omega_{\wp}$.

Definition 7.3. We define the tautological class $\gamma := \wp_*\big(c_2(\mathfrak{E})\big) \in CH^1\big(\mathcal{SU}_{13}^{\sharp}(2,\omega,8)\big)$.

We aim to determine the push-forward to $\mathcal{M}_{13}^{\sharp}$ of the class γ in terms of λ and δ_0. To that end, we begin with the following:

Proposition 7.4. *The push-forward $\wp_*(\mathfrak{E})$ is a locally free sheaf of rank 8 and*

$$c_1\big(\wp_*(\mathfrak{E})\big) = \vartheta^*(\lambda) - \frac{\gamma}{2} \in CH^1\big(\mathcal{SU}_{13}^{\sharp}(2,\omega,8)\big).$$

Proof. The fact that $\wp_*(\mathfrak{E})$ is locally free follows from [Ha]. We apply Grothendieck-Riemann-Roch to the curve $\wp\colon \mathcal{C}_{13}^{\sharp} \to \mathcal{SU}_{13}^{\sharp}(2,\omega,8)$ and to the vector bundle $\mathfrak{E}$ to obtain:

$$\mathrm{ch}\big(\wp_!(\mathfrak{E}\big) = \wp_*\Big[\Big(2 + c_1\big(\mathfrak{E}\big) + \frac{c_1^2(\mathfrak{E}) - 2c_2(\mathfrak{E})}{2} + \cdots\Big)\cdot\Big(1 - \frac{c_1(\Omega_{\wp}^1)}{2} + \frac{c_1^2(\Omega_{\wp}^1) + c_2(\Omega_{\wp}^1)}{12} + \cdots\Big)\Big].$$

We consider the degree one terms in this equality. Using [HM, page 49], observe that

$$c_1(\Omega_{\wp}^1) = c_1(\omega_{\wp}) \ \text{ and } \ \wp_*\Big(\frac{c_1^2(\Omega_{\wp}^1) + c_2(\Omega_{\wp}^1)}{12}\Big) = \vartheta^*(\lambda).$$

By Serre duality, observe that $R^1\wp_*(\mathfrak{E}) \cong \wp_*(\mathfrak{E})^{\vee}$, therefore one can write:

$$2c_1(\wp_*(\mathfrak{E})) = c_1(\wp_*(\mathfrak{E})) - c_1(R^1\wp_*(\mathfrak{E})) = 2\vartheta^*(\lambda) - \frac{1}{2}\wp_*\big(c_1^2(\omega_{\wp})\big) + \frac{1}{2}\wp_*\big(c_1^2(\omega_{\wp})\big) - \gamma,$$

which leads to the claimed formula. □

In view of our future applications to $\overline{\mathcal{R}}_{13}$, we introduce the rank 6 vector bundle

$$\mathcal{M}_{\mathfrak{E}} := \mathrm{Ker}\big\{\wp^*(\wp_*(\mathfrak{E})) \to \mathfrak{E}\big\}.$$

The fibre $M_E := \mathcal{M}_{\mathfrak{E}}[X,E]$ over a point $[X,E] \in \mathcal{SU}_{13}^{\sharp}(2,\omega,8)$ sits in an exact sequence

$$0 \longrightarrow M_E \longrightarrow H^0(X,E)\otimes\mathcal{O}_X \xrightarrow{\mathrm{ev}} E \longrightarrow 0. \tag{31}$$

Proposition 7.5. *The following formulas hold:* $c_1\big(\mathcal{M}_{\mathfrak{E}}\big) = \wp^*\Big(\vartheta^*(\lambda) - \frac{\gamma}{2}\Big) - c_1(\omega_{\wp})$ *and*

$$c_2\big(\mathcal{M}_{\mathfrak{E}}\big) = \wp^*c_2(\wp_*\mathfrak{E}) - c_2(\mathfrak{E}) - c_1(\omega_{\wp})\cdot\wp^*\big(\vartheta^*(\lambda) - \frac{\gamma}{2}\big) + c_1^2(\omega_{\wp}).$$

Proof. This follows from the splitting principle applied to $\mathcal{M}_{\mathfrak{E}}$, coupled with Proposition 7.4. □

7.2. **The resonance divisor in genus 13.** A general curve X of genus 13 has 3 stable vector bundles $E \in SU_X(2,\omega,8)$. In this case $h^0\big(X,\det(E)\big) = 2h^0(X,E) - 3$, which implies that requiring E to carry a subpencil defines a divisorial condition on the moduli space $\mathcal{SU}_{13}(2,\omega,8)$ and thus on $\mathcal{M}_{13}$. For a vector bundle $E \in SU_X(2,\omega)$, we denote its determinant map by

$$d\colon \bigwedge^2 H^0(X,E) \to H^0(X,\omega_X).$$

Definition 7.6. The *resonance divisor* $\mathfrak{Res}_{13}^{\sharp}$ is the locus of curves $[X] \in \mathcal{M}_{13}^{\sharp}$ for which

$$G\big(2,H^0(X,E)\big) \cap \mathbb{P}\big(\mathrm{Ker}(d)\big) \neq \emptyset,$$

for some vector bundle $E \in SU_X(2,\omega,8)$. In other words, $\mathfrak{Res}_{13}^{\sharp}$ is the locus of $[X]$ for which there exists an element $0 \neq a\wedge b \in \bigwedge^2 H^0(X,E)$ such that $d(a\wedge b) = 0$.

We set $\mathfrak{Res}_{13} := \mathfrak{Res}^{\sharp}_{13} \cap \mathcal{M}_{13}$. Note that $\mathfrak{Res}^{\sharp}_{13}$ comes with an induced scheme structure under the proper map $\vartheta\colon \mathcal{SU}^{\sharp}_{13}(2,\omega,8) \to \mathcal{M}^{\sharp}_{13}$. The points in $\mathfrak{Res}^{\sharp}_{13}$ correspond to those curves X for which a vector bundle $E \in SU_X(2,\omega,8)$ carries a subpencil (which is generated by the sections $a,b \in H^0(X,E)$ with $d(a\wedge b)=0$). The class $[\mathfrak{Res}^{\sharp}_{13}]$ can be computed in terms of certain tautological classes over $\mathcal{SU}^{\sharp}_{13}(2,\omega,8)$. On the other hand, we have a geometric characterization of points in $\mathfrak{Res}_{13}$, and it turns out that the resonance divisor coincides with $\mathfrak{D}_{13}$ away from the heptagonal locus $\mathcal{M}^1_{13,7}$.

Proof of Theorem 1.7. We show that one has the following equality of effective divisors

$$\mathfrak{Res}_{13} = \mathfrak{D}_{13} + 3\cdot \mathcal{M}^1_{13,7}$$

on $\mathcal{M}_{13}$. Indeed, let us assume $[X] \in \mathfrak{Res}_{13} \setminus \mathcal{M}^1_{13,7}$ and let $E \in SU_X(2,\omega,8)$ be the vector bundle which can be written as an extension

$$0 \longrightarrow A \longrightarrow E \longrightarrow \omega_X \otimes A^{\vee} \longrightarrow 0, \tag{32}$$

where $h^0(X,A) \geq 2$. Since $\mathrm{gon}(X)=8$, and since $8 \leq h^0(X,E) \leq h^0(X,A)+h^0(X,\omega_X\otimes A^{\vee})$, it follows that $A \in W^1_8(X)$ and $L := \omega_X \otimes A^{\vee} \in W^5_{16}(X)$. If such an extension exists, then the map ϕ_L is not surjective, therefore $[X] \in \mathfrak{D}_{13}$.

Conversely, if $[X] \in \mathfrak{D}_{13}$, there is some $L \in W^5_{16}(X)$ such that the multiplication map ϕ_L is not surjective. For $[X]$ a general point of an irreducible component of $\mathfrak{D}_{13}$, we may assume that the multiplication map ϕ_L has corank 1, for else $\varphi_L\colon X \hookrightarrow \mathbb{P}^5$ lies on a $(2,2,2)$ complete intersection in $\mathbb{P}^5$, which is a (possibly degenerate) $K3$ surface. But the locus of curves $[X] \in \mathcal{M}_{13}$ lying on a (possibly degenerate) $K3$ surface cannot exceed $g+19=32<3g-4$, a contradiction. We let

$$E \in \mathbb{P}\big(\mathrm{Ext}^1(L,\omega_X\otimes L^{\vee})\big)$$

be the *unique* vector bundle with $h^0(X,E)=h^0(X,L)+h^0(X,\omega_X\otimes L^{\vee})=8$. The argument of Proposition 6.2 shows that E is stable, otherwise there would exist an effective divisor M of degree 4 on X such that $L(-M) \in W^3_{12}(X)$. Since $\rho(13,3,12)=-3$, the locus of curves $[X] \in \mathcal{M}_{13}$ with $W^3_{12}(X) \neq \emptyset$ has codimension at least three in $\mathcal{M}_{13}$, hence this situation does not occur along a component of $\mathfrak{D}_{13}$. Summarizing, away from the divisor $\mathcal{M}^1_{13,7}$, the divisors $\mathfrak{Res}_{13}$ and $\mathfrak{D}_{13}$ coincide.

We now show that $\mathcal{M}^1_{13,7}$ appears with multiplicity 3 inside $\mathfrak{Res}_{13}$. Let X be a general 7-gonal curve of genus 13 and let $A \in W^1_7(X)$ denote its (unique) degree 7 pencil. Set $L := \omega_X \otimes A^{\vee} \in W^6_{17}(X)$. Each vector bundle $E \in SU_X(2,\omega,8)$ that has a subpencil appears as an extension

$$0 \longrightarrow A \longrightarrow E \stackrel{j}{\longrightarrow} L \longrightarrow 0. \tag{33}$$

In this case $h^0(X,E)=h^0(X,A)+h^0(X,L)-1$. That is, $V := \mathrm{Im}\big\{H^0(E) \stackrel{j}{\to} H^0(L)\big\}$ is 6-dimensional. Furthermore, the multiplication map

$$\mu_V\colon V\otimes H^0(X,L) \to H^0(X,L^{\otimes 2})$$

is not surjective. Conversely, each 6-dimensional subspace $V \subseteq H^0(X,L)$ such that μ_V is not surjective leads to a vector bundle $E \in \mathbb{P}\big(\mathrm{Ext}^1(L,A)\big)$ with $h^0(X,E)=8$. The corresponding bundle E is stable unless V is of the form $H^0(X,L(-p))$ for a point $p \in X$, in which case E can also be realized as an extension

$$0 \longrightarrow L(-p) \longrightarrow E \longrightarrow A(p) \longrightarrow 0.$$

To determine the number of such subspaces $V \subseteq H^0(X,L)$, we consider the projective space $\mathbf{P}^6 := \mathbb{P}\big(H^0(X,L)^{\vee}\big)$ and consider the vector bundle $\mathcal{A}$ on $\mathbf{P}^6$ with fibre

$$\mathcal{A}(V) = \frac{V\otimes H^0(X,L)}{\bigwedge^2 V}$$

over a point $[V] \in \mathbf{P}^6$. There exists a bundle morphism $\mu\colon \mathcal{A} \to H^0(X,L^{\otimes 2})\otimes \mathcal{O}_{\mathbf{P}^6}$ given by multiplication and the subspaces $[V] \in \mathbf{P}^6$ for which μ_V is not surjective (or, equivalently, $\mu^{\vee}$ is not injective) are precisely those lying in the degeneracy locus of μ, that is, for which $\mathrm{rk}(\mu(V)) = 21$. Applying the Porteous formula we find

$$[Z_{21}(\mu)] = c_6\Big(H^0(X, L^{\otimes 2})^\vee \otimes \mathcal{O}_{\mathbf{P}^6} - \mathcal{A}^\vee\Big) = c_6(-\mathcal{A}).$$

To compute the Chern classes of $\mathcal{A}$, we recall that via the Euler sequence the rank 6 vector bundle $M_{\mathbf{P}^6}$ on $\mathbf{P}^6$ with $M_{\mathbf{P}^6}(V) = V \subseteq H^0(X, L)$ can be identified with $\Omega_{\mathbf{P}^6}(1)$. Then $\mathcal{A}$ is isomorphic to $M_{\mathbf{P}^6} \otimes H^0(X, L)/\bigwedge^2 M_{\mathbf{P}^6}$. From the exact sequence

$$0 \longrightarrow \bigwedge^2 M_{\mathbf{P}^6} \longrightarrow \bigwedge^2 H^0(X, L) \otimes \mathcal{O}_{\mathbf{P}^7} \longrightarrow M_{\mathbf{P}^6}(1) \longrightarrow 0,$$

recalling that $c_{\mathrm{tot}}(M_{\mathbf{P}^6}) = \frac{1}{1+h}$, where $h = c_1(\mathcal{O}_{\mathbf{P}^6}(1))$, we find $c_{\mathrm{tot}}\big(\bigwedge^2 M_{\mathbf{P}^6}\big) = \frac{1+2h}{(1+h)^7}$, therefore

$$[Z_{21}(\mu)] = \Big[\frac{1}{(1+h)^7} \cdot \frac{(1+h)^7}{1+2h}\Big]_6 = \Big[\frac{1}{1+2h}\Big]_6 = 2^6 \cdot h^6 = 64.$$

From this, we subtract the excess contribution corresponding to the locus $X \overset{|L|}{\hookrightarrow} \mathbf{P}^6$, parametrizing the subspaces $V = H^0(X, L(-p))$ corresponding to unstable bundles. This locus appears in the class $[Z_{21}(\mu)]$ with a contribution of

$$c_1\Big(\mathrm{Ker}(\mu^\vee) \otimes \mathrm{Coker}(\mu^\vee) - N_{X/\mathbf{P}^6}\Big) = -5c_1\big(\mathrm{Ker}(\mu^\vee)\big) + c_1\big(\mathcal{A}^\vee_{|X}\big) - c_1(N_{X/\mathbf{P}^6}).$$

The restriction to $X \subseteq \mathbf{P}^6$ of the kernel bundle of $\mu^\vee$ can be identified with $L^\vee$, whereas $c_1\big(\mathcal{A}^\vee_{|X}\big) = -2c_1\big(M_{\mathbf{P}^6|X}\big) = 2\deg(L)$. Furthermore $c_1\big(N_{X/\mathbf{P}^6}\big) = 7\deg(L) + 2g(X) - 2$. All in all, the excess contribution to $[Z_{21}(\mu)]$ coming from X equals

$$10\deg(L) + 2\deg(L) - 7\deg(L) - 2g(X) - 2 = 5 \cdot 17 - 24 = 61.$$

Therefore, for a general curve $[X] \in \mathcal{M}^1_{13,7}$, there are $3 = 64 - 61$ vector bundles $E \in SU_X(2, \omega, 8)$ having A as a subpencil, which finishes the proof. □

We are now in a position to explain how Theorems 1.3 and 1.7 provide enough geometric information to determine the push-forward to $\mathcal{M}^\sharp_{13}$ of the class γ.

Proposition 7.7. *One has* $\vartheta_*(\gamma) = \frac{11288}{143}\lambda - \frac{1582}{143}\delta_0 \in CH^1(\mathcal{M}^\sharp_{13})$.

Proof. The divisor $\mathfrak{Res}^\sharp_{13}$ is defined as the push-forward under $\vartheta\colon \mathcal{SU}^\sharp_{13}(2, \omega, 8) \to \mathcal{M}^\sharp_{13}$ of the locus where the fibres of the morphism of vector bundles

$$d\colon \bigwedge^2 \wp_*(\mathfrak{E}) \to \wp_*(\omega_\wp)$$

contain a rank two tensor in their kernel. To compute the class of this locus, we use Proposition 7.4 in combination with [FR, Theorem 1.1][4]:

$$[\mathfrak{Res}^\sharp_{13}] = 132\Big(c_1\big(\wp_*(\omega_\wp)\big) - \frac{13}{4}c_1\big(\wp_*(\mathfrak{E})\big)\Big) = 132\Big(-\frac{9}{4}\vartheta^*(\lambda) + \frac{13}{8}\gamma\Big).$$

Using [HM] we write $[\overline{\mathcal{M}}^1_{13,7}] = 6 \cdot (48\lambda - 7\delta_0 - \cdots)$ for the class of the heptagonal locus, while the class $[\widetilde{\mathfrak{D}}_{13}]$ is computed by Theorem 1.4. Since $\deg(\vartheta) = 3$, we then find

$$\vartheta_*(\gamma) = \frac{48}{13}\Big(\frac{5059}{264}\lambda - \frac{749}{264}\delta_0 + \frac{9}{8}\lambda + \frac{3}{132}(48\lambda - 7\delta_0)\Big) = \frac{1128}{143}\lambda - \frac{1582}{143}\delta_0.$$

□

[4]The result in [FR] is stated for a morphism of vector bundles of the form $\mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$. An immediate inspection of the proof shows though that the *same formula* applies also in the setting of a morphism of the form $\bigwedge^2(\mathcal{E}) \to \mathcal{F}$.

7.3. **The class of the non-abelian Brill-Noether divisor on $\overline{\mathcal{M}}_{13}$.**

In the introduction, we defined the non-abelian Brill-Noether divisor $\mathcal{MP}^\sharp_{13}$ as the locus of curves $[X] \in \mathcal{M}^\sharp_{13}$ for which there exists $E \in SU_X(2,\omega,8)$ such that the map

$$\mu_E\colon \mathrm{Sym}^2 H^0(X,E) \to H^0(X,\mathrm{Sym}^2 E)$$

is not an isomorphism, or equivalently, the scheme $SU_X(2,\omega,8)$ is not reduced. We now compute the class of this divisor.

Proof of Theorem 1.1. The locus $\mathcal{MP}^\sharp_{13}$ is the push-forward under the proper map ϑ of the degeneracy locus of the following map of vector bundles over $\mathcal{SU}^\sharp_{13}(2,\omega,8)$:

$$\mathrm{Sym}^2 \wp_*(\mathfrak{E}) \to \wp_*\big(\mathrm{Sym}^2\mathfrak{E}\big).$$

Using Grothendieck-Riemann-Roch for $\wp\colon \mathcal{C}^\sharp_{13} \to \mathcal{SU}^\sharp_{13}(2,\omega,8)$, we compute

$$c_1\Big(p_*(\mathrm{Sym}^2\mathfrak{E})\Big) = \wp_*\Big[\Big(3+3c_1(\mathfrak{E})+\frac{5c_1^2(\mathfrak{E})-8c_2(\mathfrak{E})}{2}\Big)\cdot\Big(1-\frac{c_1(\Omega^1_\wp)}{2}+\frac{c_1^2(\Omega^1_\wp)+c_2(\Omega^1_\wp)}{12}\Big)\Big]_2.$$

Using again that $12\wp_*\Big(c_1^2(\Omega^1_\wp)+c_2(\Omega^1_\wp)\Big)=\vartheta^*(\lambda)$, we conclude that

$$c_1\Big(\wp_*(\mathrm{Sym}^2\mathfrak{E})\Big) = 3\vartheta^*(\lambda)+\wp_*\big(c_1^2(\omega_\wp)\big)-4\gamma = \vartheta^*\big(15\lambda-\delta_0\big)-4\gamma.$$

Via Proposition 7.4, we have $c_1\big(\mathrm{Sym}^2\wp_*(\mathfrak{E})\big) = 9c_1\big(\wp_*(\mathfrak{E})\big) = 9\big(\vartheta^*(\lambda)-\frac{\gamma}{2}\big)$, yielding

$$[\mathcal{MP}^\sharp_{13}] = \vartheta_*\Big(c_1\big(\wp_*(\mathrm{Sym}^2\mathfrak{E})-\mathrm{Sym}^2\wp_*(\mathfrak{E})\big)\Big) = 3(6\lambda-\delta_0)+\frac{\vartheta_*(\gamma)}{2}.$$

Substituting via Proposition 7.7, we find $[\mathcal{MP}^\sharp_{13}] = \frac{1}{143}\big(8218\,\lambda - 1220\,\delta_0\big)$. $\square$

8. THE KODAIRA DIMENSION OF $\overline{\mathcal{R}}_{13}$.

We turn our attention to showing that the Prym moduli space $\overline{\mathcal{R}}_{13}$ is a variety of general type. We begin by recalling basics on the geometry of the moduli of Prym variety, referring to [FL] for details. We denote by $\overline{\mathfrak{R}}_g := \overline{\mathfrak{M}}_g\big(\mathcal{B}\mathbb{Z}_2\big)$ the Deligne-Mumford stack of *Prym curves* of genus g classifying triples $[Y,\eta,\beta]$, where Y is a nodal curve of genus g such that each of its rational components meets the rest of the curve in at least two points, $\eta \in \mathrm{Pic}^0(Y)$ is a line bundle of total degree 0 such that $\eta_{|R} = \mathcal{O}_R(1)$ for every rational component $R \subseteq Y$ with $\big|R\cap \overline{Y\setminus R}\big| = 2$ (such a component is called *exceptional*), and $\beta\colon \eta^{\otimes 2}\to\mathcal{O}_Y$ is a morphism generically non-zero along each non-exceptional component of Y. Let $\overline{\mathcal{R}}_g$ be the coarse moduli space of $\mathfrak{R}_g$. One has a finite cover

$$\pi\colon \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g.$$

8.1. **The boundary divisors of $\overline{\mathcal{R}}_g$.** The geometry of the boundary of $\overline{\mathcal{R}}_g$ is described in [FL] and we recall some facts. If $[X_{yq} = X/y\sim q] \in \Delta_0 \subseteq \overline{\mathcal{M}}_g$ is such that $[X,y,q]\in\mathcal{M}_{g-1,2}$, denoting by $\nu\colon X\to X_{yq}$ the normalization map, there are three types of Prym curves in the fibre $\pi^{-1}\big([X_{yq}]\big)$. First, one can choose a non-trivial 2-torsion point $\eta\in\mathrm{Pic}^0(X_{yq})$. If $\nu^*(\eta)\neq\mathcal{O}_X$, this amounts to choosing a 2-torsion point $\eta_X\in\mathrm{Pic}^0(X)[2]\setminus\{\mathcal{O}_X\}$ together with an identification of the fibres $\eta_X(y)$ and $\eta_X(q)$ at the points y and q respectively. As we vary $[X,y,q]$, points of this type fill-up the boundary divisor $\Delta_0^{'}$ in $\overline{\mathcal{R}}_g$. The Prym curves corresponding to the situation $\nu^*(\eta)\cong\mathcal{O}_X$ fill-up the boundary divisor $\Delta_0^{''}$. Finally, choosing a line bundle η_X on X with $\eta_X^{\otimes 2}\cong\mathcal{O}_X(-y-q)$ leads to a Prym curve $[Y := X\cup_{y,q} R,\eta,\beta]$, where R is a smooth rational curve meeting X at y and q and $\eta\in\mathrm{Pic}^0(Y)$ is a line bundle such that $\eta_{|X}=\eta_X$ and $\eta_{|R}=\mathcal{O}_R(1)$. Points of this type fill-up the boundary divisor Δ_0^{ram} of $\overline{\mathcal{R}}_g$.

Denoting by $\delta_0^{'} := [\Delta_0^{'}]$, $\delta_0^{''} := [\Delta_0^{''}]$ and $\delta_0^{\mathrm{ram}} := [\delta_0^{\mathrm{ram}}]$ the corresponding divisor classes, one has the following relation in $CH^1(\overline{\mathcal{R}}_g)$, see [FL]:

$$\pi^*(\delta_0) = \delta_0^{'} + \delta_0^{''} + 2\delta_0^{\mathrm{ram}}.$$

The finite morphism $\pi\colon \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ being ramified only along the divisor Δ_0^{ram}, one has

$$K_{\overline{\mathcal{R}}_g} = 13\lambda - 2(\delta_0^{'} + \delta_0^{''}) - 3\delta_0^{\mathrm{ram}} - 2\sum_{i=1}^{\lfloor \frac{g}{2} \rfloor}(\delta_i + \delta_{g-i} + \delta_{i:g-i}) - (\delta_1 + \delta_{g-1} + \delta_{1:g-1}), \tag{34}$$

where $\pi^*(\delta_i) = \delta_i + \delta_{g-i} + \delta_{i:g-i}$, see [FL, Theorem 1.5] for details.

8.2. **The universal theta divisor on $\overline{\mathcal{R}}_{13}$.**

For a semistable vector bundle $E \in SU_X(2,\omega)$ on a smooth curve X of genus g, its *Raynaud theta divisor* $\Theta_E := \bigl\{\xi \in \mathrm{Pic}^0(X) : H^0(X, E\otimes \xi) \neq 0\bigr\}$ is a 2θ-divisor inside the Jacobian of X, see [Ray].

Definition 8.1. The universal theta divisor Θ_{13} on $\mathcal{R}_{13}$ is defined as the locus of smooth Prym curves $[X,\eta] \in \mathcal{R}_{13}$ for which there exists a vector bundle $E \in SU_X(2,\omega,8)$ such that $H^0(X, E\otimes\eta) \neq 0$.

We first show that, as expected, this definition gives rise to a divisor on $\mathcal{R}_{13}$.

Proposition 8.2. *For a general Prym curve $[X,\eta] \in \mathcal{R}_{13}$ one has $H^0(X, E\otimes\eta) = 0$ for all $E \in SU_X(2,\omega,8)$. It follows that Θ_{13} is an effective divisor on $\mathcal{R}_{13}$.*

Proof. Consider the subvariety of $\mathcal{R}_{13} \times_{\mathcal{M}_{13}} \mathcal{SU}_{13}(2,\omega,8)$

$$\mathcal{Z} := \bigl\{[X,\eta,E] : H^0(X, E\otimes\eta) \neq 0\bigr\}.$$

Assume for contradiction that $\mathcal{Z}$ surjects onto $\mathcal{R}_{13}$. Then $\mathcal{Z}$ is a union of *irreducible* components of $\mathcal{R}_{13} \times_{\mathcal{M}_{13}} \mathcal{SU}_{13}(2,\omega,8)$. In particular $\mathcal{Z}$ surjects onto the irreducible variety $\mathcal{SU}_{13}(2,\omega,8)$; see Corollary 6.3. Therefore, for every pair $[X,E] \in \mathcal{SU}_{13}(2,\omega,8)$, there exists a 2-torsion point $\eta \in \mathrm{Pic}^0(X)$ with $H^0(X, E\otimes\eta) \neq 0$.

We now specialize to the case when E is a strictly semistable bundle of the type

$$E = A^{\otimes 3} \oplus (\omega_X \otimes A^{\otimes(-3)}),$$

where $[X,A]$ is a general tetragonal curve of genus 13. Note that $h^0(X, A^{\otimes 3}) = 4$, by [CM, Proposition 2.1]. In particular $h^0(X,E) = 8$. Using [BF2] the space $\mathcal{R}_{13} \times_{\mathcal{M}_{13}} \mathcal{M}^1_{13,4}$ parametrizing Prym curves over tetragonal curves of genus 13 is irreducible, therefore $H^0\bigl(X, A^{\otimes 3}\otimes\eta) \neq 0$ for *every* triple $[X,\eta,A] \in \mathcal{R}_{13} \times_{\mathcal{M}_{13}} \mathcal{M}^1_{13,4}$. We now further specialize the tetragonal curve X to a hyperelliptic curve and $A = A_0(x+y)$, where $A_0 \in W^1_2(X)$ and $x,y \in X$ are general points, whereas

$$\eta = \mathcal{O}_X(p_1+p_2+p_3+p_4-q_1-q_2-q_3-q_4) \in \mathrm{Pic}^0(X)[2],$$

with $p_1,\ldots,p_4,q_1,\ldots,q_4$ being mutually distinct Weierstrass points of X. It immediately follows that for these choices $H^0\bigl(X, A^{\otimes 3}\otimes\eta\bigr) = 0$, which is a contradiction. $\square$

We consider the open substack $\mathcal{R}^\sharp_{13} := \pi^{-1}(\mathcal{M}^\sharp_{13})$ of $\overline{\mathcal{R}}_{13}$ and identify $CH^1(\mathcal{R}^\sharp_{13})$ with the space $\mathbb{Q}\langle\lambda, \delta_0^{'}, \delta_0^{''}, \delta_0^{\mathrm{ram}}\rangle$. In what follows we extend the structure on the universal theta divisor Θ_{13} to $\mathcal{R}^\sharp_{13}$ and realize it as the push-forward of the degeneracy locus of a map of vector bundles of the same rank over the fibre product

$$\mathcal{RSU}^\sharp_{13}(2,\omega,8) := \mathcal{R}^\sharp_{13} \times_{\mathcal{M}^\sharp_{13}} \mathcal{SU}^\sharp_{13}(2,\omega,8).$$

We start with a triple $[X,\eta,E] \in \mathcal{RSU}^\sharp_{13}(2,\omega,8)$. Via Proposition 7.2 the vector bundle E is globally generated and we let $M_E := \mathrm{Ker}\bigl\{H^0(X,E)\otimes\mathcal{O}_X \to E\bigr\}$. By tensoring with η and taking cohomology in the exact sequence (31), we observe that $H^0(X, E\otimes\eta) \neq 0$ if and only if the coboundary map

$$\upsilon\colon H^1\bigl(X, M_E\otimes\eta\bigr) \to H^0(X,E) \otimes H^0(X, \omega_X\otimes\eta)^\vee \tag{35}$$

is not injective. Since clearly $H^0(X, M_E \otimes \eta) = 0$, it follows that

$$h^1(X, M_E \otimes \eta) = -\deg(M_E) + 6(g-1) = 96 = 8 \cdot 12 = h^0(X,E) \cdot h^0(X, \omega_X \otimes \eta).$$

That is, υ is a map between vector space of the same dimension.

By slightly abusing notation, we still denote by

$$\wp \colon \mathcal{RC}_{13}^{\sharp} \to \mathcal{RSU}_{13}^{\sharp}(2,\omega,8)$$

the universal curve of genus 13 over $\mathcal{RSU}_{13}^{\sharp}(2,\omega,8)$. It comes equipped with a universal rank 2 vector bundle $\mathfrak{E}$ such that $\bigwedge^2 \mathfrak{E} \cong \omega_{\wp}$ and $\wp_*(\mathfrak{E})$ is locally free of rank 8 (cf. Proposition 7.4), as well as with a universal Prym line bundle $\mathcal{L}$ with $\mathcal{L}_{|\wp^{-1}([X,\eta,E])} \cong \eta$, for any point $[X,\eta,E] \in \mathcal{RSU}_{13}^{\sharp}(2,\omega,8)$.

We consider the rank 6 vector bundle $\mathcal{M}_{\mathfrak{E}}$ on $\mathcal{RC}_{13}^{\sharp}$ defined by the exact sequence

$$0 \longrightarrow \mathcal{M}_{\mathfrak{E}} \longrightarrow \wp^*\big(\wp_*\mathfrak{E}\big) \longrightarrow \mathfrak{E} \longrightarrow 0,$$

then introduce the following sheaves over $\mathcal{RSU}_{13}^{\sharp}(2,\omega,8)$:

$$\mathcal{A} := R^1\wp_*\big(\mathcal{M}_{\mathfrak{E}} \otimes \mathcal{L}\big) \quad \text{and} \quad \mathcal{B} := \wp_*(\mathfrak{E}) \otimes \wp_*\big(\omega_{\wp} \otimes \mathcal{L}\big)^{\vee}.$$

Both $\mathcal{A}$ and $\mathcal{B}$ are locally free of the same rank 96, and there exists a morphism

$$\upsilon \colon \mathcal{A} \to \mathcal{B} \tag{36}$$

whose fibre restrictions are the maps (35). Recall that the forgetful map $\vartheta \colon \mathcal{RSU}_{13}^{\sharp}(2,\omega,8) \to \mathcal{R}_{13}^{\sharp}$ is generically finite of degree 3. We denote by $\Theta_{13}^{\sharp}$ the push-forward to $\mathcal{R}_{13}^{\sharp}$ of the degeneracy locus of the morphism υ given by (36). Observe that $\Theta_{13}^{\sharp} \cap \mathcal{M}_{13} = \Theta_{13}$.

Theorem 8.3. *The class of the universal theta divisor $\Theta_{13}^{\sharp}$ on $\mathcal{R}_{13}$ is given by*

$$[\Theta_{13}^{\sharp}] = \frac{1}{143}\Big(10430\,\lambda - 1582\,(\delta_0^{'} + \delta_0^{''}) - \frac{5899}{2}\delta_0^{\mathrm{ram}}\Big) \in CH^1\big(\mathcal{R}_{13}^{\sharp}\big).$$

Proof. From Proposition 8.2 it follows that υ is generically non-degenerate, therefore

$$[\Theta_{13}^{\sharp}] = c_1\big(\mathcal{B} - \mathcal{A}\big).$$

Computing the class $c_1(\mathcal{B})$ is straightforward. We find that $c_1\big(\wp_*(\omega_{\wp} \otimes \mathcal{L})\big) = \vartheta^*\big(\lambda - \frac{\delta_0^{\mathrm{ram}}}{4}\big)$, using [FL, Proposition 1.7]. Then via Proposition 7.4, we compute

$$\begin{aligned} c_1(\mathcal{B}) = 12c_1\big(\wp_*\mathfrak{E}\big) - 8c_1\big(\wp_*(\omega_{\wp} \otimes \mathcal{L})\big) &= 12\Big(\vartheta^*(\lambda) - \frac{\gamma}{2}\Big) - 8\Big(\vartheta^*\big(\lambda - \frac{\delta_0^{\mathrm{ram}}}{4}\big)\Big) \\ &= \vartheta^*\big(4\lambda + 2\delta_0^{\mathrm{ram}}\big) - 6\gamma. \end{aligned}$$

To determine $c_1(\mathcal{A})$ we apply Grothendieck-Riemann-Roch to the morphism $\wp$:

$$\begin{aligned} \mathrm{ch}\big(\wp_!(\mathcal{M}_{\mathfrak{E}} \otimes \mathcal{L})\big) = \wp_*\Big[&\Big(6 + c_1(\mathcal{M}_{\mathfrak{E}} \otimes \mathcal{L}) + \frac{c_1^2(\mathcal{M}_{\mathfrak{E}} \otimes \mathcal{L}) - 2c_2(\mathcal{M}_{\mathfrak{E}} \otimes \mathcal{L})}{2} + \cdots\Big) \\ &\cdot\Big(1 - \frac{c_1(\Omega_{\wp}^1)}{2} + \frac{c_1^2(\Omega_{\wp}^1) + c_2(\Omega_{\wp}^1)}{12} + \cdots\Big)\Big]. \end{aligned} \tag{37}$$

Observe by direct calculation that the following formulas hold:

$$c_1(\mathcal{M}_{\mathfrak{E}} \otimes \mathcal{L}) = c_1(\mathcal{M}_{\mathfrak{E}}) + 6c_1(\mathfrak{L}), \quad c_2(\mathcal{M}_{\mathfrak{E}} \otimes \mathcal{L}) = c_2(\mathcal{M}_{\mathfrak{E}}) + 5c_1(\mathcal{M}_{\mathfrak{E}}) \cdot c_1(\mathcal{L}) + 15c_1^2(\mathcal{L}),$$

therefore

$$\begin{aligned} \wp_*\Big(\frac{c_1^2(\mathcal{M}_{\mathfrak{E}} \otimes \mathcal{L}) - 2c_2(\mathcal{M}_{\mathfrak{E}} \otimes \mathcal{L})}{2}\Big) &= \wp_*\Big(\frac{c_1^2(\mathcal{M}_{\mathfrak{E}}) - 2c_2(\mathcal{M}_{\mathfrak{E}})}{2} + c_1(\mathcal{M}_{\mathfrak{E}}) \cdot c_1(\mathcal{L}) + 3c_1^2(\mathcal{L})\Big) \\ &= \gamma - \frac{1}{2}\wp_*(c_1^2(\omega_{\wp})) = \gamma - \frac{1}{2}\Big(\vartheta^*\big(12\lambda - \delta_0^{'} - \delta_0^{''} - 2\delta_0^{\mathrm{ram}}\big)\Big), \end{aligned}$$

where in the last formula we have used Proposition 7.5, Mumford's formula [HM] for the class $\wp_*(c_1^2(\omega_{\wp}))$, and $2\wp_*\big(c_1^2(\mathcal{L})\big) = -\vartheta^*(\delta_0^{\mathrm{ram}})$; see [FL, Proposition 1.6].

Substituting in the equation (37), coupled with Proposition 7.5 and also using that via the push-pull formula $\wp_*\big(\wp^*(\vartheta^*(\lambda)-\frac{\gamma}{2})\cdot c_1(\omega_\wp)\big)=(g-1)\cdot\big(\vartheta^*(\lambda)-\frac{\gamma}{2}\big)$, we obtain

$$c_1(\mathcal{A})=-7\gamma+\vartheta^*(6\lambda+\frac{3}{2}\delta_0^{\mathrm{ram}}).$$

Putting everything together we find

$$[\Theta_{13}^\sharp]=\vartheta_*c_1(\mathcal{B}-\mathcal{A})=\vartheta_*\Big(\gamma-2\lambda+\frac{\delta_0^{\mathrm{ram}}}{2}\Big)=2\vartheta_*(\gamma)-6\lambda+\frac{3}{2}\delta_0^{\mathrm{ram}}.$$

Finally Proposition 7.7 gives $143\,\vartheta_*(\gamma)=11288\lambda-1582(\delta_0^{'}+\delta_0^{''}+2\delta_0^{\mathrm{ram}})$ and the conclusion follows. □

We can now complete the proof that $\overline{\mathcal{R}}_{13}$ is of general type.

Proof of Theorem 1.2. It is shown in [FL, Theorem 6.1] that any g pluricanonical forms defined on $\overline{\mathcal{R}}_g$ automatically extend to any resolution of singularities, therefore $\overline{\mathcal{R}}_g$ is of general type if and only if the canonical class $K_{\overline{\mathcal{R}}_g}$ is big. To that end, we shall use apart from the closure $\overline{\Theta}_{13}$ of the universal theta divisor, the divisor $D_{13:2}$ on $\mathcal{R}_{13}$ consisting of pairs $[X,\eta]$ where the 2-torsion point η lies in the divisorial *difference variety*

$$X_6-X_6=\Big\{\mathcal{O}_X(D-E):D,E\in X_6\Big\}\subseteq \mathrm{Pic}^0(X).$$

It is shown in [FL, Theorem 0.2] that up to a positive rational constant, the closure of $D_{13:2}$ inside$\overline{\mathcal{R}}_{13}$ is given by $[\overline{D}_{13:2}]=19\lambda-3(\delta_0^{'}+\delta_0^{''})-\frac{13}{4}\delta_0^{\mathrm{ram}}-\cdots\in CH^1(\overline{\mathcal{R}}_{13})$. We then consider the following effective divisor on $\overline{\mathcal{R}}_{13}$:

$$\mathcal{D}:=\frac{65}{674}[\overline{\Theta}_{13}]+\frac{1153}{3707}[\overline{D}_{13:2}]=\frac{4362}{337}\lambda-2(\delta_0^{'}+\delta_0^{''})-3\delta_0^{\mathrm{ram}}-\sum_{i=1}^{12}a_i\delta_i-\sum_{i=1}^{6}a_{i,13-i}\delta_{i:13-i}.$$

By a simple argument using pencils on $K3$ surfaces, one can show that each of the coefficients $a_1,\ldots,a_{12}$ or $a_{1,12},\ldots,a_{6,7}$ is at least equal to 3, see [FL, Proposition 1.9]. Since $\frac{4362}{337}=12.943...<13$, comparing the class of $\mathcal{D}$ to the one of $K_{\overline{\mathcal{R}}_{13}}$ given in (34), we conclude that $K_{\overline{\mathcal{R}}_{13}}$ can be written as a positive combination of $[\mathcal{D}]$ and a multiple of λ, hence it is big. □

8.3. **The Kodaira dimension of $\overline{\mathcal{M}}_{13,n}$.** We indicate how our results on divisors on $\overline{\mathcal{M}}_{13}$ can be used to determine the Kodaira dimension of the moduli space $\overline{\mathcal{M}}_{13,n}$.

Proof of Theorem 1.6. It suffices to show that $\overline{\mathcal{M}}_{13,9}$ is of general type to conclude that the same holds for $\overline{\mathcal{M}}_{13,n}$ when $n\geq 10$. We use the divisor $\mathcal{D}_{13:2^4,1^5}$ considered by Logan [Log] and defined as the $\mathfrak{S}_9$-orbit (under the action permuting the marked points) of the locus of pointed curves $[X,p_1,\ldots,p_9]\in\mathcal{M}_{13,9}$ such that

$$h^0\big(X,\mathcal{O}_X(2p_1+\cdots+2p_4+p_5+\cdots+p_9)\big)\geq 2.$$

Up to a positive constant the class of the closure in $\overline{\mathcal{M}}_{13,9}$ of $D_{13:2^4,1^5}$ equals

$$[\overline{\mathcal{D}}_{13:2^4,1^5}]=-\lambda+\frac{17}{9}\sum_{i=1}^{9}\psi_i-\frac{25}{6}\delta_{0:2}-\cdots\in CH^1(\overline{\mathcal{M}}_{13,9}).$$

(See [F] or [Log] for the standard notation on the generators of $CH^1(\overline{\mathcal{M}}_{g,n})$.) If $\pi\colon\overline{\mathcal{M}}_{13,9}\to\overline{\mathcal{M}}_{13}$ is the map forgetting the marked points, a routine calculation shows that the canonical class $K_{\overline{\mathcal{M}}_{13,9}}$ can be expressed as a positive linear combination of $[\overline{\mathcal{D}}_{13:2^4,1^5}]$ and $\pi^*([D])$, where $D\in\mathrm{Eff}(\overline{\mathcal{M}}_{13})$ if and only if $2s(D)-\frac{9}{17}<13$. Observe that the class of the non-abelian Brill-Noether divisor $[\overline{\mathcal{MP}}_{13}]$ verifies this inequality, and the result follows. □

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