ANALYTIC CURVES AND TROPICAL CURVES

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1. Analytic curves: a tropical description

Let K be an algebraically closed field which is complete with respect to a nontrivial, nonarchimedean valuation $\mathrm{val} : K \to \mathbf{R} \cup \{\infty\}$. Let R be the valuation ring of K and let k be its residue field. For notational simplicity we assume that $\mathrm{val}(K^\times) = \mathbf{R}$.

There is an analytification functor $X \mapsto X^{\mathrm{an}}$ from the category of locally-finite-type K-schemes to the category of K-analytic spaces in the sense of Berkovich [Ber90]. The K-analytic space X^{an} is closely analogous to the complex analytic space $X(\mathbf{C})$ associated to a complex variety X. An important difference is that in the nonarchimedean setting, the set of points of X^{an} is identified with the set of morphisms $\mathrm{Spec}(L) \to X$, where L/K is an extension of valued fields (with a real-valued valuation), up to the evident equivalence relation generated by towers of valued field extensions. An advantage of using Berkovich's theory of analytic spaces is that X^{an} has very nice point-set topological properties: for example, X is connected if and only if X^{an} is path-connected, X is proper if and only if X^{an} is compact and Hausdorff, etc. An analytic space also comes equipped with a sheaf of *analytic* or *holomorphic* functions, locally modeled on (quotients of) rings of convergent power series in finitely many variables.

Now let X be a smooth, proper, connected K-curve. In this case the analytification X^{an} is a kind of infinite metric graph. We regard X^{an} as the canonical "intrinsic" tropicalization of X. Here is one way to make this statement precise.

Let Γ be a metric graph with infinite leaves: this is an ordinary (finite) metric graph onto which we have attached some finite number of infinite tails, i.e. completed rays isometric to $[0,\infty]$. The leaf vertex at the end of such a ray is called an *infinite vertex*. An *elementary tropical modification* of Γ is a graph Γ' obtained from Γ by potentially subdividing an edge and attaching another infinite tail to a vertex of Γ. By retracting the attached edge one obtains a retraction map $\tau : \Gamma' \to \Gamma$, and there is an obvious inclusion map $\Gamma \hookrightarrow \Gamma'$. For our purposes, a *tropical modification* of Γ is a graph Γ' obtained by performing a sequence of elementary tropical modifications. The set of all modifications forms a directed system with respect to inclusion and an inverse system with respect to projection, so it makes sense to take the limits $\varinjlim \Gamma'$ and $\varprojlim \Gamma'$.

The above constructions are closely related to the analytification of a curve X. Let $\mathfrak{X}$ be a semistable R-model of X. The *incidence graph* of $\mathfrak{X}$ is the graph $\Gamma_{\mathfrak{X}}$ whose vertices are the irreducible components of $\mathfrak{X}_k$, and whose edges are the nodes in $\mathfrak{X}_k(k)$; a node connects the two (or one) irreducible component(s) containing it. If $x \in \mathfrak{X}_k(k)$ is a node, then by semistability we have

$$\widehat{\mathscr{O}}_{\mathfrak{X},x} \cong R[\![X,Y]\!]/(XY - \varpi)$$

for some $\varpi \in K$ with $0 < \mathrm{val}(\varpi) < \infty$. We define the length of the edge e_x corresponding to x to be $\ell(e_x) = \mathrm{val}(\varpi)$. This enriches $\Gamma_{\mathfrak{X}}$ with the structure of a metric graph.

Fact 1.1. *There is a canonical embedding $\Gamma_{\mathfrak{X}} \hookrightarrow X^{\mathrm{an}}$. Moreover, there is a metric on the target with respect to which this embedding is an isometry. There is a deformation retraction $\tau : X^{\mathrm{an}} \twoheadrightarrow \Gamma_{\mathfrak{X}}$.*

The subset $\Gamma_{\mathfrak{X}} \subset X^{\mathrm{an}}$ is called a *skeleton* of $\mathfrak{X}$. This notion can be extended to a marked curve (X, D): namely, let $\mathfrak{X}$ be a semstable model of (X, D) in the sense that the points of D reduced to distinct smooth points of $\mathfrak{X}_k$. Let $\Gamma_{(\mathfrak{X},D)}$ be the graph obtained from $\Gamma_{\mathfrak{X}}$ by adding one infinite tail for each marked point $x \in D$, attached to the vertex corresponding to the irreducible component of $\mathfrak{X}_k$ containing the reduction of X. Then $\Gamma_{(\mathfrak{X},D)}$ again embeds into X^{an}, and there is a deformation retraction $\tau : X^{\mathrm{an}} \twoheadrightarrow \Gamma_{(\mathfrak{X},D)}$. We again call $\Gamma_{(\mathfrak{X},D)}$ the *skeleton* associated to $(\mathfrak{X}, D)$.

A *skeleton* of a curve X (resp. of a marked curve (X, D)) is by definition a subset of the form $\Gamma_{(\mathfrak{X},D)}$ as above. Any two skeleta are related by a sequence of elementary tropical modifications.

Fact 1.2. *The canonical map to the inverse limit of all skeleta*

$$X^{\mathrm{an}} \longrightarrow \varprojlim \Gamma$$

is a homeomorphism. The canonical map

$$\varinjlim \Gamma \longrightarrow X^{\mathrm{an}}$$

is injective, and its image is the set of all points of X^{an} of types $1, 2,$ and 3.

In this sense, the construction of the topological space X^{an} from X can be understood in tropical terms.

Example 1.3. Let X/K be an elliptic curve with j-invariant $j(X)$. Suppose that $\mathrm{val}(j(X)) = -r < 0$. Then X has bad (multiplicative) reduction, so by the reduction theory of elliptic curves, X admits a semistable model $\mathfrak{X}$ whose special fiber is isomorphic to a nodal cubic. If $x \in \mathfrak{X}_k(k)$ is the node then it follows from Tate's nonarchimedean uniformization theory that $\widehat{\mathscr{O}}_{\mathfrak{X},x} \cong R[\![X, Y]\!]/(XY - \varpi)$ with $\mathrm{val}(\varpi) = r$. Therefore $\Gamma_{\mathfrak{X}}$ is a circle of circumference r, and X^{an} is obtained from $\Gamma_{\mathfrak{X}}$ by taking the inverse limit over all tropical modifications of this circle.

2. Meromorphic functions

Let X be a curve as above, and let $f \in K(X)^\times$. In the complex setting, $F = -\log|f|$ is a harmonic function on $X(\mathbf{C})$. This remains true in the nonarchimedean setting.

Definition 2.1. Let Γ be a metric graph with infinite tails and let $F : \Gamma \to \mathbf{R} \cup \{\pm\infty\}$ be a continuous function such that F takes infinite values only on infinite vertices.

1. (1) We say that F is *piecewise affine with integer slopes* if its restriction to each edge (identified with an interval or ray) has the form $F(x) = mx + b$ for $m \in \mathbf{Z}$ and $b \in \mathbf{R}$, and if there are only finitely many points at which F is not differentiable.
2. (2) For $x \in \Gamma$ we let T_x denote the set of tangent directions at x, and for $v \in T_x$ we let $d_v F(x)$ denote the outgoing slope of F in the v-direction. We say that F is *harmonic at* $x \in \Gamma$ provided that $\sum_{v \in T_x} d_v F(x) = 0$, and that F is *harmonic* if it is harmonic at all points $x \in \Gamma$ except potentially at the infinite vertices.

The following is a reformulation of Thuillier's nonarchimedean Poincaré-Lelong formula. See [Thu05].

Theorem 2.2. *Let $\mathfrak{X}$ be a semistable model of (X, D) and let $\Gamma = \Gamma_{(\mathfrak{X},D)}$. Let $f \in K(X)^\times$ be a meromorphic function supported on D and let $F = -\log|f| : X^{\mathrm{an}} \to \mathbf{R} \cup \{\pm\infty\}$. Then*

1. (1) *F factors through the retraction $\tau : X^{\mathrm{an}} \to \Gamma$.*
2. (2) *F is harmonic and differentiable on the edges of Γ.*
3. (3) *If $x \in D$ and v is the unique tangent direction at x then $d_v F(x) = \mathrm{ord}_x(f)$.*

One can show that F is the unique function on Γ, up to additive translation, which satisfies properties (1)–(3) of the Theorem.

Example 2.3. Let X be the Tate curve of Example 1.3, and let $\tau : X^{\mathrm{an}} \to \Gamma$ be the retraction onto the circle. Choose an origin $O \in X(K)$, so (X, O) is an elliptic curve. It is known that a divisor $D = \sum n_x(x)$ is principal if and only if $\deg(D) = \sum n_x = 0$ and $\sum n_x \cdot x = O$ in the group law on $X(K)$. Let $\zeta \in X[3]$ be a 3-torsion point. Assuming $\mathrm{char}(k) \neq 3$, one can show using Tate's uniformization theory that there exists a choice of ζ such that $\tau(\zeta) \neq \tau(O)$, and that the three points $\tau(O), \tau(\zeta), \tau(2\zeta)$ are equidistant on the circle Γ. The divisor $D = (\zeta) + (2\zeta) - 2(O)$ is principal; let $f \in K(X)$ be a meromorphic function with divisor D. Let Γ' be the tropical modification of Γ

obtained by adding infinite tails at the points $\tau(O), \tau(\zeta), \tau(2\zeta)$. This is a skeleton of $(X, \{O, \zeta, 2\zeta\})$. Then $F = -\log|f|$ is the harmonic function on Γ' depicted in Figure 1.

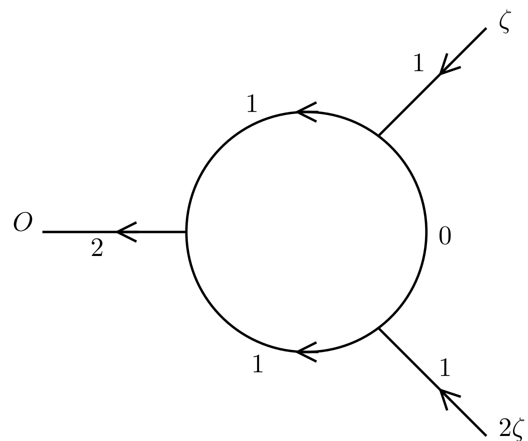

FIGURE 1. The skeleton Γ' and the harmonic function F of Example 2.3. The numbers indicate the slope of F along an edge, and the arrows indicate direction in which F is increasing.

We have the following basic fact about the relationship between meromorphic functions on X and piecewise affine functions on a skeleton. This theorem can be used as an analytic tool to construct algebraic functions from combinatorial data.

Theorem 2.4. (Baker-Rabinoff) *Let $\mathfrak{X}$ be a semistable model of X and let $\Gamma = \Gamma_{\mathfrak{X}}$. Any piecewise affine function with integer slopes $F : \Gamma \to \mathbf{R}$ has the form $F = -\log|f|$ for some $f \in K(X)^{\times}$.*

In terms of divisiors, Theorem 2.4 says that every principal divisor D' on the graph Γ is the image of a principal divisor D on X under the map $\tau_* : \mathrm{Div}(X) \to \mathrm{Div}(\Gamma)$ obtained from τ by extending linearly. (Recall that we are assuming $\mathrm{val}(K^{\times}) = \mathbf{R}$.) In the statement of Theorem 2.4 it is possible that D could have many more zeros and poles than D' which cancel out under the retraction map; however one can show that given D', it is possible to choose D retracting to D' and which has at most $g(X)$ extra zeros.

3. MAPS TO TORI AND EMBEDDED TROPICALIZATIONS

Let X be a curve as above, let $f_1, \ldots, f_n \in \Gamma(X \setminus D, \mathscr{O}_X)^{\times}$, and let $F_i = -\log|f_i|$. These functions define a morphism

$$f = (f_1, \ldots, f_n) \ : \ X \setminus D \longrightarrow \mathbf{T} := \mathbf{G}_m^n.$$

Composing with the tropicalization map $\mathrm{trop} : \mathbf{T}^{\mathrm{an}} \to \mathbf{R}^n$, we obtain a map

$$\mathrm{trop} = (F_1, \ldots, F_n) \ : \ X^{\mathrm{an}} \setminus D \longrightarrow \mathbf{T}^{\mathrm{an}} \xrightarrow{\mathrm{trop}} \mathbf{R}^n.$$

By Theorem 2.2, if Γ is a skeleton of (X, D) then trop factors through the retraction $\tau : X^{\mathrm{an}} \to \Gamma$, hence is controlled by the map $\mathrm{trop} : \Gamma \to \mathbf{R}^n$. This map should be thought of as a morphism of the abstract tropical curve Γ onto the embedded tropical curve $\mathrm{Trop}(X) := \mathrm{trop}(X^{\mathrm{an}} \setminus D) \subset \mathbf{R}^n$. If $e \subset \Gamma$ is an edge then F_i is an affine function on e with integer slope $m_i \in \mathbf{Z}$, so $\mathrm{trop}(e)$ is a line segment or ray with rational slope. Hence $\mathrm{trop}(e)$ has a natural metric, namely, the lattice length, and the map $e \mapsto \mathrm{trop}(e)$ expands distances by the *expansion factor* $d_e = \gcd(m_1, \ldots, m_n)$.

The following is one of the main theorems of [BPR11].

Theorem 3.1. (Baker-Payne-Rabinoff) *Let $\mathfrak{X}$ be a semistable model of X and let $\Gamma = \Gamma_{\mathfrak{X}}$. Then there exists a collection of meromorphic functions $f_1, \ldots, f_n \in K(X)^{\times}$ such that the associated tropicalization* *map* $\mathrm{trop} : X^{\mathrm{an}} \setminus D \to \mathbf{R}^n$ *takes* Γ *homeomorphically and isometrically onto its image, where* D *contains the support of each* $\mathrm{div}(f_i)$.

Example 3.2. Let X be as in Examples 1.3 and 2.3. Let D' be the principal divisor $2(\zeta) - (2\zeta) - (O)$, let f' be a meromorphic function with divisor D', and let $F' = -\log|f'|$. The behavior of F' on the skeleton Γ' is depicted in Figure 2. Since we know the behavior of $\mathrm{trop} = (F, F')$ on each edge of Γ', we can calculate the image of Γ' under trop, and hence calculate $\mathrm{Trop}(X)$ (up to translation). For example, F has slope 0 on the oriented edge from B to C, and F' has slope 1, so $\mathrm{trop}(\overline{BC})$ is a vertical segment; since the expansion factor on this edge is equal to 1, the length of this segment is $r/3$. See Figure 3.

In this example, all expansion factors are equal to 1, so trop takes all of Γ' isometrically onto its image. In particular, the loop in $\mathrm{Trop}(X)$ has lattice length $r = -\mathrm{val}(j(X))$. This is an example of a phenomenon first noticed by Katz-Markwig-Markwig [KMM08].

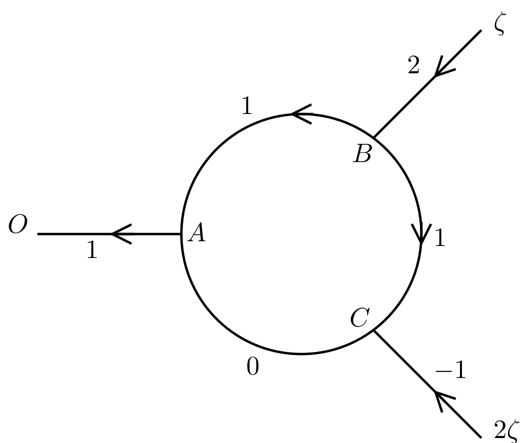

FIGURE 2. The skeleton Γ' and the harmonic function F' of Example 3.2. Cf. Figure 1.

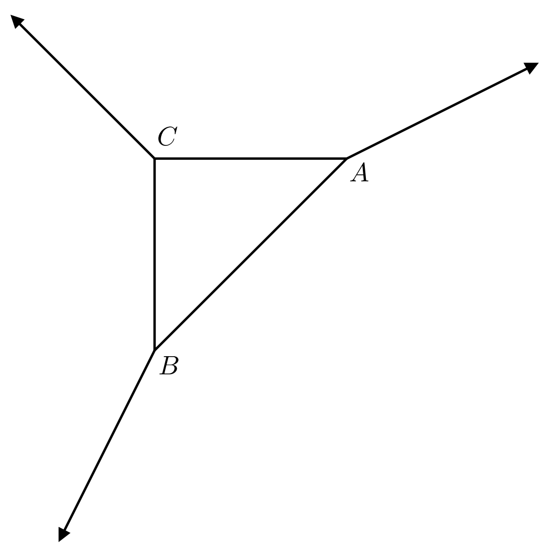

FIGURE 3. The tropicalization of the Tate curve X of Example 3.2.

Thus we see that a good understanding of a skeleton of X allows one to get information about tropicalizations of X, and conversely, by Theorem 3.1 there exist tropicalizations which are very nice from a skeletal point of view. It is natural to ask whether there is any way to verify that a given tropicalization is an isometric embedding on its preimage in some skeleton of X, using only "tropical" calculations.

For the rest of this paper, we assume for simplicity that $f = (f_1, \ldots, f_n) : X \setminus D \to \mathbf{T}$ is a closed immersion.

Theorem 3.3. (Baker-Payne-Rabinoff) *Let Γ be a skeleton of (X, D). Choose sets of vertices on Γ and $\mathrm{Trop}(X)$ such that for every edge $e \subset \Gamma$, either $\mathrm{trop}(e)$ is an edge of $\mathrm{Trop}(X)$ or is a vertex. Let $e \subset \Gamma$ be an edge. Then for all v in the interior of e, we have*

$$m_{\mathrm{Trop}}(v) = \sum_{e' \mapsto e} d_{e'}.$$

In other words, the tropical multiplicity of v is equal to the sum of the expansion factors of all edges of Γ mapping to e.

Corollary 3.4. *Let e be an edge of $\mathrm{Trop}(X)$ and let v be in the interior of e. If the initial degeneration $\mathrm{in}_v(X)$ is an integral scheme then there exists a unique edge $e' \subset \Gamma$ mapping onto e, and the map $e' \to e$ is an isometry. If $\Gamma' \subset \mathrm{Trop}(X)$ is a subgraph such that $\mathrm{in}_v(X)$ is integral for all $v \in \Gamma'$ then there exists a unique isometric section $\Gamma' \to \Gamma$ of the tropicalization map.*

Example 3.5. Let X be a Tate curve as in Example 1.3. Suppose that $X \setminus D$ is a closed subscheme of $\mathbf{G}_m^2$ cut out by a Laurent polynomial g whose Newton complex is a unimodular triangulation with an interior lattice point. Then the hypotheses of Corollary 3.4 are verified for the entire tropicalization, and hence there is a section of the tropicalization map $\mathrm{Trop}(X) \hookrightarrow X^{\mathrm{an}}$ which is an isometry onto a skeleton. In particular, the lattice length of the loop in $\mathrm{Trop}(X)$ is equal to $-\mathrm{val}(j(X))$. This gives another proof of the theorem of Katz-Markwig-Markwig.

There are many other applications of this set of tools, found in [BPR11]. For example:

(1) One can prove a generalized version of Speyer's "well-spacedness" condition.
(2) There are applications to tropical elimination theory.
(3) Much of the technology used in the proofs in [BPR11] works equally well in higher dimensions, and can be used to prove e.g. the Sturmfels-Tevelev multiplicity formula in the non-constant valuation setting.

References

[Ber90] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990.

[BPR11] M. Baker, S. Payne, and J. Rabinoff, *Non-Archimedean geometry, tropicalization, and metrics on curves*, 2011, Preprint available at `http://arxiv.org/abs/1104.0320`.

[KMM08] E. Katz, H. Markwig, and T. Markwig, *The tropical j-invariant*, 2008, Preprint available at `http://arxiv.org/abs/0803.4021v2`.

[Thu05] A. Thuillier, *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov*, Ph.D. thesis, University of Rennes, 2005, Preprint available at `http://tel.ccsd.cnrs.fr/documents/archives0/00/01/09/90/index.html`.