ON WILD COVERINGS OF BERKOVICH CURVES

MICHAEL TEMKIN

Abstract. These notes present an extended version of an author's talk given at the Simons Symposium on Non-Archimedean and Tropical Geometry held in May 2017 at Schloss Elmau. It gives a brief overview of results and methods of recent works [CTT16] and [Tem14] and a work in preparation [BT] on the structure of residually wild morphisms between Berkovich curves.

1. Introduction

1.1. **Conventiones.** In this notes k denotes an algebraically closed complete real-valued field with a non-trivial valuation of residual characteristic $p = \mathrm{char}(\widetilde{k})$. By a *nice curve* X we always mean a connected compact separated quasi-smooth strictly k-analytic curve. For any point $x \in X^{(2)}$ of type 2, C_x denotes the reduction curve of X at x (or the germ reduction of X at x). Recall that C_x is the smooth $\widetilde{k}$-curve with $k(C_x) = \widetilde{\mathcal{H}(x)}$ which parameterizes the branches of X at x.

Furthermore, by $f\colon Y \to X$ we always mean a finite generically étale morphism of nice curves, and then $n_y\colon Y \to \mathbf{N}$ is the multiplicity function of f given by $n_f(y) = [\mathcal{H}(y) : \mathcal{H}(x)]e_y$, where e_y is the classical ramification index. For $d \in \mathbf{N}$ we denote by N_d and $N_{\geq d}$ the set of points $y \in Y$ such that $n_f(y) = d$ and $n_f(y) \geq d$, respectively. Finally, for any $y \in Y^{(2)}$ by $\widetilde{f}_y\colon C_y \to C_{f(y)}$ we denote the corresponding reduction morphism of f.

1.2. **Skeletons of curves.** In this notes, a skeleton Γ of a nice curve X is a topological graph $\Gamma_X \subset X$ with finitely many edges and vertices such that $X \setminus \Gamma$ is a disjoint union of open discs and the set of vertices $V(\Gamma)$ consists of points of type 2 and 1 and contains at least one point of type 2. It follows that Γ is connected and $V(\Gamma) = \Gamma^{(1)} \coprod V(\Gamma)^{(2)}$, where $\Gamma^{(1)}$ is the set of points of Γ of type 1. We call vertices of types 2 and 1 *finite* and *infinite*, respectively.

Remark 1.2.1. One can often assign to analytic spaces invariants on four levels: a topological level, a tropical or combinatorial level, a reduction level, and a log reduction level. Some invariants are strictly more informative than the other ones, as illustrated by the following scheme.

$$\begin{tikzcd} \mathrm{Top} \arrow[r, hook] \arrow[d, hook] & \mathrm{Trop} \arrow[d, hook] \\ \widetilde{k}-\mathrm{Sch} \arrow[r, hook] & \widetilde{k}-\mathrm{LogSch}. \end{tikzcd}$$

In our definition, a skeleton Γ of X is an invariant of the topological level. It can be enriched to other levels as follows:

2010 *Mathematics Subject Classification.* 14E22, 11S15, 14G22.

- The combinatorial level: provide each finite vertex y with the genus $g_y = g(C_y)$, and provide each edge of Γ with the exponential metric induced by the radii of annuli.
- The reduction level: consider the semistable $\widetilde{k}$-curve $\mathfrak{X}_s$ with the divisor $\mathfrak{D}_s$, where $\mathfrak{X}$ is the semistable formal model associated with Γ and $\mathfrak{D}$ is the divisor associated with $\Gamma^{(1)}$.
- The log reduction level: adds to $\mathfrak{X}_s$ a log structures obtained by restricting the log structure of generic units on $\mathfrak{X}$.

Remark 1.2.2. (i) In fact, this log structure is determined by the lengthes of the edges of Γ. So, the diagram behaves as a pushout diagram: combinatorial and reduction information agree on the topological level, and their union is equivalent to the information encoded in the log reduction level.

(ii) An equivalent way to work with the reduction level was suggested in [ABBR15]: a *metrized complex of curves* is obtained by assigning a reduction curve C_x to any finite vertex x of a metrized graph Γ so that the edges from x correspond to some points of C_x. In the sequel we will work with such objects.

1.3. **Skeletons of morphisms.** By a *skeleton* of $f\colon Y \to X$ we mean a pair of compatible skeletons of X and Y and the induced map $\Gamma_f\colon \Gamma_Y \to \Gamma_X$ such that Γ_Y contains all ramification points of f. It exists by the simultaneous semistable reduction theorem, which, in fact, follows rather simply from the usual semistable reduction of curves. Note that for any edge $e = [y_1, y_2] \in \Gamma_Y$ the induced map $e \to f(e)$ is monomial of degree n_e and for $i = 1, 2$ we have that n_e is the degree of $\widetilde{f}_{y_i}$ at the point corresponding to e. On the combinatorial level, Γ_f satisfies various natural restrictions. Even in the residually tame case, these restrictions do not ensure that a map of graphs lifts to a map of curves, but the following result of [ABBR15] shows that the only obstacle is to lift it to a log reduction level

Theorem 1.3.1. *(A) If f is residually tame then Γ_f is a finite harmonic morphism of metrized complexes of curves.*

(B) If X is a nice curve with a skeleton Γ_X and $h\colon \Gamma \to \Gamma_X$ is a finite generically separated harmonic map of metrized complexes of curves then there exists a residually tame lifting $f\colon Y \to X$ such that $h = \Gamma_f$.

1.4. **Goals of this project.** This project aims to define finer invariants of morphisms that will provide an adequate description also in the residually wild case. In particular, we aim to obtain some lifting results. In the sequel we assume that $p > 0$ as otherwise f is automatically residually tame.

These notes describe main results of the following works:

- [CTT16] constructs a new combinatorial invariant of f, the different function $\delta_f\colon Y \to [0, 1]$.
- [Tem14] constructs a profile function ϕ_f, which is a stronger combinatorial invariant of f, and perhaps encodes any other combinatorial invariant.
- [BT] refines δ_f to the log reduction level by associating to f reduction differential forms ϕ_y on C_y for any finite vertex y.

2. The different function

2.1. **Different of extensions.**

2.1.1. *The real-valued case.* Recall that for a finite separable extension of real-valued fields L/K the different $\delta_{L/K}$ is defined to be the absolute value of the zeroth Fitting ideal of Ω_{L°/K°. In our case, the latter module will always coincide with the annihilator of Ω_{L°/K°, and hence $\delta_{L/K} = |\mathrm{Ann}(\Omega_{L^\circ/K^\circ})|$. One can also define a logarithmic different, but it coincides with $\delta_{L/K}$ whenever $|K^\times|$ is not discrete.

2.1.2. *The discrete-valued case.* Same definition applies to the case when F/E is a finite separable extension of discrete valuation fields with perfect residue fields. In this case we will use the additive notation: $\delta_{F/E} = \mathrm{ord}_F(\mathrm{Ann}(\Omega_{F^\circ/E^\circ}))$. In the same way, the logarithmic different is defined as $\delta^{\log}_{F/E} = \mathrm{ord}_F(\mathrm{Ann}(\Omega^{\log}_{F^\circ/E^\circ}))$. The two differents are related by

$$\delta^{\log}_{F/E} = \delta_{F/E} - \mathrm{ord}_F(\pi_E) + \mathrm{ord}_F(\pi_F) = \delta_{F/E} - e_{E/F} + 1.$$

Note that $\delta^{\log}_{F/E} \geq 0$ and the equality is achieved if and only if F/E is tame. Thus, the different is a qualitative invariant measuring the "wildness" of F/E.

2.2. **The different function.** In a joint work [CTT16] with A. Cohen and D. Trushin we study the *different function* $\delta_f\colon Y^{\mathrm{hyp}} \to [0,1]$ defined by $\delta_f(y) = \delta_{\mathcal{H}(y)/\mathcal{H}(f(y))}$ for a point y of type 2, 3 or 4.

2.2.1. *Basic properties.* First, one shows that δ_f is pm (i.e. piecewise monomial) and describes the range its values and slopes may attain:

Theorem 2.2.2. *For a finite generically étale $f\colon Y \to X$ and a skeleton $\Gamma_f\colon \Gamma_Y \to \Gamma_X$ one has that:*

(i) δ_f is a pm function on Y^{hyp} and it is monoial on each edge of Γ_Y,
(ii) $|n_f(y)| \leq |\delta_f(y)| \leq 1$,
(iii) for any $y \in Y^{(2)}$ and $v \in C_y$ the slope $s = \mathrm{slope}_v(\delta_f)$ satisfies

$$|n_f(v)| \leq |\delta_f(y)| \leq |n_f(v) + s|.$$

Moreover, (ii) and (iii) are the only restrictions on δ and s, and any allowed combination can be already obtained for étale morphisms $A_1 \to A_2$ of two annuli given by $x = t^n + ct^m$.

Remark 2.2.3. This is essentially a claim about étale morphisms between annuli, so computations are simple. One should use that $\delta = |\frac{dx}{dt}| \cdot |t| \cdot |x|^{-1}$ on the skeleton of A_1 and $\frac{dx}{dt} \in \mathcal{O}(A_1)^\times$. In particular, in the example one takes y to be a boundary point and achieves that $n = n_v$, $s = n - m$ and $\delta_f(y) = |c|$.

Remark 2.2.4. The conditions (ii) and (iii) of the theorem are most restrictive in the mixed characteristic case. For example, if $n = p$ then $|p| \leq \delta_f \leq 1$, $s \notin p\mathbf{Z} \setminus \{0\}$ and $s = 0$ can happen only when $\delta_f(y) = 1$ or $\delta_f(y) = |p|$.

2.3. **Main results.** The following result describes the restrictions δ_f satisfies locally at a point $y \in Y$.

Theorem 2.3.1. *For a finite generically étale $f\colon Y \to X$ and a skeleton $\Gamma_f\colon \Gamma_Y \to \Gamma_X$ one has that:*

(i) For any point $y \in Y^{(1)}$ of type 1 one has that $\mathrm{slope}_y(\delta_f) = \delta^{\log}_{\mathcal{O}_y/\mathcal{O}_{f(y)}}$ *and* $|\delta_f(y)| = |n_y|$.

(ii) For a non-boundary point $y \in Y$ of type 2 with $x = f(y)$ one has that

$$2g(y) - 2 - 2n_y(g(x) - 1) = \sum_{v \in C_y} (-\mathrm{slope}_v \delta_f + n_v - 1).$$

In particular, almost all slopes of δ_f at y equal the inseparability degree n_y^i of $\widetilde{\mathcal{H}(y)}/\widetilde{\mathcal{H}(x)}$.

(iii) The different behaves trivially outside of Γ_Y in the sense that $\mathrm{slope}_v \delta_f = n_v - 1$ *for any direction v not pointing towards Γ_Y.*

Remark 2.3.2. (i) Part (i) of the theorem indicates that δ_f is, in fact, the log different function. This does not affect its values at the points of Y but gives a better interpretation of formulas involving differents of discretely valued fields.

(ii) Part (ii) reduces to the Riemann-Hurwitz formula for $C_y \to C_x$ when f is residually tame at y. In general, it is related to the Ogg-Shafarevich formula for certain connections.

(iii) The theorem implies the global Riemann-Hurwitz formula for f (which includes a correction term at the boundary when X and Y are not proper). In particular, it gives a local analytic proof of non-existence of finite étale coverings of $\mathbf{P}^1_k$. Note that this is a non-trivial fact, since $\mathbf{P}^1_k$ does have infinite étale covers.

2.4. **The degree-p case.** If f is of degree p then $\delta_f(y) < 1$ can happen only when $n_y = p$. Hence it follows from Theorem 2.3.1(iii) that δ_f increases outside of Γ_Y with constant slope $p - 1$ until it attains the maximal possible value 1. In particular, N_p is the radial neighborhood of the subgraph $\Gamma_p = N_p \cap \Gamma_Y$ of Γ_Y of radius $\delta_f^{1/(p-1)}|_{\Gamma_p}$.

3. Radialization and profile function

3.1. **The sets N_d.** In general, $\delta_f(y)$ is determined by its restriction onto Γ_Y and the multiplicity function n_f, but the latter can be relatively complicated. So, it is a natural question if one can describe the sets N_d in any constructive way. This description was found in [Tem14]:

Theorem 3.1.1. *For any skeleton Γ_f of f and a natural number $d \in \mathbf{N} \setminus p^{\mathbf{N}}$ the set N_d is a subgraph of Γ_Y. Furthermore, if Γ_f is large enough then each set $N_{\geq p^n}$ is radial around $\Gamma_{\geq p^n} := \Gamma_Y \cap N_{\geq p^n}$ of radius r_n, where $r_n \colon \Gamma_{\geq p^n} \to [0, 1]$ is a continuous function monomial on the edges of $\Gamma_{\geq p^n}$.*

Remark 3.1.2. In fact, any skeleton works fine if f is tame, Galois, or of degree p. In general, it suffices to choose any skeleton of the Galois closure of f and project it onto Y.

The proof of this theorem is pretty simple. It is trivial for residually tame morphisms, and the case when f is of degree p is established in §2.4. If f is Galois then one can locally split it into compositions of morphisms of the above two types. Finally, the general case is dealt with by passing to the Galois closure.

3.2. **The profile function.** In principle, a radializing skeleton Γ_f of f (i.e. a skeleton as in Theorem 3.1.1) and pm functions $r_1, \ldots, r_n$ on Γ_f provide a full combinatorial description of f (in particular, they determine δ_f). However, there is an equivalent but more convenient way to organize this combinatorial information which uses a so-called profile function. For any interval $l = [z, y] \in Y$ with z of type 1 and $[z,y]\cap\Gamma_Y=\{y\}$ we have that y is of type 2, and hence both l and $f(l)$ can be identified with the interval $[0,1]$ by the exponential distance function. In particular, the restriction of f onto l can be viewed as a pm automorphism of $[0,1]$, say $\varphi_l\in\mathrm{Aut}^{\mathrm{pm}}([0,1])$. If Γ_f is radializing then φ_l depends only on the end-point $y\in\Gamma_Y$ and we obtain a map $\varphi_\Gamma^{(2)}\colon\Gamma_Y^{(2)}\to\mathrm{Aut}^{\mathrm{pm}}([0,1])$ which assigns to y the profile function φ_y.

Remark 3.2.1. One can easily express φ_y in terms of the radii $r_i(y)$ and vice versa. The main advantage of φ_y is that it is obviously compatible with compositions.

Clearly, $\varphi_\Gamma^{(2)}$ is compatible with extensions of Γ, hence the profile φ_y at a point of type 2 is independent of the skeleton, and we obtain a profile map $\varphi^{(2)}\colon Y^{(2)}\to\mathrm{Aut}^{\mathrm{pm}}([0,1])$.

Theorem 3.2.2. *(i) $\varphi^{(2)}$ extends to a pm profile map $\varphi_f\colon Y^{\mathrm{hyp}}\to\mathrm{Aut}^{\mathrm{pm}}([0,1])$.*

(ii) The profile maps are compatible with compositions of morphisms and algebraically closed extensions of the ground field.

(iii) $\varphi_f(y)$ is, in fact, the classical Herbrand function $\varphi_{\mathcal{H}(y)/\mathcal{H}(f(y))}$ of the extension $\mathcal{H}(y)/\mathcal{H}(f(y))$ of the completed residue fields.

Remark 3.2.3. (i) The proof is again via local splitting of f into a composition of residually tame morphisms and morphisms of degree p.

(ii) In order to make this rigorous, one had to extend the classical theory of Herbrand functions to certain non-discrete-valued cases, but this is rather straightforward.

(iii) Our geometric definition of the profile function directly applies to non-normal and even inseparable extensions.

4. Reduction forms

Our last aim is to refine the combinatorial invariants δ_f and φ_f to the log reduction level. This is a new approach developed with U. Brezner in [BT]. To outline it we will have first to recall some arguments from [CTT16], and to illustrate the main ideas we will even start with the case of algebraic varieties over $\widetilde{k}$.

4.1. **A baby case.** In this section, $h\colon C\to S$ is a generically étale morphism of smooth $\widetilde{k}$-curves, $v\in C$ is a closed point and $u=h(v)$. Note that Ω_C defines a natural order on the space of meromorphic forms on C. Namely, $\mathrm{ord}_v(fdt_v)=\mathrm{ord}_v(f)$, where t_v is a regular parameter at v. In the same way $\Omega_C^{\log}$ defines another order, that we call the logarithmic order. Note that $\mathrm{logord}_v(g\frac{dt_v}{t_v})=\mathrm{ord}_v(g)$ and $\mathrm{logord}_v(\omega)=\mathrm{ord}_v(\omega)+1$.

Finally, we define a *p-order* on meromorphic functions by

$$\mathrm{ord}_v(f)=\max_{c\in k(C)}\mathrm{ord}_v(f-c^p).$$

Clearly, it is determined by the image of f in the completion of $k(C)$ at v.

Lemma 4.1.1. *Keep the above notation. Then*

(i) The different measures the difference between the differential orders on C and S, namely $\delta_{v/u}=\mathrm{ord}_v(h^*\omega)-n_v\mathrm{ord}_u(\omega)$ for any non-zero meromorphic differential form ω on S.

(ii) In the same way, $\delta_{v/u}^{\log}=\mathrm{logord}_v(h^*\omega)-n_v\mathrm{logord}_u(\omega)$.

(iii) $\mathrm{pord}_v(t) = \mathrm{logord}_v(dt)$ *for any* $t \in k(C)$.
(iv) If t_u *is a regular parameter at* u *then*

$$\delta_v = \mathrm{ord}_v(dt_u) - n_v = \mathrm{pord}_v(t_u) - n_v + 1.$$

The proof is a simple exercise.

4.2. **k-analytic analogues.** Let X be a nice curve. Then the $\mathcal{O}_X^\circ$-semilattice $\Omega_X^\diamond = \mathcal{O}_X^\circ d(\mathcal{O}_X^\circ)$ of Ω_X defines a seminorm $\| \ \|_\Omega$ called the *Kähler seminorm*, see [Tem16]. For any form ω on X the function $\|\omega\|_\Omega$ is pm, and, as in the baby case, given a morphism $f\colon Y \to X$ and a non-zero form ω on X we have that $\delta_f = \|f^*\omega\|_Y \cdot f^*\|\omega\|_X^{-1}$. This formula with $\omega = dt_X$ for a tame parameter t_X on X will be our main tool for computing δ_f. Note that $\|dt_X\|_X = r_{t_x}$ is the radius function of t_X.

In addition, we will consider a *p-seminorm* $| \ |_p$ on functions on Y given by $|h|_{p,y} = \inf_{c \in \mathcal{O}_y} |h - c^p|_y$.

4.3. **Reduction of differential forms.** The following theorem from [CTT16] computes the reduction of $\Omega_X^\diamond$.

Theorem 4.3.1. *For any point* $x \in X$ *of type 2,* $\Omega^\diamond_{X_G,C_x}/k^{\circ\circ}\Omega^\diamond_{X_G,C_x} = \Omega^{\log}_{C_x}$.

Corollary 4.3.2. *For any point* $x \in X$ *of type 2,* $\Omega^\diamond_{X_G,x}/k^{\circ\circ}\Omega^\diamond_{X_G,x} = \Omega^{\log}_{C_x,x}$. *In particular, for any form* ω *with* $\|\omega\|_x \le 1$ *we obtain a reduction* $\widetilde{\omega}$*, which is a meromorphic differential form on* C_x *and for any* $v \in C_x$ *we have that*

$$\mathrm{slope}_v\|\omega\| = -\mathrm{logord}_v(\widetilde{\omega}).$$

4.4. **Reduction of the different.** Consider now a morphism of nice curves $f\colon Y \to X$ with points $y \in Y$ and $x = f(y)$. Then any $a \in k$ with $|a| = \delta_f(y)^{-1}$ induces an isometry $f^*\Omega_{X,x} \xrightarrow{\sim} \Omega_{Y,y}$ and passing to reductions we obtain an isomorphism

$$\phi_y\colon \widetilde{f}_y^*\Omega_{C_x,x} \to \Omega_{C_y,y}.$$

This isomorphism is the main tool in [CTT16] to prove the balancing condition on δ_f at y.

Remark 4.4.1. ϕ_y can be viewed as a meromorphic section of $D_y := \widetilde{f}_y^*\Omega^{-1}_{C_x,x} \otimes \Omega_{C_y,y}$ and the proof in [CTT16] boils down to computing $\deg(D_y)$ by summing up the orders of zeros and poles of ϕ_y.

Remark 4.4.2. Note that D_y is an invariant of the morphism $\widetilde{f}_y\colon C_y \to C_x$, namely it is the determinant sheaf of the cotangent complex $\mathbf{L}_{C_y/C_x}$.

Finally, one observes in [BT] that ϕ_y is defined rather uniquely.

Theorem 4.4.3. *For any* $v \in C_y$ *one has that* $\mathrm{slope}_v(\delta_f) = -\mathrm{ord}_v(\phi_y) + n_v - 1$, *and this determines* ϕ_y *up to a scalar* $\widetilde{c} \in \widetilde{k}^\times$.

4.5. **Restrictions on ϕ_y.** Clearly, ϕ_y can not be an arbitrary meromorphic section of D_y because its zeros and poles must satisfy some restrictions, see Theorem 2.2.2(iii). To describe them we need a more explicit way to compute ϕ_y. The main tool here will be the following result contained in [Tem10].

Theorem 4.5.1. *If* $x \in X$ *is of type 2 or 3,* $h \in \mathcal{H}(x)$ *and* $|h|_p \ge |ph|$ *then there exists* $c \in \mathcal{H}(x)$ *with* $|h|_p = |h - c^p|$. *Furthermore, in this case* $t = h - c^p$ *is a tame parameter at* x.

This decomposition easily implies the following theorem from [BT].

Theorem 4.5.2. *Assume that $f\colon Y \to X$ is generically étale of degree p, $y \in Y^{(2)}$, and $x = f(y)$. Choose a tame parameter z at x and find a presentation $f^*z = c^p + t'$ in $\mathcal{H}(y)$ such that $|t'| = |f^*z|_p$ if $|f^*z|_p \geq |pz|$ and $|t'| \leq |f^*z|_p$ otherwise. Then*

(i) If $|f^*z|_p > |pz|$ then $\phi_y = d\widetilde{t} \otimes (\widetilde{f}^*_y d\widetilde{z})^{-1}$, where $t = t'/a$ for an $a \in k$ with $|a| = |t'|$.

(ii) If $|f^*z|_p \leq |pz|$ then $\phi_y = (d\widetilde{t} + \widetilde{c}^{p-1} d\widetilde{c}) \otimes (\widetilde{f}^*_y d\widetilde{z})^{-1}$, where $t' = t/p$.

In addition, the case (ii) takes place if and only if $\delta(y) = |p|$, in particular, k is of mixed characteristic.

Remark 4.5.3. This theorem implies all properties of the different function at points of type 2. In particular, it is easy to see that all slopes are non-negative in case (ii) due to the "logarithmic" term $\widetilde{c}^{p-1} d\widetilde{c} \otimes (\widetilde{f}^*_y d\widetilde{z})^{-1}$. Note that $\widetilde{c} = \widetilde{z}^{1/p}$ so this term can be viewed as something like $(\frac{\widetilde{z}}{d\widetilde{z}})^{p-1}$, in particular, it has no zeros.

Finally, one proves in [BT] a lifting theorem for morphisms of degree p.

Theorem 4.5.4. *If $h\colon \Gamma_Y \to \Gamma_X$ is a finite morphism of metrized complexes of curves enriched by a different function $\Gamma_Y \to [0,1]$ and reduction forms ϕ_y at finite vertices that satisfy the conditions of Theorems 4.5.2 and 4.4.3 then there exists a morphism of nice curves $Y \to X$ whose skeleton with the different function and reduction forms is h.*

Remark 4.5.5. We cannot fix X in this theorem, but suspect that this should be possible. The main idea is, similarly to [ABBR15], to lift maps over star neighborhoods of vertices (by explicit formulas suggested by the above theorem) and then glue the morphism over the annuli. The main point is that an étale morphism of annuli $A_1 \to A_2$ of degree p is determined by the different up to $\mathrm{Aut}(A_1) \times \mathrm{Aut}(A_2)$. Unlike the tame case, we also have to use $\mathrm{Aut}(A_2)$, and this forces us to play with X as well.

Remark 4.5.6. Finally, one might wonder what is the situation with an arbitrary degree. We expect that locally at $y \in Y$ one should consider l reduction forms, where $n^i_y = p^l$ is the inseparability degree of $\widetilde{\mathcal{H}(y)}/\widetilde{\mathcal{H}(x)}$. They are constructed by splitting f into a composition and satisfy the same conditions as above. We expect that any consistent data consisting of a finite map of metrized complexes of curves, a profile function on Γ_Y, and reduction forms at finite vertices, should admit a lifting to a morphism of nice curves.

References

[ABBR15] Omid Amini, Matthew Baker, Erwan Brugallé, and Joseph Rabinoff, *Lifting harmonic morphisms I: metrized complexes and Berkovich skeleta*, Res. Math. Sci. **2** (2015), Art. 7, 67. MR 3375652

[BT] Uri Brezner and Michael Temkin, *Liftings of moderately wild coverings of berkovich curves*, in preparation.

[CTT16] Adina Cohen, Michael Temkin, and Dmitri Trushin, *Morphisms of Berkovich curves and the different function*, Adv. Math. **303** (2016), 800–858.

[Tem10] Michael Temkin, *Stable modification of relative curves*, J. Algebraic Geom. **19** (2010), no. 4, 603–677. MR 2669727 (2011j:14064)

[Tem14] ———, *Metric uniformization of morphisms of Berkovich curves*, ArXiv e-prints (2014), http://arxiv.org/abs/1410.6892.

[Tem16] ———, *Metrization of differential pluriforms on Berkovich analytic spaces*, Nonarchimedean and Tropical Geometry, Simons Symposia, Springer, 2016, pp. 195–285.

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Giv'at Ram, Jerusalem, 91904, Israel
E-mail address: temkin@math.huji.ac.il