NOTES OF MY TALK AT THE 2017 SIMONS SYMPOSIUM ON NON-ARCHIMEDEAN AND TROPICAL GEOMETRY

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Title: The Frobenius structure conjecture in dimension two (joint work with S. Keel)

Plan:

(1) The Frobenius structure conjecture of Gross-Hacking-Keel
(2) Main theorem
(3) Structure constants via non-archimedean geometry
(4) Finiteness theorems
(5) Compactification and extension

Acknowledgement: I would like to thank Matt Baker and Sam Payne for the invitation and the organization.

1. The Frobenius structure conjecture of Gross-Hacking-Keel

Setup:

- Geometric data:
  - Y connected smooth projective variety $/\mathbb{C}$
  - $D \in |-K_Y|$ effective snc divisor, containing at least one 0-stratum, supporting an ample divisor
  - $U \coloneqq Y \setminus D$, called log Calabi-Yau variety with maximal boundary
- Combinatoric data:
  - B: dual intersection cone complex of D
  - $B(\mathbb{Z})$: integer points (thought of as divisorial valuations on the function field of Y)
- Algebraic data:
  - $R \coloneqq \mathbb{Z}[\mathrm{NE}(Y)]$, $\mathrm{NE}(Y)$: monoid of curves in Y modulo numerical equivalence

Date: July 15, 2017.

- A: free R-module generated by $B(\mathbb{Z})$,

$$A \coloneqq \bigoplus_{P \in B(\mathbb{Z})} R \cdot \theta_P$$

GHK observe a natural R-multilinear map

$$\langle , \dots , \rangle \colon A^n \to R, \quad \text{for each } n \geq 2,$$

by counting rational curves in Y.

Let me first state their conjecture before describing the R-multilinear map.

Conjecture (GHK)**.** *(1) The R-multilinear map $\langle , \dots , \rangle$ is non-degenerate.*

(2) The R-multilinear map comes from a Frobenius algebra structure, i.e., $\exists!$ commutative associative R-algebra structure on A such that
- $1_A = \theta_0$
- *Coefficient of θ_0 in $a_1 \cdots a_n$ is equal to $\langle a_1, \dots, a_n \rangle_n$.*

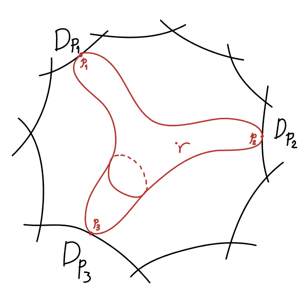
(3) $\operatorname{Spec} A \to \operatorname{Spec} R$ *restricted to* $T_{\operatorname{Pic}(Y)} (\coloneqq \operatorname{Pic}(Y) \otimes \mathbb{G}_{\mathrm{m}}) \subset \operatorname{Spec} R$ *is a family of affine log Calabi-Yau varieties (of same dimension as U) with maximal boundary, (called the mirror family of U).*

Now let me describe the R-multilinear map:

Given $P_1, \dots, P_n \in B(\mathbb{Z})$, $\beta \in \mathrm{NE}(Y)$, choose a toric blowup $\pi \colon (\widetilde{Y}, \widetilde{D}) \to (Y, D)$, (composition of blowups of boundary strata, $\widetilde{D} = \widetilde{Y} \setminus U$), such that P_i has divisorial center $D_{P_i} \subset \widetilde{D}$.

Let $c(P_1, \dots, P_n, \beta) \coloneqq$ number of $\Big(\mathbb{P}^1, (p_1, \dots, p_n, r)\Big) \xrightarrow{f} \widetilde{Y}$ such that

- $f^*(D_{P_i}) = m_i \cdot p_i$, where $P_i = m_i \cdot P_i^{\text{prim}}$
- $f_*[\mathbb{P}^1] = \beta$
- domain fixed general modulus
- $f(r) =$ a fixed general point $y \in \widetilde{Y}$.



Proposition. *The set above is a finite set. So the number is well-defined.*

Remark. Although the conjecture is phrased in terms of the R-multilinear map, it is really about the geometry of rational curves in Y.

2. Main theorem

Theorem 1. *The conjecture holds in dimension two.*

3. Structure constants via non-archimedean geometry

We construct the multiplication on A by counting non-archimedean holomorphic disks (based on the techniques developed in my thesis).

For $P_1, \ldots, P_n \in B(\mathbb{Z})$, write

$$\theta_{P_1} \cdot \theta_{P_2} \cdots \theta_{P_n} = \sum_{Q \in B(\mathbb{Z})} \left(\sum_{\gamma \in \mathrm{NE}(Y)} \chi(P_1, \ldots, P_n, Q, \gamma) z^\gamma \right) \theta_Q$$

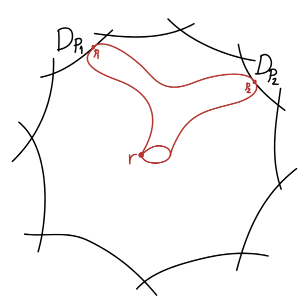
Goal: Define $\chi(P_1, \ldots, P_n, Q, \gamma)$ using non-archimedean geometry.

Let $k = \mathbb{C}(\!(t)\!)$, $U_k \coloneqq U \otimes_{\mathbb{C}} k$, and U_k^{an} k-analytic space (in the sense of Berkovich).

We have embedding $B \hookrightarrow U_k^{\mathrm{an}}$, and retraction $\tau \colon U_k^{\mathrm{an}} \to B$ (by Berkovich, Thuillier, Gubler-Rabinoff-Werner).

$\chi(P_1, \ldots, P_n, Q, \gamma)$ counts holomorphic disks in $\widetilde{Y}_k^{\mathrm{an}}$, $\big(\Delta, (p_1, \ldots, p_n, r)\big) \xrightarrow{f} \widetilde{Y}_k^{\mathrm{an}}$, such that

- $f^*(D_{P_i}) = m_i \cdot p_i$
- $(\tau \circ f)(\partial \Delta) =$ a fixed point $b \in B$ near $\overrightarrow{OQ}$
- $(\tau \circ f)(\text{neighborhood of } \partial \Delta) =$ a segment starting from b in the direction Q
- class $= \gamma$
- domain fixed general modulus
- $f(r) =$ a fixed general point $y \in \tau^{-1}(b)$.



Big trouble: The space of all such holomorphic disks is ∞-dimensional.

Solution from my thesis: Impose a *regularity condition* on the boundary of our holomorphic disks. We ask: by analytic continuation at the boundary, our holomorphic disks extend all straight (i.e. its image in B is straight with respect to the $\mathbb{Z}$-affine structure on B).

Theorem 2. *The space of holomorphic disks in* $\widetilde{Y}_k^{\mathrm{an}}$ *satisfying all the conditions above (including the boundary regularity condition) is a finite set.*

Trouble 2: Extending straight on the left side of $\overrightarrow{OQ}$ may differ from extending straight on the right side of $\overrightarrow{OQ}$. So our counts depend on the choice of b being on the left or right of $\overrightarrow{OQ}$?

Solution: Define another regularity condition: by analytic continuation at the boundary, our holomorphic disks extend all straight with respect to a toric model $\pi\colon Y \to \overline{Y}$, with $\overline{Y}$ toric.

Theorem 3. *Counts using "left regularity condition"*
=Counts using "toric regularity condition"
=Counts using "right regularity condition".

Corollary. *The counts* $\chi(P_1, \dots, P_n, Q, \gamma)$ *is well-defined.*

Now the product $\theta_{P_1} \cdots \theta_{P_n}$ is well-defined. Natural question: commutativity? associativity?

Commutativity: easy from definition. Associativity: difficult theorem.

Theorem 4. *The multiplication is associative.*

4. Finiteness theorems

Natural question: Are the two sums in the multiplication formula finite? Otherwise we will only get a formal algebra instead of a genuine algebra.

Theorem 5 (Finiteness I)**.** *Given* $P_1, \dots, P_n \in B(\mathbb{Z})$, $\exists$ *finitely many* (Q, γ), *such that* $\chi(P_1, \dots, P_n, Q, \gamma) \neq 0$.

Corollary. *A is a commutative associative R-algebra.*

Theorem 6 (Finiteness II)**.** *A is a finitely generated R-algebra.*

5. Compactification and extension

We have constructed a family: $\operatorname{Spec} A \to \operatorname{Spec} R$.

In order to obtain more information about $\operatorname{Spec} A$, we need to compactify the fibers, and extend the family over a larger base.

Let us start with compactification:

Fix F ample divisor on Y such that $\operatorname{supp} F = D$.

$\rightsquigarrow$ Filtration on A: $A_{\leq s} := \bigoplus_{\{F^{\mathrm{trop}}(Q) \leq s\}} R \cdot \theta_Q$.

$\rightsquigarrow$ Subalgebra $\widetilde{A} \subset A[T]$ generated by $\{\, a \cdot T^s \mid a \in A_{\leq s} \,\}$ as submodule.

$\rightsquigarrow$ $\mathcal{X} := \operatorname{Proj}(\widetilde{A}) \to \operatorname{Spec} R$ is a fiberwise compactification of $\operatorname{Spec} A \to \operatorname{Spec} R$.

Now we will extend our family over a larger base.

Note $R = \mathbb{Z}[\mathrm{NE}(Y)]$. Since the dual of the cone of curves is the nef cone, we have $\operatorname{Spec} R = \mathrm{TV}(\mathrm{Nef}(Y))$.

This has a natural extension: $\mathrm{TV}(\mathrm{Nef}(Y)) \subset \mathrm{TV}(\mathrm{MoriFan}(Y))$, where $\mathrm{MoriFan}(Y)$ is a fan in $N^1(Y, \mathbb{R})$ defined as follows: For every birational map $\pi\colon Y \to Y'$ with Y' normal, we have a full dimensional cone

$$\mathrm{Bogus}(\pi) := \Big\{ \pi^* N + E \mid N \in \mathrm{Nef}(Y', \mathbb{R}), E \text{ effective divisor} \\ \text{supported on the exceptional locus} \Big\} \subset N^1(Y, \mathbb{R}).$$

We define

$$\mathrm{MoriFan}(Y) := \bigcup_{\pi\colon Y \to Y'} \mathrm{Bogus}(\pi) \subset N^1(Y, \mathbb{R}).$$

Theorem 7. *The compactified family $\mathcal{X} \to \operatorname{Spec} R$ extends over $\mathrm{TV}(\mathrm{MoriFan}(Y))$.*

Theorem 8 (Final theorem of the talk)**.** *The extended family $\big(\mathcal{X}, \mathcal{Z} = (T = 0)\big) \to \mathrm{TV}(\mathrm{MoriFan}(Y))$ is a flat projective family of surfaces with effective Weil divisor, satisfying*

(1) $\mathcal{Z} \to \mathrm{TV}(\mathrm{MoriFan}(Y))$ is a trivial family of cycles of rational curves.

(2) The fibers (X, Z) are semi-log canonical, and $K_X + Z$ is trivial.

(3) The fibers (X, Z) over $T_{\mathrm{Pic}(Y)} \subset \mathrm{TV}(\mathrm{MoriFan}(Y))$ are log canonical. The interior $V := X \setminus Z$ is an affine canonical log Calabi-Yau surface with maximal boundary.

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