CHIRAL ALGEBRAS

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INTRODUCTION

В настоящей работе не содержатся
сведения, которые могли бы составить
предмет изобретения или открытия

Анон., "Акт Экспертизы", 1956 (?)+

0.0. This work is an exposition of basic chiral algebra theory. Chiral algebras take their origin in mathematical physics; they lie at the heart of conformal field theory. Mathematicians usually look at them through the formalism of vertex operator algebras (see, e.g., [Bor], [FLM], [FHL], [DL], [LZ], [K]), which incorporates chiral algebras in representation theory of infinite-dimensional algebras. We follow a different approach which tastes more of algebraic geometry than of representation theory.

0.1. Chiral algebras are "quantum" objects. Let us describe first the corresponding "classical" objects; we call them *coisson* algebras ("coisson" may be considered as an abbreviation for "chiral Poisson" or "compound Poisson", the word "compound" being related with the notion of compound tensor category which we discuss in 0.4). In fact "coisson algebra" is a new name for a well known class of objects. Informally a coisson algebra is defined by a local Poisson bracket on a space of classical fields. In the simplest case where

+ This article does not contain any information that could be the subject of an invention or a discovery. Anon., "Expert Certification", 1956 (?)

"classical field" means "a function $u(x)$" a mathematical physicist would define a local Poisson bracket by a formula like

$$\{u(x), u(y)\} = \sum_{i=1}^{n} \varphi_i(x, u(x), u'(x), \dots)\delta^{(i)}(x-y), \quad \varphi_i \in A. \tag{0.1.1}$$

Here A denots the algebra of functions of $x, u, u', \dots, u^{(k)}$ (k is not fixed). Of course in the l.h.s. of (0.1.1) $u(x)$ is understood as a functional $u \mapsto u(x)$ on the space of classical fields. The bracket should be skew-symmetric and satisfy the Jacobi identity, so the collection of functions φ_i should satisfy certain differential equations. For $f \in A$ and every x we have the functional $\ell_{f,x}$ on the space of classical fields defined by $\ell_{f,x}(u) = f(x, u(x), u'(x), \dots)$. It follows from (0.1.1) that

$$\{\ell_{f,x}, \ell_{g,y}\}(u) = \sum_i \varphi_i^{f,g}(x, u(x), u'(x), \dots)\delta^{(i)}(x-y) \tag{0.1.2}$$

for some $\varphi_i^{f,g} \in A$. Formula (0.1.2) has a more symmetric version:

$$\{\ell_{f,x}, \ell_{g,y}\}(u) = \sum_{i,j} \partial_x^i \partial_y^i \{\psi_{ij}^{f,g}(x, u(x), u'(x), \dots)\delta(x-y)\}, \quad \psi_{ij}^{f,g} \in A \tag{0.1.3}$$

Of course $\psi_{ij}^{f,g}$ are not uniquely determined by $\varphi_i^{f,g}$; we must take into account the relations

$$(\partial_x + \partial_y)(a\delta) = (\partial_x a) \cdot \delta, \quad a \in A, \tag{0.1.4}$$

where $\delta = \delta(x-y)$.

Now from the algebraic point of view the primary object is the algebra A rather than the "space of classical fields". For instance one can take $A = \mathbb{C}[x, u, u', u'', \dots]$. More generally, A can be any commutative differential unital algebra over $\mathbb{C}[x]$, and a "classical field" is a homomorphism of differential unital $\mathbb{C}[x]$-algebras $A \to B$ where B is, e.g., the algebra of C^∞-functions of $x \in \mathbb{R}$. A choice of a set of generators and defining relations for the differential $\mathbb{C}[x]$-algebra A identifies classical fields with collections of functions satisfying certain systems of differential equations.

Let us try to formulate in terms of A what is a "local Poisson bracket on the space of classical fields". A glance at (0.1.4) and the r.h.s. of (0.1.3) shows that such an object is defined by a mapping

$$A \otimes_{\mathbb{C}} A \to V \tag{0.1.5}$$

where V is the module over $\mathbb{C}[\partial_x, \partial_y]$ generated by the symbols $a\delta$, $a \in A$, with the defining relations (0.1.4). We prefer another interpretation of V: V is the module over $\mathbb{C}[x, y, \partial_x, \partial_y]$ generated by the symbols $a\delta$, $a \in A$, with the defining relations (0.1.4) and also the following ones:

$$f(x,y) \cdot (a\delta) = (f(x,x)a) \cdot \delta, \quad f \in \mathbb{C}[x,y], \; a \in A. \tag{0.1.6}$$

The mapping (0.1.5) should satsify certain properties. First of all it should be a morphism of $\mathbb{C}[x, y, \partial_x, \partial_y]$-modules (notice that $A \otimes A$ is a module over $\mathbb{C}\left[x, \frac{\partial}{\partial_x}\right] \otimes \mathbb{C}[x, \partial_y] = \mathbb{C}[x, y, \partial_x, \partial_y]$). Before discussing the other properties let us rewrite (0.1.5) in geometrical terms. An algebra A over $\mathbb{C}[x]$ is the same as a quasicoherent $\mathcal{O}_X$-algebra $\mathcal{A}$ where $X := \operatorname{Spec} \mathbb{C}[x]$ is the affine line. Differential $\mathbb{C}[x]$-algebras correspond to $\mathcal{D}_X$-*algebras* (see 0.2). The $\mathcal{D}_{X \times X}$-module corresponding to the $\mathbb{C}[x, y, \partial_x, \partial_y]$-module V from (0.1.5) is canonically isomorphic to $\Delta_* \mathcal{A}$ where $\Delta \colon X \to X \times X$ is the diagonal embedding. So (0.1.5) is equivalent to a $\mathcal{D}_{X \times X}$-module morphism

$$\mathcal{A} \boxtimes \mathcal{A} \to \Delta_* \mathcal{A}. \tag{0.1.7}$$

We denote by $\boxtimes$ the external tensor product, i.e., $\mathcal{A} \boxtimes \mathcal{A} := p_1^* \mathcal{A} \otimes p_2^* \mathcal{A}$ where p_1 and p_2 are the projections $X \times X \to X$.

Notice that (0.1.7) makes sense for an arbitrary smooth variety X. In 0.6 we will define a coisson algebra on X as a commutative (= commutative associative unital) $\mathcal{D}_X$-algebra $\mathcal{A}$ with a $\mathcal{D}$-module morphism (0.1.7) satisfying certain properties which actually mean that (0.1.3) is a Poisson bracket on the "space(s) of classical fields." To formualte these properties in a natural way we need some polylinear algebra in the category of $\mathcal{D}_X$-modules (see 0.2 – 0.5).

Remarks. (i) The notions of Poisson $\mathcal{O}_X$-algebra or Poisson $\mathcal{D}_X$-algebra are inadequate for expressing the idea of "Poisson bracket on the space(s) of classical fields": a Poisson bracket on $\mathcal{A}$ is a morphism $\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{A}\to\mathcal{A}$ while (0.1.7) is an object of different nature. The reason is that $\operatorname{Spec}\mathcal{A}$ is the space of "jets of classical fields", not the space of classical fields.

(ii) The reader can compare our notion of coisson algebra with other ways of formalizing the notion of "Poisson bracket on the space of classical fields" (see [DG1], [DG2], [Ku], [KuM], [Ma] and references therein). As far as we understand, our approach is essentially equivalent to that of [KuM] and [Ma].

0.2. Let X be a smooth complex algebraic variety. Consider the category $\mathcal{M}^\ell(X)$ of left $\mathcal{D}$-modules on X (see e.g. [B]). The $\mathcal{O}_X$-tensor product $\otimes$ makes it a tensor category with unit object $\mathcal{O}_X$. Commutative algebras in $\mathcal{M}^\ell(X)$ (*commutative $\mathcal{D}_X$-algebras*) are just quasicoherent commutative $\mathcal{O}_X$-algebras equipped with a flat connection along X. We call an X-scheme equipped with a flat connection along X a *$\mathcal{D}_X$-scheme*; so the spectra of commutative $\mathcal{D}_X$-algebras are $\mathcal{D}_X$-schemes affine over X.

A standard example: for any X-scheme $Y\to X$ the space $\mathcal{J}(Y/X)$ of ∞-jets of sections of Y/X is a $\mathcal{D}_X$-scheme. Horizontal sections of $\mathcal{J}(Y/X)$ are the same as arbitrary sections of Y/X. One may view the (closed) $\mathcal{D}_X$-subschemes of $\mathcal{J}(Y/X)$ as systems of differential equations on sections of Y/X (see [G]).

What makes the tensor category $\mathcal{M}^\ell(X)$ substantially different from, say, the tensor category of $\mathcal{O}_X$-modules is the absence of duals. Precisely, for $F\in\mathcal{M}^\ell(X)$ the dual object in the tensor category sense (see e.g. [D]) exists iff F is coherent as an $\mathcal{O}_X$-module (i.e., M is a vector bundle with an integrable connection). For example, for a group $\mathcal{D}_X$-scheme G one has its Lie coalgebra $CoLie(G)$ but we cannot dualize it to get the Lie algebra (unless G is finite dimensional as a usual scheme). We will see that the category $\mathcal{M}^\ell(X)$ carries a richer structure (that of *compound tensor category*) which remedies the above flaw.

0.3. Consider the category $\mathcal{M}(X)$ of right $\mathcal{D}$-modules on X. Let I be a finite non-empty set; denote by $\Delta^{(I)}\colon X\hookrightarrow X^I$ the diagonal embedding. For an I-family of $\mathcal{D}$-modules

$L_i \in \mathcal{M}(X)$ and $M \in \mathcal{M}(X)$ set

$$P_I^*(\{L_i\}, M) := \mathrm{Hom}(\boxtimes L_i, \Delta_*^{(I)} M) \tag{0.3.1}$$

Elements of $P_I^*(\{L_i\}, M)$ are called $*$ *I-operations*; they are X-local (since $\Delta_*^{(I)} M$ sits on the diagonal). The $*$ operations compose in a natural way[1], just as polylinear maps between vector spaces do (i.e., for a surjective map $\pi: J \to I$, a J-family $\{K_j\}$ of $\mathcal{D}$-modules, and $\varphi \in P_I^*(\{L_i\}, M)$, $\psi_i \in P_{\pi^{-1}(i)}^*(\{K_j\}, L_i)$ we have $\varphi(\psi_i) \in P_J^*(\{K_j\}, M)$). The composition is associative; if $|I| = 1$ then $P_I^* = \mathrm{Hom}$. We call such data of operations (or "polylinear maps") between the objects of a category a *pseudo-tensor structure.* Thus $\mathcal{M}(X)$ carries a canonical pseudo-tensor structure; $\mathcal{M}(X)$ equipped with this structure is denoted by $\mathcal{M}(X)^*$.

One may view pseudo-tensor categories as a straightforward generalization of operads (see e.g. [L]): an operad is just a pseudo-tensor category with single object.

The notions of algebras, modules over them, etc., make perfect sense in any pseudo-tensor category. For example, a Lie algebra in $\mathcal{M}(X)^*$ (or simply a *Lie* algebra on* X) is a $\mathcal{D}$-module L together with a binary $*$ operation $[\ \ ] \in P_2^*(\{L, L\}, L)$ which is skew-symmetric and satisfies Jacobi identity.

For $M \in \mathcal{M}(X)$ set $h(M) := M \underset{\mathcal{D}_X}{\otimes} \mathcal{O}_X$ (the sheaf of middle de Rham cohomology). There is a canonical map $P_I^*(\{L_i\}, M) \to \mathrm{Hom}(\otimes h(L_i), h(M))$ compatible with composition of operations. Therefore h sends Lie* algebras to the sheaves of usual Lie algebras. In fact, many important Lie algebras (including the Virasoro and affine Kac-Moody algebras) arise from Lie* algebras.

0.4. Let us identify the categories of left and right $\mathcal{D}$-modules by the usual equivalence $\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}^\ell(X)$, $M \mapsto M^\ell = M\omega_X^{-1} = M \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}$ (here $\omega_X = \Omega_X^{\dim X}$). Therefore $\mathcal{M}(X)$ has the tensor product $M \otimes^! L := (M^\ell \otimes L^\ell)\omega_X$. Now the ! tensor product of two $*$ operations is defined according to the following pattern. Let I, J be finite sets, $i_0 \in I$, $j_0 \in J$; denote by $I \vee J$ the disjoint union of I, J with i_0, j_0 identified. Then there is a

[1]We switched to right $\mathcal{D}$-modules to make the "sign rule" for the $*$ operations obvious.

natural map

$$\otimes^{I,J}_{i_0,j_0}\colon P^*_I(\{M_i\},L)\otimes P^*_J(\{N_j\},K)\to P^*_{I\vee J}(\{M_i,N_j,M_{i_0}\otimes^! N_{j_0}\}_{\substack{i\neq i_0\\ j\neq j_0}},L\otimes^! K)$$

If $I=\{i_0\}$, $J=\{j_0\}$ this is the usual tensor product of morphisms.

A *compound tensor structure* on a category is the data of a tensor and a pseudo-tensor structure related as above by the $\otimes^{I,J}_{i_0,j_0}$ maps which are associative and commutative in the obvious sense. Thus $\mathcal{M}(X)$ (and $\mathcal{M}^\ell(X)$) is a compound tensor category.

0.5. The compound tensor structure allows to do basic differential geometry in $\mathcal{M}(X)$ implementing the principle that the "functions" multiply according to $\otimes^!$ format, while duals and the action of infinitesimal symmetries must be understood in the $*$ sense. In other words, the geometric objects we consider are $\mathcal{D}_X$-schemes, the Lie algebras are actually Lie* algebras, and one defines easily what the action of a Lie* algebra on a $\mathcal{D}_X$-scheme means. Note that an action of a Lie* algebra L on a $\mathcal{D}_X$-scheme $\mathcal{Y}$ yields an action on $\mathcal{Y}$ of the usual Lie algebra $h(L)$.

For example, let G be a group $\mathcal{D}_X$-scheme such that $CoLie(G)$ is a coherent $\mathcal{D}_X$-module. Set $Lie(G):=\operatorname{Hom}_{\mathcal{D}_X}(CoLie(G),\mathcal{D}_X)$ (this usual $\mathcal{D}$-module duality is actually the duality for the $*$ pseudo-tensor structure). Then $Lie(G)$ is a Lie* algebra. A G-action on a $\mathcal{D}_X$-scheme $\mathcal{Y}$ yields a $CoLie(G)$-coaction on $\mathcal{O}_{\mathcal{Y}}$ (in the $\underset{\mathcal{O}_X}{\otimes}$ sense). Dualizing, we get a $Lie(G)$-action on $\mathcal{Y}$.

0.6. Now a *coisson algebra* is simply a Poisson algebra in the compound setting. Namely, this is a commutative $\mathcal{D}_X$-algebra A^ℓ together with a Lie* algebra structure on A (*the coisson bracket*) such that the adjoint action is compatible with the multiplicative structure on A^ℓ. ¿From this we get an action of $h(A)$ on A^ℓ in the usual sense. Derivations of A^ℓ that come from $h(A)$ are called hamiltonian vector fields. Coisson brackets localize nicely so one knows what is a coisson structure on any $\mathcal{D}_X$-scheme.

0.7. Let $\mathcal{Y}=\operatorname{Spec}A^\ell$ be an affine $\mathcal{D}_X$-scheme. The space of horizontal sections $X\to\mathcal{Y}$ is an ind-affine ind-scheme; if X is compact this is actually a scheme. We denote the corresponding (pro) algebra as $\langle A\rangle=H_0^{ch}(X,A)$. The derived version of this construction yields a graded commutative superalgebra $H_\cdot^{ch}(X,A)$.

The above construction also makes sense locally. For example, assume that X is an affine curve; let $X \subset \bar{X}$ be a smooth compactification. For $z \in X^\infty = \bar{X} \smallsetminus X$ the space of horizontal sections of $\mathcal{Y}$ over the (formal) punctured disc at z is again an ind-affine scheme $\operatorname{Spec} A_{((z))}$. The restriction of a global section to the punctured discs defines a surjective morphism $A_{((\infty))} := \widehat{\underset{X^\infty}{\otimes}} A_{((z))} \twoheadrightarrow H_0^{ch}(X, A)$. Its kernel is the ideal generated by the image of certain canonical map $r_{((\infty))}\colon\ H^0(X, h(A)) \to A_{((\infty))}$.

If A is a coisson algebra finitely generated as $\mathcal{D}_X$-algebra then $A_{((z))}$ are Poisson algebras, $H^0(X, h(A))$ is a Lie algebra, and $r_{((\infty))}$ commutes with brackets. Therefore $r_{((\infty))}$ is a Hamiltonian action of $H^0(X, h(A))$ on $\operatorname{Spec} A_{((\infty))}$, and $\operatorname{Spec}\langle A\rangle$ is the zero fiber of the momentum map.

0.8. Now let us pass to chiral algebras. There are two complementary (equivalent) approaches to this notion: Lie algebra style and commutative algebra style. We begin with the "Lie algebra" approach; for the "commutative algebra" one see 0.12.

¿From now on we assume that $\dim X = 1$. Let $j^{(I)}\colon U^{(I)} \hookrightarrow X^I$ be the complement to the diagonal divisor. Set

$$P_I^{ch}(\{L_i\}, M) := \operatorname{Hom}(j_*^{(I)} j^{(I)*} \boxtimes L_i, \Delta_*^{(I)} M). \tag{0.8.1}$$

These are *chiral I-operations*. One defines their composition in the obvious manner. We get the *chiral pseudo-tensor structure* on $\mathcal{M}(X)$; the corresponding pseudo-tensor category is denoted by $\mathcal{M}(X)^{ch}$.

Now a *chiral algebra* on X is simply a Lie algebra in $\mathcal{M}(X)^{ch}$ with an additional property (existence of unit), which we prefer to skip at the moment.

0.9. We want to show that the notion of chiral algebra is a quantization of that of coisson algebra. Let us consider first the chiral structure itself.

For a finite non-empty set I denote by $Q(I)$ the set of all quotient sets of I (= equivalence relations on I), so for $S \in Q(I)$ there is a canonical decomposition $I = \coprod_{s\in S} I_s$. The $\mathcal{D}$-module $j_*^{(I)} j^{(I)*} \boxtimes L_i$ on X^I carries a canonical filtration with associated graded module identified with a quotient of $\underset{S\in Q(I)}{\oplus} \underset{S}{\boxtimes} (\Delta_*^{(I_s)} (\underset{I_s}{\otimes^!} L_i) \otimes \mathcal{L}ie_{I_s}^*)$ (the Lie operad spaces appear here as top homology of spaces of configurations of points on a plane, see [C]). Set

$$P^c_I(\{L_i\},M) := \underset{S\in Q(I)}{\oplus} P^*_S(\{\overset{!}{\underset{i\in I_s}{\otimes}} L_i\},M)\otimes(\underset{s\in S}{\otimes}\mathcal{L}ie_{I_s}). \tag{0.9.1}$$

We see that the above filtration on $j^{(I)}_* j^{(I)*}\boxtimes L_i$ yields a filtration on $P^{ch}_I(\{L_i\},M)$ with $\mathrm{gr}P^{ch}_I(\{L_i\},M)$ identified with a subspace of $P^c_I(\{L_i\},M)$; if L_i are locally free $\mathcal{D}_X$-modules and X is affine, then $\mathrm{gr}\,P^{ch} = P^c$. The filtration on P^{ch} is compatible with the composition of chiral operations, so it defines composition of operations from $\mathrm{gr}\,P^{ch}$ and P^c.

Therefore c operations are symbols, or classical limits, of chiral operations. Now it is easy to see that c operations and their composition are defined purely in terms of the compound tensor structure (and, in fact, determine this structure uniquely). So the chiral pseudo-tensor structure is a *quantization* of the compound tensor structure on $\mathcal{M}(X)$.

0.10. Notice that a binary c operation is the same as a pair that consists of a binary ! operation and a binary $*$ operation. One checks easily that such c operation is a Lie bracket if and only if the corresponding ! operation is commutative and associative, and the $*$ operation is a coisson bracket with respect to this commutative algebra structure. Thus Lie^c algebras are the same as coisson algebras.

Now let A be a chiral algebra. The chiral Lie bracket $\mu\in P^{ch}_2(\{A,A\},A)$ ("chiral product") yields a $*$ bracket $[\ \ ]_\mu\in P^*_2(\{A,A\},A)$, so a chiral algebra is automatically a Lie^* algebra. We say that μ is *commutative* if $[\ \ ]_\mu=0$. As above, such μ is equivalent to a commutative $\mathcal{D}_X$-algebra structure on A^ℓ. Now assume we have a family $A_\hbar$ of chiral algebras that depend on a parameter $\hbar\in\mathbb{C}$ such that A_0 is commutative. Then $\{\ \ \}:=\left(\hbar^{-1}[\ \ ]_{\mu_\hbar}\right)\big|_{\hbar=0}$ is a coisson bracket on A^ℓ_0. We have shown that chiral algebras are quantizations of coisson algebras as promised.

The problem of quantization of coisson algebras is fairly interesting. We do not know how to solve it in general.[2] In the main body of this work we treat the simplest cases of

[2] Could it be that Kontsevich's formulas can be naturally explained in the framework of chiral algebras?

linear brackets. In particular, we construct chiral enveloping algebras of Lie* algebras and chiral algebras of twisted differential operators.

In a sense, in the "chiral world" chiral algebras are parallel to usual associative algebras while Lie* algebras and commutative $\mathcal{D}_X$-algebras play, respectively, the roles of usual Lie and commutative algebras.

0.11. The constructions from 0.7 generalize to arbitrary chiral algebras. For a chiral algebra A one defines its *chiral homology* $H_i^{ch}(X, A)$. Here $H_0^{ch}(X, A)$ is what mathematical physicists usually call "the space of conformal blocks". For any $z \in X^\infty$ we have an associative algebra $A_{((z))}$; if X is affine then $H_0^{ch}(X, A)$ identifies canonically with a quotient of $A_{((\infty))} = \widehat{\bigotimes}_{X^\infty} A_{((z))}$ modulo a right ideal. In the classical limit we return to the objects from 0.7.

0.12. Let us sketch now the "commutative algebra" style description of chiral algebras. This approach is essential for certain subjects, in particular, for the definition of chiral homology. Let A^ℓ be a left $\mathcal{D}_X$-module. A *factorization algebra* structure on A^ℓ is[3] a datum of left $\mathcal{D}$-modules $A^\ell_{X^n}$ on X^n, $n > 0$, such that for any $(x_1, .., x_n) \in X^n$ the fiber $A^\ell_{(x_1,..,x_n)}$ equals to $\bigotimes_{s \in S} A^\ell_s$ where $S := \{x_1, .., x_n\} \subset X$ (factorization property). We demand that the identifications are compatible with natural symmetries. Therefore $A^\ell_X = A^\ell$, $A^\ell_{X\times X}$ coincides with $A^\ell \boxtimes A^\ell$ over the complement U to the diagonal, and its pull-back to the diagonal equals A^ℓ, etc.

It appears that factorization algebra structure on $\mathcal{A}^\ell$ amounts to chiral algebra structure on $\mathcal{A}$. Namely, the chiral product corresponding to the above factorization algebra structure is the composition $j_* j^* A \boxtimes A = j_* j^* A_{X\times X} \to \Delta_* \Delta^! A_{X \times X} = \Delta_* A$ where the equalities are structure identifications and the arrow is a canonical morphism. Notice that A is commutative if and only if $A^\ell_{X\times X} \supset A^\ell \boxtimes A^\ell$.

0.13. Physicists usually describe the chiral algebra structure in terms of *operator product expansions*, OPE for short (see e.g. [BPZ]). This may be viewed as a generalization of the above description of commutative chiral algebras.

[3] As before, we do not worry about the unit.

To explain what OPE is one needs to consider some non-quasicoherent $\mathcal{D}$-modules. Thus, for a left $\mathcal{D}_X$-module A^ℓ let $\Delta_*^\wedge A^\ell$ be a sheaf of left $\mathcal{D}_{X\times X}$-modules supported on the diagonal $X \hookrightarrow X \times X$ which is I-adically complete and satisfies $\Delta_*^\wedge A^\ell / I\Delta_*^\wedge A^\ell = A^\ell$ (here $I \subset \mathcal{O}_{X\times X}$ is the ideal of the diagonal). Such $\Delta_*^\wedge A^\ell$ exists and is unique. Note that for any local section $a \in A^\ell$ there is a unique section $a^{(1)} \in \Delta_*^\wedge A^\ell$ such that $a^{(1)}$ is horizontal along the second variable and $a^{(1)} \bmod I\Delta_*^\wedge A^\ell = a$. Therefore, if t is a local coordinate on X, one may write any section φ of $\Delta_*^\wedge A^\ell$ as a formal power series $\sum\limits_{i\geq 0} a_i^{(1)}(t_2 - t_1)^i$, $a_i = (i!^{-1}\partial_{t_2}^i \varphi) \bmod I\Delta_*^\wedge A^\ell$. Now let $\Delta_*^\sim A^\ell = j_*\mathcal{O}_U \underset{\mathcal{O}_{X\times X}}{\otimes} \Delta_*^\wedge A^\ell$ be localization of $\Delta_*^\wedge A^\ell$ with respect to the equation of the diagonal. A section of $\Delta_*^\sim A^\ell$ is a Laurent power series $\sum\limits_{i\gg -\infty} a_i^{(1)}(t_2 - t_1)^i$, i.e. we may write $\Delta_*^\sim A^\ell$ as $A^\ell((t_2 - t_1))$.

There is a canonical isomorphism of right $\mathcal{D}_{X\times X}$-modules

$$\mathrm{Hom}_{\mathcal{O}_{X\times X}}(j_*\mathcal{O}_U, \Delta_* A) \xrightarrow{\sim} (\Delta_*^\sim A^\ell)\omega_{X\times X}$$

identifying $\varphi: j_*\mathcal{O}_U \to \Delta_* A$ with $\sum a_i^{(1)}(t_2 - t_1)^i dt_1 \wedge dt_2$, $a_i dt = \mathrm{tr}_{p_1}(\varphi((t_2 - t_1)^{-i-1}) \in A$; here $\mathrm{tr}_{p_1}: p_1.\Delta_* A \to A$ is the canonical trace map for the projection $p_1: X \times X \to X$. Therefore, by adjunction, a morphism $\circ: A^\ell \boxtimes A^\ell \to \Delta_*^\sim A^\ell$ of left $\mathcal{D}_{X\times X}$-modules is the same as a morphism $\mu = \mu_\circ: j_*j^*A \boxtimes A = (A \boxtimes A) \underset{\mathcal{O}_{X\times X}}{\otimes} j_*\mathcal{O}_U \to \Delta_* A$ of right $\mathcal{D}$-modules on $X \times X$, i.e., a chiral binary operation (we use the identification $\omega_X \boxtimes \omega_X \xrightarrow{\sim} \omega_{X\times X}$, $\nu_1 \boxtimes \nu_2 \mapsto p_1^*\nu_1 \wedge p_2^*\nu_2$). The morphism $\circ = \circ_\mu$, $a \boxtimes b \mapsto \sum (a \underset{i}{\circ} b)^{(1)}(t_2 - t_1)^i$, is the *operator product expansion* disguise of μ.

It is clear what commutativity of $\circ$ means, and it amounts exactly to skew commutativity of $\mu_\circ$. To see what associativity of $\circ$ means take $a, b, c \in A^\ell$ and consider the iterated product

$$a \circ (b \circ c) = \sum_{i\gg -\infty} \sum_{j\gg -\infty} (a \underset{j}{\circ} (b \underset{i}{\circ} c))^{(1)}(t_2 - t_1)^j (t_3 - t_2)^i \in A^\ell((t_2 - t_1))((t_3 - t_2))$$

Informally speaking this series converges in the domain $0 < |t_3 - t_2| \ll |t_2 - t_1| \ll 1$. Just as with μ and $\circ$, one may see that this iterated product amounts exactly to the operation $\mu(id_A, \mu) \in P_3^{ch}(\{A, A, A\}, A)$. Similarly, we may consider $(a \circ b) \circ c \in A^\ell((t_3 - t_1))((t_2 - t_1))$.

Note that $(a \circ b) \circ c$ and $a \circ (b \circ c)$ converge in different domains so to compare them one needs analytic continuation. Precisely, both iterated Laurent power series spaces contain the subspace $A^\ell[\![t_2 - t_1, t_3 - t_2]\!][(t_1 - t_2)^{-1}, (t_1 - t_3)^{-1}, (t_2 - t_3)^{-1}]$, and associativity of $\circ$ means that both iterated products lie in this subspace and coincide (as its elements).

A simple Cousin type argument shows that for commutative $\circ$ its associativity amounts to Jacobi identity for μ, i.e., chiral algebras are the same as commutative and associative OPE algebras.

¿From the point of view of 0.12 OPE is non other than the glueing isomorphism that reconstructs $A^\ell_{X\times X}$ from its restriction to U (equal $A^\ell \boxtimes A^\ell|_U$) and to the formal neighbourhood of the diagonal (equal to $\overset{\wedge}{\Delta_*} A^\ell$).

0.14. A tiny heuristic comment about the origin of chiral algebras in mathematical physics (see [BPZ]). Consider a quantum field theory on X. It unfolds within the algebra of local fields. If we play with an operator version of the theory then the local fields form a sheaf R of functions on X with values in (unbounded) operators acting on a space of states. This R is closed under the action of differential operators with C^∞ coefficients. If $\{x_i\} \in X$ is a finite set of pairwise distinct points on X, then for any local fields r_i the operators $r_i(x_i)$ commute and their product is well-defined.

Denote by $A^\ell \subset R$ the subsheaf of holomorphic local fields (passing to A^ℓ we begin to deal with "holomorphic quantum mechanics" rather than with $2D$ field theory), so A^ℓ is a left $\mathcal{D}$-module on X. Assume further the property that if $x, y \in X$, $x \neq y$, are close points then for $f, g \in A^\ell$ the product $f(x)g(y)$ may be written as Laurent power series $\sum\limits_{i \ll -\infty} a_i(x)(y-x)^i$, $a_i \in A^\ell$. This series is a commutative and associative OPE product on A^ℓ.

0.15. To see an example of a non-commutative chiral algebra let us describe in geometric terms the chiral enveloping algebra of a Kac-Moody Lie algebra.

Let G be an algebraic group. For $x \in X$ consider the set of pairs $(\mathcal{F}, \alpha)$, $\mathcal{F}$ is a G-bundle on X, α is a trivialization of $\mathcal{F}$ on $X \smallsetminus \{x\}$. This is the set of points of a formally smooth ind-scheme $\mathcal{GR}_x$; a choice of a parameter t_x at x identifies $\mathcal{GR}_x$ with $G(\mathbb{C}((t)))/G(\mathbb{C}[\![t]\!])$.

Our $\mathcal{GR}_x$ are fibers of an ind-scheme $\mathcal{GR}_X$ over X called the *affine Grassmannian*. This is a $\mathcal{D}_X$-ind-scheme: when x varies infinitesimally $X \smallsetminus \{x\}$, hence $\mathcal{GR}_x$, do not change.

The trivialized G-bundle $(\mathcal{F}, \alpha)$ on X defines the section $e: X \to \mathcal{GR}$ (which is the only horizontal section of $\mathcal{GR}$). Denote by A^ℓ_x the vector space of distributions on $\mathcal{GR}_x$ supported at $e(x)$, i.e., A^ℓ_x is the (topological) dual to the formal completion of the local ring $\mathcal{O}_{\mathcal{GR}_x, e(x)}$. When x varies A^ℓ_x become the fibers of a left $\mathcal{D}_X$-module A^ℓ_X.

Let us show that A is a chiral algebra in a natural way. We follow format 0.12. For any $n > 0$ we have a $\mathcal{D}$-ind-scheme $\mathcal{GR}_{X^n}$ over X^n with fibers $\mathcal{GR}_{(x_1,..,x_n)}$ equal to the space of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is a G-bundle on X, α is a trivialization of $\mathcal{F}$ on $X \smallsetminus \{x_1, ..x_n\}$. As in case $n = 1$ one has a canonical horizontal section e of $\mathcal{GR}_{X^n}$, and we set $A^\ell_{(x_1,..,x_n)}$ to be the vector space of distributions on $\mathcal{GR}_{(x_1,..,x_n)}$ supported at e. These are fibers of a left $\mathcal{D}_{X^n}$-module $A^\ell_{X^n}$. The factorization property for $A^\ell_{X^n}$ follows from the similar property of $\mathcal{GR}_{X^n}$ itself: there is a canonical identification $\mathcal{GR}_{(x_1,..x_n)} = \prod_{s \in S} \mathcal{GR}_s$ where $S = \{x_1, .., x_n\} \subset X$.

To get the chiral envelope of a Kac-Moody algebra of non-zero level one should twist A^ℓ by some canonical line bundle on $\mathcal{GR}$.

0.16. A vertex operator algebra from references in 0.0 is essentially a chiral algebra on the "coordinate disc" $\operatorname{Spec} \mathbb{C}[[t]]$ equipped with an action of the group ind-scheme $\operatorname{Aut} \mathbb{C}[[t]]$ of its symmetries. As usual, such object is the same as a "universal" chiral algebra (= a law that assigns to any curve a chiral algebra on it in a way compatible with étale localization). Similarly, "universal" Lie* algebras are the same as objects dubbed "conformal algebras" in the book [K].

In some versions of VOA axioms it is assumed that the action of the Lie algebra $\operatorname{Der} \mathbb{C}[[t]]$ comes from a stress-energy tensor which amounts to a morphism from Virasoro Lie* algebra to our chiral algebra.

0.17. Our work contains no new deep results. Its mere purpose was to understand the ingenious formal power series manipulations that haunt the books on vertex operator algebras.

A really challenging problem is to find out what is a chiral algebra on higher dimensional X (recall that coisson algebras live in any dimension). The definitions from 0.9, 0.10 formally work also for $\dim X > 1$, but to make them sensible one must plunge into derived categories. This brings the uneasiness which we had no courage to overcome.

0.18. The work consists of three chapters. In the first one we discuss some relevant abstract nonsense. The classical (coisson) part of the picture is treated in the second chapter. The third chapter deals with chiral algebras proper.

As usual, distilled axioms are pretty indigestible. We do suggest the reader to follow complicated alembics of Chapter One simultaneously with respective sections of more wholesome Chapters Two and Three.

0.19. Our sincere gratitude is due to Sasha Belavin and Pierre Deligne. Apart from an obvious influence of their ideas, it was in their apartments where the prime part of this work was done. The first author would also like ot thank IHES, MPI, and IAS for their hospitality, and his wife for her being able to endure the rewritings of the draft. We are grateful to Dottie Phares and Richard A. Lloyd for the careful typing of the first part of the first version of the manuscript.

0.20. A few words about notation commonly used below. "$x \in Y$" means that either x is an element of a set Y, or x is an object of a category Y, or x is a local section of a sheaf Y. For a category $\mathcal{M}$ we denote by $\mathcal{M}^\circ$ the dual category; $Sets$ denotes the category of all sets. For a smooth variety X we denote by Θ_X and $\mathcal{D}_X$ the tangent sheaf and the algebra of differential operators.

CHAPTER ONE: AXIOMATIC PATTERNS

Да поразит тя пуще грома
Ужасна, сильна аксиома.

А.А.Нахимов, "Поэт и Математик",
Харьковский Демокрит, 1816 +

§1 Pseudo-tensor categories

1.1.1. Denote by $\mathcal{S}$ the category of finite non-empty sets and surjective maps. For a morphism $\pi\colon J \twoheadrightarrow I$ in $\mathcal{S}$ and $i \in I$ set $J_i := \pi^{-1}(i) \subset J$; let $\cdot \in \mathcal{S}$ be the one element set.

Definition. [4] A *pseudo-tensor category* is a class of objects $\mathcal{M}$ together with the following data:

(a) For any $I \in \mathcal{S}$, an I-family of objects $L_i \in \mathcal{M}$, $i \in I$, and an object $M \in \mathcal{M}$ one has the set $P_I^{\mathcal{M}}(\{L_i\}, M) = P_I(\{L_i\}, M)$; its elements are called I-operations.

(b) For any map $\pi\colon J \twoheadrightarrow I$ in $\mathcal{S}$, families of objects $\{L_i\}_{i\in I}$, $\{K_j\}_{j\in J}$ and an object M one has the composition map

$$P_I(\{L_i\}, M) \times \prod_I P_{J_i}(\{K_j\}, L_i) \to P_J(\{K_j\}, M)$$

$(\varphi, (\psi_i)) \mapsto \varphi(\psi_i)$. Here in the notation $P_{J_i}(\{K_j\}, L_i)$ we assume that $j \in J_i$.

+ Let thee be smitten as by thunderbolt
By a dreadful, strong axiom.
A.A. Nakhimov, "A Poet and a Mathematician",
Kharkovski Democritus, 1816

[4] This kind of objects were considered sporadically in mathematical literature under different names. See e.g. [La] ("multicategories") or [Li] ("multilinear categories").

These data should satisfy the following properties.

(i) The composition is associative: if $H \twoheadrightarrow J$ is another surjective map, $\{F_h\}$ is an H-family of objects, $\chi_j \in P_{H_j}(\{F_h\}, K_j)$ then $\varphi(\psi_i(\chi_j)) = (\varphi(\psi_i))(\chi_j) \in P_H(\{F_h\}, M)$.

(ii) For any $M \in \mathcal{M}$ there is an element $id_M \in P_\cdot(\{M\}, M)$ such that for any $\varphi \in P_I(\{L_i\}, M)$ one has $id_M(\varphi) = \varphi(id_{L_i}) = \varphi$. $\square$

We will often use the notation $P_{J/I}(\{K_j\}, \{L_i\}) = P_\pi(\{K_j\}, \{L_i\})$ for $\prod\limits_I P_{J_i}(\{K_j\}, L_i)$ to write the composition map as $P_I \times P_{J/I} \to P_J$.

1.1.2. Let $\mathcal{M}$ be a pseudo-tensor category. If $I = \{1, \dots, n\}$ we write P_n for P_I. For $L, M \in \mathcal{M}$ set $\mathrm{Hom}(L, M) = P_1(\{L\}, M)$. The composition of morphisms comes from (b), therefore $\mathcal{M}$ becomes a usual category. The composition with morphisms makes P_I a functor $(\mathcal{M}^\circ)^I \times \mathcal{M} \to \mathit{Sets}$. Therefore, we may consider the pseudo-tensor categories as usual categories equipped with extra *pseudo-tensor structure* formed by a collection of functors P_I, $|I| \geq 2$, together with composition morphisms.

1.1.3. Assume that a family $\{L_i\}_{i \in I}$ has the property that the functor $M \mapsto P_I(\{L_i\}, M)$ is representable. We denote the corresponding object of $\mathcal{M}$ as $\underset{I}{\otimes} L_i$ and call it the *pseudo-tensor product* of L_i's. If the pseudo-tensor product exists for any family of objects, then we call our pseudo-tensor structure *representable*. In other words, a representable pseudo-tensor structure on a category $\mathcal{M}$ is a rule that assigns to any $I \in \mathcal{S}$ a functor $\underset{I}{\otimes}: \mathcal{M}^I \to \mathcal{M}$ and to any map $\pi: J \twoheadrightarrow I$ a natural compatibility morphism

$$\varepsilon_\pi: \underset{J}{\otimes} K_j \to \underset{I}{\otimes}(\underset{J_i}{\otimes} K_j)$$

The morphisms ε_π must behave naturally with respect to composition of the maps π, and $\underset{\cdot}{\otimes}$ is the identity functor.

If all the compatibility morphisms ε_π are isomorphisms, then we arrive to the usual notion of tensor (= symmetric monoidal) category.

1.1.4. A pseudo-tensor category $\mathcal{B}$ with single object is the same as an *operad* (see e.g. [May], [L], [GK], [Kap]). Namely, such $\mathcal{B}$ amounts to a collection of sets $\mathcal{B}_n = P_n^{\mathcal{B}}$, $n \geq 1$, equipped with the action of symmetric group Σ_n, together with appropriate "composition of operations" maps between them, and our axioms coincide here with those of an operad.

Example. The linear orders operad Σ: one has $\Sigma_I = \{\text{linear orders on } I\}$, the composition is defined by the lexicographic rule.

1.1.5. Let $\mathcal{M}$, $\mathcal{N}$ be pseudo-tensor categories. A *pseudo-tensor functor* $\tau\colon \mathcal{N} \to \mathcal{M}$ is a law that assigns to any object $G \in \mathcal{N}$ an object $\tau(G) \in \mathcal{M}$ and to $\{F_i\}_{i\in I}$, $G \in \mathcal{N}$ a map $\tau_I\colon P_I^{\mathcal{N}}(\{F_i\}, G) \to P_I^{\mathcal{M}}(\{\tau(F_i)\}, \tau(G))$ so that τ_I are compatible with composition and $\tau_\cdot(id_G) = id_{\tau(G)}$. One defines a morphism between pseudo-tensor functors in the obvious way, therefore the pseudo-tensor functors $\mathcal{N} \to \mathcal{M}$ form a category. The composition of pseudo-tensor functors is a pseudo-tensor functor in the obvious way.

Any pseudo-tensor functor τ defines a usual functor between the corresponding usual categories. We call τ a *pseudo-tensor extension* of this usual functor.

An *adjoint pair* is a pair of pseudo-tensor functors $\mathcal{M} \underset{\tau}{\overset{\nu}{\rightleftarrows}} \mathcal{N}$ together with identifications $a\colon P_I^{\mathcal{N}}(\{\nu(M_i)\}, G) \xrightarrow{\sim} P_I^{\mathcal{M}}(\{M_i\}, \tau(G))$ for $M_i \in \mathcal{M}$, $G \in \mathcal{N}$, compatible with composition of operations.[5] We say that ν is left adjoint to τ, and τ is right adjoint to ν. The corresponding usual functors are obviously adjoint.

Exercise. Show that for a given pseudo-tensor functor its left (resp. right) adjoint pseudo-tensor functor is determined uniquely up to a unique isomorphism (if it exists).

1.1.6 Examples. (i) A *pseudo-tensor subcategory* of a pseudo-tensor category $\mathcal{M}$ is given by a subclass of objects $\mathcal{T} \subset \mathcal{M}$ together with subsets $P_I^{\mathcal{T}}(\{L_i\}, M) \subset P_I^{\mathcal{M}}(\{L_i\}, M)$ for L_i, $M \in \mathcal{T}$ such that $P_I^{\mathcal{T}}$ are closed under the composition of operations, and $id_M \in P_\cdot^{\mathcal{T}}(\{M\}, M)$ for any $M \in \mathcal{T}$. Then $(\mathcal{T}, P^{\mathcal{T}})$ is a pseudo-tensor category, and $\mathcal{T} \hookrightarrow \mathcal{M}$ is a pseudo-tensor functor. If $P_I^{\mathcal{T}}(\{L_i\}, M) = P_I^{\mathcal{M}}(\{L_i\}, M)$ for any L_i, $M \in \mathcal{P}$, then we call $\mathcal{T}$ a *full* pseudo-tensor subcategory of $\mathcal{M}$.

Any subclass of objects of $\mathcal{M}$ may be considered as a full pseudo-tensor subcategory of $\mathcal{M}$. In particular, for any $M \in \mathcal{M}$ we have the full pseudo-tensor subcategory with single

[5] Precisely, we demand that for $\varphi \in P_I^{\mathcal{N}}(\{\nu(M_i)\}, G)$, $\psi_i \in P_{J_i}^{\mathcal{M}}(\{L_j\}, M_i)$ one has $a(\varphi(\nu(\psi_i))) = a(\varphi)(\psi_i)$ and for $\chi \in P_K^{\mathcal{N}}(\{G_k\}, H)$, $\varphi_k \in P_{I_k}^{\mathcal{N}}(\{\nu(M_i)\}, G_k)$ one has $a(\chi(\varphi_k)) = \tau(\chi)(a(\varphi_k))$.

object M. We denote by $P(M) = P^{\mathcal{M}}(M)$ the corresonding operad.

(ii) Assume that $\mathcal{M}$, $\mathcal{N}$ have representable pseudo-tensor structures. Then a pseudo-tensor functor $\mathcal{N} \to \mathcal{M}$ is the same as a usual functor $\tau: \mathcal{N} \to \mathcal{M}$ together with natural morphisms $\nu_I: \underset{I}{\otimes} \tau(N_i) \to \tau(\underset{I}{\otimes} N_i)$ for any finite families of objects $N_i \in \mathcal{N}$, such that ν_I commute with the compatibility morphisms ε_π (i.e., for any $\pi: J \twoheadrightarrow I$ and a J-family $K_j \in \mathcal{N}$ the diagram

$$\begin{tikzcd}
\underset{J}{\otimes}\tau(K_j) \arrow[r, "\nu_J"] \arrow[d, "\varepsilon_\pi^{\mathcal{M}}"] & \tau(\underset{J}{\otimes}K_j) \arrow[dd, "\tau(\varepsilon_\pi^{\mathcal{N}})"] \\
\underset{I}{\otimes}(\underset{J_i}{\otimes}\tau(K_j)) \arrow[d, "\underset{I}{\otimes}\nu_{J_i}"] & \\
\underset{I}{\otimes}\tau(\underset{J_i}{\otimes}K_j) \arrow[r, "\nu_I"] & \tau(\underset{I}{\otimes}(\underset{J_i}{\otimes}K_j))
\end{tikzcd}$$

commutes). In particular, any tensor functor between tensor categories is a pseudo-tensor functor.

(iii) If $\mathcal{B}$ is an operad then we call a pseudo-tensor functor $\mathcal{B} \to \mathcal{M}$ a $\mathcal{B}$ *algebra* in $\mathcal{M}$; we denote the category of $\mathcal{B}$ algebras in $\mathcal{M}$ as $\mathcal{B}(\mathcal{M})$. A $\mathcal{B}$ algebra in $\mathcal{M}$ amounts to an object $L \in \mathcal{M}$ together with a morphism of operads $\mathcal{B} \to P^{\mathcal{M}}(L)$.

For example, a Σ algebra in $\mathcal{M}$ is just a monoid, i.e., an object L equipped with an associative product $\cdot \in P_2(\{L, L\}, L) = P^{\mathcal{M}}(L)_2$. An algebra for a "trivial" one point operad is a commutative monoid.

(iv) For any family $\{\mathcal{M}_j\}$ of pseudo-tensor categories one defines their product $\Pi\mathcal{M}_j$ in the obvious way. A pseudo-tensor functor $\mathcal{N} \to \Pi\mathcal{M}_j$ is the same as a collection of pseudo-tensor functors $\mathcal{N} \to \mathcal{M}_j$.

(v) If $\mathcal{A}$ is a tensor category then the product functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a pseudo-tensor functor in the obvious way. Let $\mathcal{M}$ be a pseudo-tensor category. An $\mathcal{A}$-*Action on* $\mathcal{M}$ is a pseudo-tensor functor $\otimes: \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ together with an identification of two pseudo-tensor functors $\mathcal{A} \times \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, $(A_1 \otimes A_2) \otimes M \xrightarrow{\sim} A_1 \otimes (A_2 \otimes M)$, that satisfies the obvious compatibilities. In concrete terms, such Action is given by an Action of $\mathcal{A}$ on $\mathcal{M}$ as on a usual category together with natural maps $P_I^{\mathcal{M}}(\{M_i\}, N) \to P_I^{\mathcal{M}}(\{A_i \otimes M_i\}, (\otimes A_i) \otimes N)$ for $M_i, N \in \mathcal{M}$, $A_i \in \mathcal{A}$.

1.1.7. The above definitions have an obvious generalization when operations $P_I(\{L_i\}, M)$ are not sets but objects of a certain fixed tensor (or even pseudo-tensor!) category $\mathcal{A}$.[6] We call such thing a *pseudo-tensor $\mathcal{A}$-category.* The common example is $\mathcal{A} =$ the tensor category of k-modules, k is a commutative ring; this is the case of *pseudo-tensor k-categories.* Other examples (super setting, DG categories, etc.) are discussed in 1.1.16.

Explicitly, a pseudo-tensor k-category is a pseudo-tensor category $\mathcal{M}$ together with a k-module structure on the sets P_I such that the composition map is k-polylinear. For example, if $\mathcal{M}$ consists of a single object then we have a k-operad. A pseudo-tensor k-category $\mathcal{M}$ is *additive* if it is additive as a usual k-category, and $\mathcal{M}$ is *abelian* if it is abelian as a usual k-category and the polylinear functors P_I are left exact.

A pseudo-tensor functor $F: \mathcal{N} \to \mathcal{M}$ between the pseudo-tensor k-categories is *k-linear* if the maps F_I are k-linear. If $\mathcal{B}$ is a k-operad we get a notion of a $\mathcal{B}$ k-algebra in $\mathcal{M}$ (we usually call them simply $\mathcal{B}$ algebras).

For example, we have the favorite k-operads $\mathcal{A}ss$, $\mathcal{C}om$, $\mathcal{L}ie$, $\mathcal{P}ois$ which give rise, respectively, to associative, commutative and associative, Lie, and Poisson algebras. Note that $\mathcal{A}ss$, $\mathcal{C}om$ are k-envelopes of the corresponding set-theoretic operads: $\mathcal{A}ss = k[\Sigma]$, $\mathcal{C}om = k$. To avoid problems with skew-symmetry we always assume that $1/2 \in k$ when speaking about Lie or Poisson algebras.

1.1.8 Remark. Let $\mathcal{B}$ be any k-operad. For $I \in \mathcal{S}$ denote by $\mathcal{FB}(I)$ the free $\mathcal{B}$ algebra in the tensor category k mod of k-modules with generators e_i labeled by elements $i \in I$. Then the morphism of k-modules $\mathcal{B}_I \to \mathcal{FB}(I)$, $\varphi \mapsto \varphi(\underset{I}{\otimes} e_i)$, is injective, so one may consider $\mathcal{B}_I$ as the k-submodule of $\mathcal{FB}(I)$ generated by "multiplicity free monomials of degree I".

1.1.9. Let $\mathcal{N}_j$, $j \in J$, be a finite family of pseudo-tensor k-categories. Their *tensor product* $\otimes \mathcal{N}_j$ is a pseudo-tensor category defined as follows. Its objects are families of objects

[6] So the composition maps are morphisms $P_I \otimes (\otimes_I P_{J_i}) \to P_J$ in $\mathcal{A}$, etc.

$N_j \in \mathcal{N}_j$; we write the corresponding object of $\underset{J}{\otimes} \mathcal{N}_j$ as $\otimes N_j$. One has $P_I(\{\otimes N_{j_i}\}, \otimes K_j) := \underset{k\,J}{\otimes} P_I^{\mathcal{N}_j}(\{N_{j_i}\}, K_j)$; the composition maps are tensor products of those in $\mathcal{N}_j$. Note that for any pseudo-tensor k-category $\mathcal{M}$ a k-linear pseudo-tensor functor $\otimes\mathcal{N}_j \to \mathcal{M}$ is the same as a k-polylinear pseudo-tensor functor $\Pi\mathcal{N}_j \to \mathcal{M}$.

1.1.10. Let $\mathcal{M}$ be any pseudo-tensor k-category. Consider the pseudo-tensor k-category $\mathcal{M} \otimes \mathcal{L}ie$. It coincides with $\mathcal{M}$ as usual k-category and one has $P_I^{\mathcal{M}\otimes\mathcal{L}ie}(\{L_i\}, M) = P_I^{\mathcal{M}}(\{L_i\}, M) \otimes \mathcal{L}ie_I$. The following funny lemma will be of use.

Lemma. $\mathcal{L}ie(\mathcal{M} \otimes \mathcal{L}ie) = \mathcal{C}om(\mathcal{M})$.

Proof. We have to show that Lie algebras in $\mathcal{M}\otimes\mathcal{L}ie$ are the same as commutative algebras in $\mathcal{M}$. Let M be an object in $\mathcal{M}$. One has $P^{\mathcal{M}\otimes\mathcal{L}ie}(M)_2 = P^{\mathcal{M}}(M)_2 \otimes \mathcal{L}ie_2$, so a skew-symmetric binary $\mathcal{M} \otimes \mathcal{L}ie$ operation $[\quad]_M$ on M amounts to a symmetric binary $\mathcal{M}$ operation $\cdot_M$. Let us identify $\mathcal{L}ie_n$ with the space of Lie polynomials of variables $e_1, \dots, e_n$, so $[\quad]_M = \cdot_M \otimes [e_1, e_2]$. Since $\mathcal{L}ie_3$ is generated by $[e_1,[e_2,e_3]]$, $[e_3,[e_1,e_2]]$, $[e_2,[e_3,e_1]]$ with the only relation $[e_1,[e_2,e_3]] + [e_3,[e_1,e_2]] + [e_2,[e_3,e_1]] = 0$ we see that $[\quad]_M$ satisfies Jacobi relation iff $\cdot_M$ is associative. $\square$

1.1.11. Remark (will not be used in the sequel). V. Ginzburg pointed out that the above lemma is a particular case of a general duality statement for quadratic operads.

To formulate it we need some notions from [GK] §2. Assume for simplicity that k is a field. Let $\mathcal{A}$ be a quadratic k-operad (i.e., $\mathcal{A}$ is a quotient of the free k-operad generated by a finite dimensional vector space with Σ_2-action $\mathcal{A}_2$ modulo relations in $\mathcal{A}_3$). Its quadratic dual $\mathcal{A}^!$ is defined in [GK] 2.1.9 (this is a quadratic operad with $\mathcal{A}_2^!$ equal to the vector space dual to $\mathcal{A}_2$, the Σ_2-action is twisted by sgn). If $\mathcal{B}$ is another quadratic operad then we have a quadratic operad $\mathcal{A} \bullet \mathcal{B}$ defined in [GK] 2.2.5 (one has $(\mathcal{A} \bullet \mathcal{B})_2 = \mathcal{A}_2 \otimes \mathcal{B}_2$ with the Σ_2-action twisted by sgn) .

Notice that for any k-operad $\mathcal{P}$ the map $\mathrm{Hom}(\mathcal{A}, \mathcal{P}) \to \mathrm{Hom}(\mathcal{A}_2, \mathcal{P}_2)$ is injective. Let $\mathcal{P}^{quad}$ be a quadratic operad with $\mathcal{P}_2^{quad} = \mathcal{P}_2$ and relations coming from $\mathcal{P}_3$. Then the obvious morphism $\mathcal{P}^{quad} \to \mathcal{P}$ yields a bijection $\mathrm{Hom}(\mathcal{A}, \mathcal{P}^{quad}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{A}, \mathcal{P})$.

Lemma. For any quadratic operads $\mathcal{A}, \mathcal{B}$ and a pseudo-tensor k-category $\mathcal{M}$ one has

$$\mathcal{A}(\mathcal{M} \otimes \mathcal{B}) = (\mathcal{A} \bullet \mathcal{B}^!)(\mathcal{M}) = \mathcal{B}^!(\mathcal{M} \otimes \mathcal{A}^!). \tag{1.1.11.1}$$

Proof. Since $\bullet$ is symmetric it suffices to define the first identification. We may assume that $\mathcal{M}$ has single object. So we have a k-operad $\mathcal{P}$, and we want to define a canonical bijection $\mathrm{Hom}(\mathcal{A}, \mathcal{P} \otimes \mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{A} \bullet \mathcal{B}^!, \mathcal{P})$. It comes from the obvious identification $\mathrm{Hom}(\mathcal{A}_2, (\mathcal{P} \otimes \mathcal{B})_2) = \mathcal{A}_2^! \otimes \mathcal{P}_2 \otimes \mathcal{B}_2 = \mathrm{Hom}((\mathcal{A} \bullet \mathcal{B}^!)_2, \mathcal{P}_2)$. To check the relations we may replace $\mathcal{P}$ by $\mathcal{P}^{quad}$, $\mathcal{P} \otimes \mathcal{B}$ by $(\mathcal{P} \otimes \mathcal{B})^{quad} = \mathcal{P}^{quad} \circ \mathcal{B}$ (see [GK] 2.2.3), and we are done by [GK] 2.2.6(b). $\square$

If $\mathcal{A} = \mathcal{B} = \mathcal{C}om$ then $\mathcal{A}^! = \mathcal{B}^! == \mathcal{L}ie$ (see [GK] 2.1.11) and the above lemma becomes the lemma from 1.1.10.

1.1.12 Remark. Assume that our $\mathcal{M}$ is actually a tensor category with tensor product $\otimes$. Let us show that the (pseudo) tensor structure of $\mathcal{M}$ reconstructs uniquely from that of $\mathcal{M} \otimes \mathcal{L}ie$.

Note that the pseudo-tensor structure of $\mathcal{M} \otimes \mathcal{L}ie$ is representable; denote by $\otimes^{Lie}$ the corresponding tensor product. One has $\underset{I}{\otimes}^{Lie} = \mathcal{L}ie_I^* \otimes \underset{I}{\otimes}$. So one has a canonical identification $M_1 \otimes M_2 = M_1 \otimes^{Lie} M_2$, and the commutativity constraints differ by sign. It remains to recover the associativity constraint. For $M_1, M_2, M_3 \in \mathcal{M}$ consider the composition $\underset{\{1,2,3\}}{\otimes}^{Lie} M_i \to M_1 \otimes^{Lie} (M_2 \otimes^{Lie} M_3) \xrightarrow{\sim} M_1 \otimes (M_2 \otimes M_3)$; here the first arrow is compatibility morphism for $\otimes^{Lie}$ (see 1.1.3), and the second isomorphism comes from the ordering of our indices (e.g., $Lie_{\{2,3\}} \xrightarrow{\sim} k$, $[e_2, e_3] \mapsto 1$). Let $\underset{\{1,2,3\}}{\otimes}^{Lie} M_i \to M_1 \otimes (M_2 \otimes M_3) \oplus M_3 \otimes (M_1 \otimes M_2) \oplus M_2 \otimes (M_3 \otimes M_1)$ be the sum of these maps for cyclic transposition of indices; denote by K its cokernel. The projection maps each summand to K isomorphically, and the associativity constraint for $\otimes$ coincides with composition $M_1 \otimes (M_2 \otimes M_3) \xrightarrow{\sim} K \xrightarrow{\sim} M_3 \otimes (M_1 \otimes M_2) = (M_1 \otimes M_2) \otimes M_3$.

1.1.13. One may use operads to define pseudo-tensor structures on some categories of modules and their dual categories. Let us explain how to do this in the k-linear setting.

Let R be an associative k-algebra; denote by Rmod, mod$R$, the category of left (right) R-modules. Let $\mathcal{B}$ be a *strict R-operad*, i.e., a k-operad endowed with an isomorphism of

k-algebras $R \xrightarrow{\sim} \mathcal{B}_1$. So for any $I \in \mathcal{S}$ the k-module $\mathcal{B}_I$ is actually an $(R - R^{\otimes I})$-bimodule (here $\otimes$ is tensor power over k), and for $J \twoheadrightarrow I$ the composition map may be written as a morphism of $(R - R^{\otimes J})$-bimodules $\mathcal{B}_I \underset{R^{\otimes I}}{\otimes} \mathcal{B}_{J/I} \to \mathcal{B}_J$.

Such $\mathcal{B}$ defines the pseudo-tensor structures $P^* = P^{*\mathcal{B}}$ and $P^! = P^{!\mathcal{B}}$ on mod$R$ and Rmod°, respectively. Namely, for $L_i, M \in \text{mod}R$ set

$$P_I^*(\{L_i\}, M) := \text{Hom}_{\text{mod}R^{\otimes I}}\left(\otimes L_i, M \underset{R}{\otimes} \mathcal{B}_I\right)$$

The composition of $*$ operations comes from the composition law in $\mathcal{B}$. Similarly, for C_i, $D \in R\text{mod}$ set

$$P_I^!(D, \{C_i\}) := \text{Hom}_{R\text{mod}}\left(D, \mathcal{B}_I \underset{R^{\otimes I}}{\otimes} (\underset{I}{\otimes} C_i)\right)$$

here we write the operations in Rmod° as operations on R-modules with reversed order of arguments, so that $P^!_\cdot(D, \{C\}) = \text{Hom}_{R\text{mod}}(D, C)$ $(= \text{Hom}_{R\text{mod}^\circ}(C, D))$. The composition of ! operations comes in an obvious manner from the composition law in $\mathcal{B}$.

Remarks. (i) $\mathcal{B} = P^*(R) = P^!(R)$; here we consider R as right and left R-module, respectively.

(ii) The $*$ and ! structures are abelian (see 1.1.7) iff $\mathcal{B}_I$ are flat as, respectively, R- and $R^{\otimes I}$-modules.

1.1.14 Example. Assume that R has a cocommutative Hopf algebra structure. It defines a strict R-operad $\mathcal{B}^R$. Namely, $\mathcal{B}_I^R := R^{\otimes I}$, and the composition map $\mathcal{B}_I^R \otimes \mathcal{B}_{J/I}^R \to \mathcal{B}_J$ is $(\underset{I}{\otimes} r_i) \otimes \underset{I}{\otimes}(\underset{J_i}{\otimes} r_j^{(i)}) \mapsto \underset{I}{\otimes}(\Delta^{(J_i)}(r_i) \cdot (\underset{J_i}{\otimes} r_j^{(i)})) \in \underset{I}{\otimes} R^{\otimes J_i} = R^{\otimes J}$. Here $\Delta^{(J_i)}\colon R \to R^{\otimes J_i}$ is the J_i-multiple coproduct. Note that the ! pseudo-tensor structure coincides with the standard tensor category structure on Rmod.

1.1.15. The above notions easily sheafify. Namely, let X be a topological space (or a Grothendieck topology) and $\mathcal{M}$ a sheaf of categories on X (we assume that both morphisms and objects of $\mathcal{M}$ have local nature, i.e., satisfy the glueing property). A *pseudo-tensor structure* on $\mathcal{M}$ assigns to any open U a pseudo-tensor structure on $\mathcal{M}(U)$ and to any $j\colon V \to U$ a pseudo-tensor extension of the pull-back functor $j^*\colon \mathcal{M}(U) \to \mathcal{M}(V)$. We assume that this extension is compatible with composition of j's and the pseudo-tensor operations have local nature (i.e., for any L_i, $M \in \mathcal{M}(U)$ the prehseaf $\underline{P}_I(\{L_i\}, M)$ on U, $\underline{P}_I(\{L_i\}, M)(V) = P_I(\{L_i|_V\}, M|_V)$, is actually a sheaf). We call such $\mathcal{M}$ a *sheaf of pseudo-tensor categories*. All the notions we discuss in Chapter 1 have straightforward sheaf versions. We will tacitly assume them for the next chapters.

1.1.16. Sometimes we need to consider superobjects in a given category. The following "super" conventions are in force for this work:

Let $\mathcal{M}$ be a k-category. Its "super"($:= \mathbb{Z}/2$-graded) objects form a k-category $\mathcal{M}^s$. This is also a "super" category: for $M, N \in \mathcal{M}^s$ the k-module of all $\mathcal{M}$-morphisms $\mathrm{Hom}(M, N)$ has an obvious $\mathbb{Z}/2$-grading, and morphisms in $\mathcal{M}^s$ is the even part in $\mathrm{Hom}(M, N)$. If $\mathcal{M}$ is a tensor category then $\mathcal{M}^s$ is automatically a tensor (super) category according to the Koszul rule of signs.

A $\mathbb{Z}$-graded superobject M in $\mathcal{M}$ is a superobject in $\mathcal{M}$ equipped with an extra $\mathbb{Z}$-grading. For such M we set $M[1] = M \otimes k[1]$ where $k[1]$ is an odd copy of k placed in degree -1, i.e., $M[1]$ is the $\mathbb{Z}$-graded super object obtained from M by simultaneous shift of $\mathbb{Z}$- and $\mathbb{Z}/2$-gradings by 1. The shift is an automorphism of the category of $\mathbb{Z}$-graded superobjects, so we have $M[a]$ for every $a \in \mathbb{Z}$. We consider M^a (*not* $M^a[-a]$!) as a subobject of M. The category $\mathcal{M}^{s\mathbb{Z}}$ of A $\mathbb{Z}$-graded superobjects has an obvious "$\mathbb{Z}$-graded superstructure": for $M, N \in \mathcal{M}^{s\mathbb{Z}}$ we have a $\mathbb{Z}$-graded super k-module $\mathrm{Hom}(M, N)$ with components $\mathrm{Hom}(M, N)^i = \Pi\mathrm{Hom}(M^a, N^{a+i})$. If $\mathcal{M}$ is a tensor category then such is $\mathcal{M}^{s\mathbb{Z}}$ (the signs come from the "super" grading, so the forgetting of the $\mathbb{Z}$-grading is a tensor functor).

A DG superobject in $\mathcal{M}$ is a $\mathbb{Z}$-graded superobject equipped with an action of the commutative graded Lie superalgebra $k[-1]$, which is the same as an odd endomorphism d of degree 1 with square 0. DG super objects form a DG super category $\mathcal{M}^{sDG}$ in the obvious manner. This is a tensor DG super category if $\mathcal{M}$ is a tensor category.

If $\mathcal{M}$ is a pseudo-tensor k-category then $\mathcal{M}^s$ is automatically a pseudo-tensor super k-category (see 1.1.7). Namely, for $L_i, M \in \mathcal{M}^s$ the k-module $P_I(\{L_i\}, M)$ carries the obvious

$(\mathbb{Z}/2)^I \times \mathbb{Z}/2$-grading, hence it is a super k-module with respect to the total $\mathbb{Z}/2$-grading. The composition takes into account the Koszul sign rule. Thus for any $L_i, M \in \mathcal{M}^s$, and free super k-modules V_i, U of finite rank one has a canonical identification of super k-modules $P_I(\{L_i \otimes V_i\}, M \otimes U) = P_I(\{L_i\}, M) \otimes \mathrm{Hom}(\bigotimes_I V_i, U)$.[7] One can also view $\mathcal{M}^s$ as a plain pseudo-tensor k-category considering only even operations.

Similarly, $\mathcal{M}^{s\mathbb{Z}}$ is pseudo-tensor $\mathbb{Z}$-graded super k-category, and $\mathcal{M}^{sDG}$ is a pseudo-tensor DG super k-category. We have a canonical identification $P_I(\{L_i[a_i]\}, M[b]) = P(\{L_i\}, M)[b - \Sigma a_i]$.

§2 Complements

1.2.1 Inner Hom. Let $\mathcal{M}$ be a pseudo-tensor category, I a finite set, $\{L_i\}_{i \in I}$, M objects in $\mathcal{M}$. Assume we have an object $X \in \mathcal{M}$ and an element $\varepsilon \in P_{\widetilde{I}}(\{L_i, X\}, M)$ where $\widetilde{I} := I \sqcup \cdot$. Then for any $J \in \mathcal{S}$ and a J-family of objects $\{A_j\}$ we get a natural map

(1.2.1.1) $$\kappa = \kappa_J(\varepsilon) \colon P_J(\{A_j\}, X) \to P_{J \sqcup I}(\{A_j, L_i\}, M)$$

$\kappa(\varphi) := \varepsilon(\varphi, id_{L_i})$. Here "natural" means that for any $H \twoheadrightarrow J$, an H-family of objects $\{B_h\}$ and $\chi_j \in P_{H_j}(\{B_h\}, A_j)$ one has $\kappa(\varphi(\chi_j)) = \kappa(\varphi)(\chi_j, id_{L_i})$. A pair (X, ε) with the property that all the maps κ are isomorphisms is uniquely defined (if its exists); we denote it $(\mathcal{P}_I(\{L_i\}, M), \varepsilon_{I,\{L_i\},M})$ and call *inner P object*. Note that $\mathcal{P}_\phi(M) = M$, and $\mathcal{P}_1(L, M) = \mathcal{H}om(L, M)$ is the "inner Hom" object.

If well-defined, inner P objects depend on their arguments in functorial manner: for a surjection $E \twoheadrightarrow I$, $\varphi \in \mathrm{Hom}(M, N)$, and $\psi_i \in P_{E_i}(\{K_e\}, L_i)$ (as usually $E_i := \pi^{-1}(i)$) there is an obvious morphism $\mathcal{P}_I(\{L_i\}, M) \to \mathcal{P}_E(\{K_e\}, N)$ which is compatible with composition of φ's and ψ's.

1.2.2. Here is an "inner" version of the functoriality property. Let $\pi \colon E \to I$ be *any* map of finite sets, $\{K_e\}$ an E-family of objects. Assume that the objects $\mathcal{P}_I(\{L_i\}, M)$, $\mathcal{P}_E(\{K_e\}, M)$, $\mathcal{P}_{E_i}(\{K_e\}, L_i)$ exist. Then one has a canonical "composition" element

(1.2.2.1) $$c \in P_{\widetilde{I}}(\{\mathcal{P}_I(\{L_i\}, M),\ \mathcal{P}_{E_i}(\{K_e\}, L_i)\},\ \mathcal{P}_E(\{K_e\}, M))$$

[7] and this identification is compatible with composition of operations.

defined as the image of the composition $\varepsilon_{I,\{L_i\},M}\left(id_{\mathcal{P}_I(\{L_i\},M)},\varepsilon_{E_i,\{K_e\},L_i}\right)$ by the canonical isomorphism κ^{-1}. Note that if $E=\emptyset$ then $c=\varepsilon$.

One may check that the composition is associative in the obvious sense.

Example. Consider $L\in\mathcal{M}$ such that $\mathcal{E}nd\, L=\mathcal{H}om(L,L)$ exists. Then the composition $c\in P_2(\{\mathcal{E}nd\, L,\mathcal{E}nd\, L\},\mathcal{E}nd\, L)$ is associative, hence it makes $\mathcal{E}nd\, L$ a monoid. In the k-linear setting $\mathcal{E}nd\, L$ is an associative k-algebra.

1.2.3 Example. Assume we are in situation 1.1.13, so $\mathcal{M}=\mathrm{mod}R$ and the pseudo-tensor structure is given by the functors P^* defined by the operad $\mathcal{B}$. For L_i, M as above set $X:=\mathrm{Hom}_{R^{\otimes I}}(\underset{I}{\otimes}L_i, M\underset{R}{\otimes}\mathcal{B}_{\tilde{I}})$. This is a right R-module in the obvious way, and we have the canonical operation $\varepsilon\in P^*_{\tilde{I}}(\{L_i,X\},M)$. The maps $\kappa_J(\varepsilon)$ are isomorphisms for $|J|=1$, i.e., X necessarily coincides with $\mathcal{P}_I(\{L_i\},M)$ if the latter object exists. This happens in the following situations:

(i) Assume that the composition maps $\mathcal{B}_I\underset{R^{\otimes I}}{\otimes}\mathcal{B}_{J/I}\to\mathcal{B}_J$ are isomorphisms. Then $X=\mathcal{P}_I(\{L_i\},M)$ if L_i are projective modules of finite rank.

(ii) Assume in addition that all $\mathcal{B}_I$ are flat as R-modules. Then $X=\mathcal{P}_I(\{L_i\},M)$ if L_i are finitely presented R-modules.

1.2.4 Augmented pseudo-tensor structures. Denote by $\mathcal{S}^\wedge$ the category of arbitrary finite sets and maps between them, so $\mathcal{S}$ is a subcategory of $\mathcal{S}^\wedge$. If we replace in 1.1.1 $\mathcal{S}$ by $\mathcal{S}^\wedge$ then we get the definition of *augmented pseudo-tensor category.* An augmented pseudo-tensor category with single object is called an *augmented operad.*

Examples. (i) A tensor category with a unit object $\mathbf{1}$ is an augmented tensor category: set $P_I(\{L_i\},M)=\mathrm{Hom}(\underset{I}{\otimes}L_i,M)$. We assume that $\underset{\phi}{\otimes}=\mathbf{1}$.

(ii) The linear orders operad Σ (see 1.1.4) extends in the obvious way to an augmented operad $\Sigma^\wedge$ with $\Sigma^\wedge_\emptyset=\{\text{point}\}$. We also have the "trivial" augmented one-point operad, and their k-linear envelopes $\mathcal{A}ss^\wedge=k[\Sigma^\wedge]$ and $\mathcal{C}om^\wedge=k$.

1.2.5. One may consider an augmented pseudo-tensor category as a usual pseudo-tensor category equipped with some extra structure; let us describe it explicitly. We need the following definition.

Let $\mathcal{M}$ be any pseudo-tensor category. Assume we have a functor

(1.2.5.1) $$h\colon \mathcal{M} \to Sets,$$

and for any finite set I of order ≥ 2 and $i_0 \in I$ we have natural maps

(1.2.5.2) $$h_{I,i_0}\colon P_I(\{L_i\}, M) \times h(L_{i_0}) \to P_{I \smallsetminus \{i_0\}}(\{L_i\}, M)$$

(here, as usually, $\{L_i\}$ is an I-family of objects, $M \in \mathcal{M}$). We call these data an *augmentation functor* if the following compatibilities hold:

(i) Let J be a finite set of order ≥ 2, $\pi\colon J \twoheadrightarrow I$ a surjective map, $j_0 \in J$. Then for $\varphi \in P_I(\{L_i\}, M)$, $\psi_i \in P_{J_i}(\{K_j\}, L_i)$, $a \in h(K_{j_0})$ one has

$$h_{J,j_0}(\varphi(\psi_i), a) = \varphi(\psi_{i'}, h_{J_{i_0},j_0}(\psi_{i_0}, a))$$

(here $i_0 = \pi(j_0), i' \in I \smallsetminus \{i_0\}$) if $|J_{i_0}| \geq 2$, and

$$h_{J,j_0}(\varphi(\psi_i), a) = h_{I,i_0}(\varphi, \psi_{i_0}(a))(\psi_{i'})$$

if $|J_{i_0}| = 1$.

(ii) Assume that $|I| \geq 2$; let $i_0, i_1 \in I$ be two distinct elements. Then for $\varphi \in P_I(\{L_i\}, M)$, $a_{i_0} \in h(L_{i_0})$, $a_{i_1} \in h(L_{i_1})$, one has

$$h_{I \smallsetminus \{i_0\},i_1}(h_{I,i_0}(\varphi, a_0), a_1) = h_{I \smallsetminus \{i_1\},i_0}(h_{I,i_1}(\varphi, a_1), a_0) \in P_{I \smallsetminus \{i_0,i_1\}}(\{L_i\}, M)$$

if $|I| > 2$, and

$$h_{I,i_0}(\varphi, a_0)a_1 = h_{I,i_1}(\varphi, a_1)a_0 \in h(M)$$

if $|I| = 2$.

Remark. Any subfunctor of an augmentation functor is itself an augmentation functor. The functor $h(M) \equiv \emptyset$ is an augmentation functor.

Definition. We say that our augmentation functor is *non-degenerate* if all the maps $P_I(\{L_i\}, M) \to \mathcal{M}aps(h(L_{i_0}), P_{I \smallsetminus \{i_0\}}(\{L_i\}, M))$ coming from 1.2.5.2 are injective.

Now let $\mathcal{M}^{\wedge}$ be an augmented pseudo-tensor category, $\mathcal{M}$ the corresponding usual pseudo-tensor category. Then $\mathcal{M}$ carries an augmentation functor $h = P_{\emptyset}$, h_{I,i_0} is the composition map for the embedding $I \smallsetminus \{i_0\} \hookrightarrow I$.

1.2.6 Lemma. The above construction yields $1-1$ correspondence between augmented pseudo-tensor categories and usual pseudo-tensor categories equipped with an augmentation functor. □

1.2.7. Note that any augmentation functor $h\colon \mathcal{M} \to \mathit{Sets}$ is automatically a pseudo-tensor functor (we consider Sets as a tensor category with $\otimes = \Pi$), i.e., we have natural maps

$$h_I\colon P_I(\{L_i\}, M) \to \mathrm{Hom}(\underset{I}{\otimes} h(L_i), h(M))$$

To see this we may assume, by the above lemma, that h came from an augmented pseudo-tensor structure. Then h_I is the composition map for $\emptyset \to I$. Equivalently, for $\varphi \in P_n(\{L_i\}, M)$, $a_i \in h(L_i)$ one has (here $h_n := h_{\{1,\dots,n\}}$)

$$h_n(\varphi)(a_1 \otimes \cdots \otimes a_n) = h_{\{n-1,n\},n-1}(\dots(h_{\{1,\dots,n\},1}(\varphi, a_1)\dots), a_{n-1})a_n$$

In the situation 1.2.1 there is a canonical map

$$\bar{h}\colon h(\mathcal{P}_I(\{L_i\}, M)) \to P_I(\{L_i\}, M), \quad \lambda \longmapsto h_{\tilde{I},\cdot}(\varepsilon_{I,\{L_i\},M}, \lambda). \tag{1.2.7.1}$$

For c as in 1.2.2 the $\bar{h}$ arrows intertwine $h_{\tilde{I}}(c)$ and the usual composition map for P operations.

We say that $\mathcal{P}_I(\{L_i\}, M)$ is an *inner P object in the augmented sense* if (1.2.7.1) is an isomorphism. This amounts to the fact that the canonical morphism (1.2.2.1) (for $X = \mathcal{P}_I(\{L_i\}, M)$) is an isomorphism for $J = \emptyset$ as well.

1.2.8. One defines an *augmented pseudo-tensor functor* $\tau^\wedge\colon \mathcal{N}^\wedge \to \mathcal{M}^\wedge$ between augmented pseudo-tensor categories in the obvious way. Such $\tau^\wedge$ amounts to a usual pseudo-tensor functor $\tau\colon \mathcal{N} \to \mathcal{M}$ together with a morphism of augmentation functors $h^\wedge_{\mathcal{N}} \to h^\wedge_{\mathcal{M}}\tau$ that are compatible in the obvious manner. For example, given an augmented operad $\mathcal{B}^\wedge$ we have the notion of $\mathcal{B}^\wedge$ algebra in any augmented pseudo-tensor category $\mathcal{M}^\wedge$.

Examples. $\Sigma^\wedge$ algebras in $\mathcal{M}^\wedge$ are semi-groups; algebras for the trivial augmented operad are commutative semi-groups. In the k-linear setting $\mathcal{A}ss^\wedge$ algebras are unital associative algebras, $\mathcal{C}om^{\wedge}$ algebras are unital commutative algebras. If A is a unital associative algebra then the image $1_A \in h(A)$ of $1 \in k = \mathcal{A}ss_{\emptyset}$ is called *the unit element* in $h(A)$.

Remark. 1_A is the unit of the associative algebra $h(A)$. So a unital associative algebra in $\mathcal{M}^{\wedge}$ is the same as an associative algebra in $\mathcal{M}$ such that the associative algebra $h(A)$ is unital and the left and right product with the unit element of $h(A)$ is identity morphism of A.

1.2.9. Following the pattern of 1.1.13, one may use augmented operads to define augmented pseudo-tensor structures on some categories of modules and their duals.

Let R be an associative k-algebra and $\mathcal{B}^{\wedge}$ an augmented strict R-operad, i.e., $\mathcal{B}$ is an augmented k-operad equipped with an isomoprhism $R \underset{\sim}{\longrightarrow} \mathcal{B}_1^{\wedge}$. Such $\mathcal{B}^{\wedge}$ is the same as a usual strict R-operad $\mathcal{B}$ together with a left R-module $h = \mathcal{B}_0$ and a system of $R^{\otimes I \smallsetminus \{i\}}$-linear morphisms $h_{I,i_0}: \mathcal{B}_I \underset{R}{\otimes} \mathcal{B}_0 \to \mathcal{B}_{I \smallsetminus \{i_0\}}$ (here $|I| \geq 2$, $i_0 \in I$, and $\mathcal{B}_I$ is considered as a right R-module via the i_0-th action) that satisfy obvious compatibilities.

According to 1.1.13 $\mathcal{B}$ defines the $*$ and $!$ pseudo-tensor structures on mod$R$ and Rmod°, respectively. Now h defines the augmentation functors h^*, $h^!$ for these structures. Namely, for $M \in \text{mod}R$, $D \in R\text{mod}$ set

$$h^*(M) := M \underset{R}{\otimes} h, \qquad h^!(D) := \mathrm{Hom}_{R\mathrm{mod}}(D, h)$$

The morphisms $h^*_{I,i_0}: P_I^{*\mathcal{B}}(\{L_i\}, M) \otimes h^*(L_{i_0}) \to P^{*\mathcal{B}}_{I \smallsetminus \{i_0\}}(\{L_i\}, M)$, $h^!_{I,i_0}: P_I^{!\mathcal{B}}(D, \{C_i\}) \otimes h^!(C_{i_0}) \to P^!_{I \smallsetminus \{i_0\}}(D, \{C_i\})$ come from h_{I,i_0} maps in the obvious manner. Compatibilities 1.2.5 (i), (ii) are clear. Note that $\mathcal{B}^{\wedge} = P^{*\wedge}(R) = P^{!\wedge}(R)$.

1.2.10 Example. Assume that R has a cocommutative Hopf algebra structure with counit $\varepsilon: R \to k$. It defines an augmented strict R-operad $\mathcal{B}^{R\,\wedge}$. Namely, we already defined $\mathcal{B}^R$ in 1.1.14. Set $h = k$ (the R-module structure is defined by ε), and let h_{I,i_0} be the obvious isomorphisms $R^{\otimes I} \underset{R}{\otimes} k = R^{\otimes I \smallsetminus \{i_0\}} \otimes (R \underset{R}{\otimes} k) \underset{\sim}{\longrightarrow} R^{\otimes I \smallsetminus \{i_0\}}$. We leave it to the reader to check the compatibilities. Note that k is a unit object for the standard tensor structure on Rmod, and ! augmented structure is the corresponding augmented pseudo-tensor structure (see example 1.2.4).

1.2.11 Modules over "operadic" algebras. Let $\mathcal{C}$ be a pseudo-tensor category which has two objects $\bullet$ and $*$, and $P_I^{\mathcal{C}}(\{L_i\}, M) = \emptyset$ whenever either ≥ 2 of L_i are equal to $*$, or exatly one of the objects $\{L_i, M\}$ is equal to $*$. In other words, the only (possibly) non-empty sets of operations are the operad $\mathcal{B} := P^{\mathcal{C}}(\bullet)$ (see 1.1.6 (iii)) and $\mathcal{C}_I := P_{\tilde{I}}^{\mathcal{C}}(\{\bullet_i, *\}, *)$, I is a finite set. We will say that $\mathcal{C}$ is a $\mathcal{B}$ *module operad.* We leave it to the reader to spell explicitly the definition of $\mathcal{B}$ module operad in terms of the composition data for $\mathcal{C}$- and $\mathcal{B}$-operations.

The above definition may be rewritten in the k-linear setting in the obvious way.

1.2.12 Examples. (i) Let $\mathcal{B}$ be an operad. Consider the pseudo-tensor category $\mathcal{B}'$ with two objects $\{\bullet, *\}$ which is equivalent to $\mathcal{B}$. Let $\mathcal{B}^m \subset \mathcal{B}'$ be the pseudo-tensor category with the same set of objects as $\mathcal{B}'$ and $P_I^{\mathcal{B}^m}(\{L_i\}, M) = P_I^{\mathcal{B}'}(\{L_i\}, M) = \mathcal{B}_I$ if either $L_i = M = \bullet$, or $M = \bullet$ and exactly one of L_i equals to $\bullet$; otherwise $P_I^{\mathcal{B}^m} = \emptyset$ (or 0 in the k-linear version). Then $\mathcal{B}^m$ is a $\mathcal{B}$ module operad, so $\mathcal{C}_I = \mathcal{B}_{I \sqcup \{*\}}$.

(ii) Consider the operad Σ (see 1.1.4). The Σ module operad Σ^m contains two Σ module suboperads Σ^m_ℓ, $\Sigma^m_r \subset \Sigma^m$. Namely, $\Sigma^m_I := \Sigma_{I \sqcup \{*\}}$ = set of linear orders on $I \sqcup \{*\}$, and $\Sigma^m_{\ell I}$ (resp. Σ^m_{rI}) is the subset of orders such that $*$ is the maximal (resp. the minimal) element.

(iii) Let $\mathcal{C}_i$ be $\mathcal{B}$ module operads, $i \in I$. Denote by $\prod\limits_I{}^0 \mathcal{C}_i \subset \prod\limits_I \mathcal{C}_i$ the full pseudo-tensor subcategory with objects $\bullet := (\bullet_i)$, $* := (*_i)$. This is a $\prod\limits_I \mathcal{B}_i$ module operad.

1.2.13. Let $\mathcal{C}$ be a $\mathcal{B}$ module operad, $\mathcal{M}$ a pseudo-tensor category. Let $\tau: \mathcal{C} \to \mathcal{M}$ be a pseudo-tensor functor. Then $L := \tau(\bullet)$ is a $\mathcal{B}$ algebra in $\mathcal{M}$. We call $M := \tau(*)$ an *L-module with respect to* $\mathcal{C}$, or simply *L-module* (if $\mathcal{C}$ is fixed). Explicitly, a structure of L-module on an object $M \in \mathcal{M}$ is given by a system of compatible "action" maps $\mathcal{C}_I \to P_{\tilde{I}}(\{L_i, M\}, M)$.

Denote by $\mathcal{C}(\mathcal{M})$ the category of pseudo-tensor functors $\tau: \mathcal{C} \to \mathcal{M}$ (see 1.1.5); its objects are pairs (L, M), L is a $\mathcal{B}$ algebra, M is an L-module. If L is a fixed $\mathcal{B}$ algebra, then we denote by $\mathcal{M}(L) = \mathcal{M}(L, \mathcal{C}) \subset \mathcal{C}(\mathcal{M})$ the subcategory of L-modules: its objects are pairs (L, M), the morphisms are identity on L.

Example. Let L be an object of a pseudo-tensor category $\mathcal{M}$ such that $\mathcal{E}nd L \in \mathcal{M}$ exists.

Then L is an $\mathcal{E}nd\,L$-module with respect to Σ_ℓ^m.

The above definition may be rewritten in the k-linear setting in the obvious way. For a k-operad $\mathcal{B}$, a $\mathcal{B}$ module k-operad $\mathcal{C}$, and a $\mathcal{B}$ algebra L in a pseudo-tensor k-category $\mathcal{M}$ the category Lmod is a k-category.

1.2.14 Lemma. If $\mathcal{M}$ is an abelian pseudo-tensor category, then $\mathcal{M}(L)$ is an abelian k-category. □

1.2.15 Examples. (i) $\mathcal{B} = \mathcal{L}ie$, $\mathcal{C} = \mathcal{B}^m$. Then $\mathcal{M}(L)$ is the usual category of modules over the Lie algebra L: its object is $M \in \mathcal{M}$ equipped with a pairing $\cdot \in P_2(\{L, M\}, M)$ that satisfies the condition $\ell_1 \cdot (\ell_2 \cdot m) - \ell_2 \cdot (\ell_1 \cdot m) = [\ell_1, \ell_2] \cdot m$.

(ii) $\mathcal{B} = \mathcal{A}ss = k[\Sigma]$. Let L be an associative algebra. If $\mathcal{C} = \mathcal{B}^m = k[\Sigma^m]$ then $\mathcal{M}(L, \mathcal{C})$ is the category of L-bimodules. If $\mathcal{C} = k[\Sigma_\ell^m]$ (resp. $\mathcal{C} = k[\Sigma_r^m]$) then $\mathcal{M}(L, \mathcal{C})$ is the category of left (resp. right) L-modules.

1.2.16. Let $\mathcal{B}$ be an operad, $\mathcal{C}$ a $\mathcal{B}$ module operad, and L a $\mathcal{B}$ algebra in the tensor category $Sets$ (with $\otimes = \Pi$, see 1.2.7). We have the obvious "forgetting of L-module structure" functor o: $Sets(L, \mathcal{C}) \to Sets$. Denote by $\mathcal{C}(L)$ the monoid of endormorphisms of this functor.

Here is an explicit description of $\mathcal{C}(L)$. It is generated by elements $((\ell_i), \varphi)$, I is a finite set, $\ell_i \in L$ is an I-family of elements of L, $\varphi \in \mathcal{C}_I$. The relations for $\mathcal{C}(L)$ come from surjective maps $\pi: \widetilde{J} \to \widetilde{I}$, $\pi(\cdot) = \cdot$, and collections $\psi_i \in \mathcal{B}_{J_i}$, $i \in I$. The relation corresponding to $\pi, (\psi_i)$ is

$$((\ell_j), \varphi_1(\varphi_2, \psi_i)) = ((\psi_i(\ell_j)_{j \in J_i}, \varphi_1) \cdot ((\ell_j)_{j \in J_\cdot}, \varphi_2)$$

Here (ℓ_j) is a J-family of elements of L, $\varphi_1 \in \mathcal{C}_I$, $\varphi_2 \in \mathcal{C}_{J_\cdot}$, where $J_\cdot := J \cap \pi^{-1}(\cdot)$. The element $1 = (\emptyset, id.)$ is the unit in $\mathcal{C}(L)$. Denote by $\mathcal{C}(L)$mod the category of sets equipped with $\mathcal{C}(L)$-action. The functor o lifts to the functor $\tilde{o}$: $Sets_{L\mathcal{C}} \to \mathcal{C}(L)$mod which is an equivalence of categories. More generally, one may define a $\mathcal{C}$ action of L on an object of any category, and this is the same as a $\mathcal{C}(L)$-action.

The above construction has its k-linear version. Namely, for a k-operad $\mathcal{B}$, a $\mathcal{B}$ module operad $\mathcal{C}$, and a $\mathcal{B}$ algebra L in $k$mod (or in the category of sheaves of k-modules on some space) we get an associative k-algebra with unit $\mathcal{C}(L)$ (or a sheaf of k-algebras) such that the category of L-modules coincides with the one of $\mathcal{C}(L)$-modules. We call $\mathcal{C}(L)$ the *enveloping algebra* of L. In situation 1.2.15 (i) it coincides with the usual enveloping algebra.

1.2.17. Let now $\mathcal{M}^{\wedge} = (\mathcal{M}, h)$ be an augmented pseudo-tensor category, $\mathcal{B}$, $\mathcal{C}$ as above. Let L be a $\mathcal{B}$ algebra in $\mathcal{M}$, and $M \in \mathcal{M}(L, \mathcal{C})$ an L-module. Then $h(L)$ is a $\mathcal{B}$ algebra in *Sets* (since h is a pseudo-tensor functor), and M (considered as an object of the usual category $\mathcal{M}$) carries a canonical $\mathcal{C}(h(L))$-action. Explicitly, a generator $((\ell_i), \varphi) \in \mathcal{C}(h(L))$ acts as $h_I(a_I(\varphi))(\ell_i) \in \operatorname{End} M$. Therefore we have a canonical "identity" functor

$$\mathcal{M}(L, \mathcal{C}) \to \mathcal{C}(h(L))\text{-modules in } \mathcal{M}. \tag{1.2.17.1}$$

1.2.18 Example. Let $\mathcal{M}$ be a pseudo-tensor k-category, L a Lie algebra in $\mathcal{M}$. Let $\{M_i\}_{i\in I}$, N be a finite collection of L-modules. An operation $\varphi \in P_I(\{M_i\}, N)$ is *L-compatible* if $\cdot_N(id_L, \varphi) = \sum\limits_{i_0 \in I} \varphi(\cdot_{M_{i_0}}, id_{M_i})_{i\neq i_0} \in P_{\tilde{I}}(\{L, M_i\}, N)$.[8] Let $P_{LI}(\{M_i\}, N) \subset P_I(\{M_i\}, N)$ be the k-submodule of L-compatible operations. It is easy to see that composition of L-linear operations is L-linear. Therefore P_{LI} is a pseudo-tensor structure on $\mathcal{M}(L)$. If $\mathcal{M}$ was an abelian pseudo-tensor category, then $\mathcal{M}(L)$ is also an abelian pseudo-tensor category (see 1.2.14). If h is a k-linear augmentation functor on $\mathcal{M}$ then $h(L)$ is a Lie k-algebra, and for every $M \in \mathcal{M}(L)$ $h(M)$ is an $h(L)$-module. The functor h^L of $h(L)$-invariants, $M \mapsto h^L(M) := h(M)^{h(L)}$, is an augmentation functor on $\mathcal{M}(L)$.

If $\{F_j\}_{j\in J}$, G are $h(L)$-modules in $\mathcal{M}$ then we define the subspace $P_J^{h(L)}(\{F_j\}, G) \subset P_J(\{F_j\}, G)$ of $h(L)$-linear operations in the obvious way. Therefore $h(L)$-modules form a pseudo-tensor k-category, and it is easy to see that the "identity" functor

$$\mathcal{M}(L) \to h(L)\text{-modules in } \mathcal{M} \tag{1.2.18.1}$$

[8] Here $\cdot_N \in P_2(\{L, N\}, N)$, $\cdot_{M_i} \in P_2(\{L, M_i\}, M_i)$ are the L-actions.

is a pseudo-tensor functor.

For a k-operad $\mathcal{B}$ a $\mathcal{B}$ algebra in $\mathcal{M}(L)$ is the same as a $\mathcal{B}$ algebra A in $\mathcal{M}$ on which L acts by derivations. According to (1.2.18.1) such A carries a canonical $h(L)$-action by derivations.

§3 Compound tensor categories

We recommend the reader to consider our basic example of compound tensor category (see §2 of ch.2) before, or while, reading this section. He may also restrict himself to compound tensor categories (see 1.3.12 - 1.3.15) instead of general compound pseudo-tensor categories (1.3.4 - 1.3.7), but in this case the duality 1.3.8(i) will be lost.

1.3.1. We have to fix some notation. For a finite non-empty set I denote by $Q(I)$ the set of all equivalence relations on I. We identify an element of $Q(I)$ with the corresponding quotient map $\pi_S\colon I \to S$ and denote it by (S, π_S) or simply S by abuse of notation; as usual for $s \in S$ we set $I_s := \pi_S^{-1}(s)$. The set $Q(I)$ is ordered: we write $S_1 \leq S_2$ iff S_1 is a quotient of S_2. It is a lattice: for any $S, T \in Q(I)$ there exist $\inf\{S,T\} \in Q(I)$ and $\sup\{S,T\} \in Q(I)$. For any $S \in Q(I)$ we have the obvious identifications $Q(I)_{\leq S} := \{S' \in Q(I)\colon S' \leq S\} = Q(S)$, $Q(I)_{\geq S} := \{S'' \in Q(I)\colon S'' \geq S\} = \prod_S Q(I_s)$.

We say that $S, T \in Q(I)$ are *transversal* if $|S| + |T| = |I| + |\inf\{S,T\}|$; S and T are called *complementary* if they are transveral and $|\inf\{S,T\}| = 1$.

1.3.2 Proposition. (i) Let V be a non-zero vector space. Let us assign to $S \in Q(I)$ the corresponding "diagonal" subspace $V^S \subset V^I$. Then S,T are transversal iff so are the subspaces V^S, $V^T \subset V^I$ (i.e., $V^S + V^T = V^I$), and S, T are complementary iff, in addition, $V^S \cap V^T = V^{\cdot} \subset V^I$.

(ii) Consider the graph G with vertices labeled by $S \sqcup T$ and edges labeled by I, such that an edge $i \in I$ connects $\pi_S(i)$ and $\pi_T(i)$. Then S and T are complementary (resp. transversal) iff G is a tree (resp. forest).

Proof. (i) Note that $|S| = \dim V^S/\dim V$, and for any S, T one has $V^{\inf\{S,T\}} = V^S \cap V^T$, $S = T$ iff $V^S = V^T$. Therefore $|S| + |T| \leq |\sup\{S,T\}| + |\inf\{S,T\}|$, and the equality holds iff $V^{\sup\{S,T\}} = V^S + V^T$. Hence $|S| + |T| \leq |I| + |\inf\{S,T\}|$, and equality holds iff $V^I = V^S + V^T$.

(ii) Note that $|\inf\{S,T\}|$ coincides with the number of components of G, and compute the Euler characteristic of G. □

1.3.3 Corollary. a. Assume that $S, T \in Q(I)$ are transversal. Then

(i) for any $\alpha \in \inf\{S,T\}$ the equivalence relations induced by S, T on I_α are complementary;

(ii) for any $J \in Q(I)$ such that $J \geq S$ the pair $T, J \in Q(I)$ is transversal as well as $\inf\{T,J\}$, $S \in Q(J)$. The map $Q(I)_{\geq S} \to Q(T)$, $J \mapsto \inf\{T,J\}$ is injective.

b. Assume we have $S, P \in Q(I)$ such that $\inf\{S,P\} = \cdot$. Then $P = \inf\limits_{T \in \mathcal{T}} \{T\}$ where $\mathcal{T}$ is the set of all $T \in Q(I)$ such that T, S are complementary, and $T \geq P$. □

1.3.4. Let $\mathcal{M}$ be a category. For any functors $F\colon \mathcal{M} \to \mathit{Sets}$, $G\colon \mathcal{M}^\circ \to \mathit{Sets}$ we denote by $\langle F, G\rangle$ the quotient of the disjoint union of sets $F(X) \times G(X)$, $X \in \mathcal{M}$, modulo the relations $(\varphi(f), g) = (f, {}^t\varphi(g))$, where $f \in F(X)$, $g \in G(Y)$, $\varphi \in \mathrm{Hom}(X,Y)$ (and X, Y are objects in $\mathcal{M}$). We tacitly assume that $\langle F, G\rangle$ is a well-defined set; to assure this it suffices to know that isomorphism classes of objects in $\mathcal{M}$ form a set, or that either F or G is a representable functor (if, say, $F = \mathrm{Hom}(X_F, \cdot)$ then $\langle F, G\rangle = G(X_F)$).

If we are in the k-linear situation, i.e., $\mathcal{M}$ is a k-category and F, G are k-linear functors with values in k-modules, then we denote by $\langle F, G\rangle_k$ the k-module which is the quotient of $\bigoplus\limits_{X \in \mathcal{M}} F(X) \underset{k}{\otimes} G(X)$ modulo the relations $\varphi(f) \otimes g = f \otimes {}^t\varphi(g)$. We have an obvious map of sets $\langle F, G\rangle \to \langle F, G\rangle_k$ which is bijection if either F or G is representable.

1.3.5. Now assume that both categories $\mathcal{M}$ and $\mathcal{M}^\circ$ carry the pseudo-tensor structures given by functors P_I^* and $P_I^!$, respectively. It will be convenient to write the variables for $P_I^!$ in reverse order, i.e., as $P_I^!(M, \{L_i\})$ so that $P_\cdot^!(M, L) = \mathrm{Hom}_{\mathcal{M}}(M, L)$.

For any $I, J \in \mathcal{S}$ and I- and J-families of objects $\{L_i\}, \{M_j\}$, respectively, set

$$P_{I,J}^{*!}(\{L_i\}, \{M_j\}) := \left\langle P_I^*(\{L_i\}, \cdot), P_J^!(\cdot, \{M_j\})\right\rangle$$

Our $P_{I,J}^{*!}$ is a functor; moreover, the $*$ and $!$ operations act on $P^{*!}$. Precisely, for any surjective maps $H \twoheadrightarrow J$, $G \twoheadrightarrow I$ and families of objects $\{N_h\}$, $\{K_g\}$ we have the composition map

$$P^!_{H/J}(\{M_j\},\{N_h\}) \times P^{*!}_{I,J}(\{L_i\},\{M_j\}) \times P^*_{G/I}(\{K_g\},\{L_i\}) \to P^{*!}_{G,H}(\{K_g\},\{N_h\})$$

Note that $P^{*!}_{I,\cdot} = P^*_I$, $P^{*!}_{\cdot,J} = P^!_J$.

1.3.6. Assume that for any $I \in \mathcal{S}$ and a complementary pair $S, T \in Q(I)$ we have a natural map

$$\langle\ \ \rangle^I_{S,T} : P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \to P^{*!}_{T,S}(\{N_t\},\{M_s\})$$

$((\varphi_s),(\psi_t)) \mapsto \langle(\varphi_s),(\psi_t)\rangle^I_{S,T}$. Here $\{L_i\}$, $\{M_s\}$, $\{N_t\}$ are arbitrary families of objects parametrized by I, S, T, respectively, and "natural" means that $\langle\ \ \rangle^I_{S,T}$ is functorial with respect to M_s and N_t variables. Note that $\langle\ \ \rangle^I_{S,T}$ extends uniquely to the family of maps

$$\langle\ \ \rangle^I_{S,T} : \prod_S P^{*!}_{I_s,H_s}(\{L_i\},\{B_h\}) \times \prod_T P^{*!}_{G_t,I_t}(\{A_g\},\{L_i\}) \to P^{*!}_{G,H}(\{A_g\},\{B_h\})$$

labeled by surjecitons $H \twoheadrightarrow S$, $G \twoheadrightarrow T$, in a way compatible with composition of $*$ and $!$ operations (acting on A_g and B_h, respectively).

1.3.7 Definition. The data $(P^*, P^!, \langle\ \ \rangle)$ form a *compound pseudo-tensor structure* on $\mathcal{M}$ if the following compatibility axioms hold (here I, S, T, $\{L_i\}$, $\{M_s\}$, $\{N_t\}$ are as above):

(i) Take $H \in Q(I)$ such that $H \geq S$; set $V := \inf\{H, T\}$. Then for any H-family of objects $\{K_h\}$ the following diagram commutes:

$$\begin{tikzcd}
P^*_{H/S}(\{K_h\},\{M_s\}) \times P^*_{I/H}(\{L_i\},\{K_h\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \arrow[r] \arrow[d, "\prod_V \langle\ \ \rangle^{I_v}_{H_v,T_v}"] & P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \arrow[d, "\langle\ \ \rangle^I_{S,T}"] \\
P^*_{H/S}(\{K_h\},\{M_s\}) \times \prod_V P^{*!}_{T_v,H_v}(\{N_t\},\{K_h\}) \arrow[r, "\langle\ \ \rangle^H_{S,V}"] & P^{*!}_{T,S}(\{N_t\},\{M_s\})
\end{tikzcd}$$

Here the upper horizontal arrow is the product of composition maps for $I_s \twoheadrightarrow H_s$.

(ii) Take $G \in Q(I)$ such that $G \geq T$; set $U := \inf\{S, G\}$. Then for any G-family of objects $\{R_g\}$ the following diagram commutes:

$$\begin{CD}
P^*_{I/S}(\{L_i\},\{M_s\})\times P^!_{I/G}(\{R_g\},\{L_i\})\times P^!_{G/T}(\{N_t\}\{R_g\}) @>>> P^*_{I/S}(\{L_i\},\{M_s\})\times P^!_{I/T}(\{N_t\},\{L_i\})\\
@V{\prod\limits_U <\ \ >^{I_u}_{S_u,G_u}}VV @VV{<\ \ >^I_{S,T}}V\\
\prod\limits_U P^{*!}_{G_u,S_u}(\{R_g\},\{M_s\})\times P^!_{G/T}(\{N_t\},\{R_g\}) @>{<\ \ >^G_{U,T}}>> P^{*!}_{T,S}(\{N_t\},\{M_s\})
\end{CD}$$

The upper arrow is the product of composition maps for $I_t \twoheadrightarrow G_t$.

(iii) If $S = I$, $T = .$, $M_i = L_i$, then $\langle (id_{L_i}), \psi\rangle^I_{I,.} = \psi$ for any $\psi \in P^!_I(N, \{L_i\})$

(iv) If $S = .$, $T = I$, $N_i = L_i$, then $\langle \varphi, (id_{L_i})\rangle^I_{.,I} = \varphi$ for any $\varphi \in P^*_I(\{L_i\}, M)$ $\square$

We call a category $\mathcal{M}$ equipped with a compound pseudo-tensor structure a *compound pseudo-tensor category*. We usually denote it by $\mathcal{M}^{*!}$, while $\mathcal{M}^*$ and $\mathcal{M}^{o!}$ denote $\mathcal{M}$ and $\mathcal{M}^o$ considered as plain pseudo-tensor categories (with the operations P^* and $P^!$, respectively).

1.3.8 Remarks. (i) The axioms (i) – (ii) and (iii) – (iv) are mutually dual, therefore a compound pseudo-tensor structure on $\mathcal{M}$ amounts to that on $\mathcal{M}^\circ$.

(ii) Any *tensor* structure on $\mathcal{M}$ defines a compound pseudo-tensor structure: set $P^*_I(\{L_i\}, M) = \mathrm{Hom}(\underset{I}{\otimes} L_i, M)$, $P^!_I(N, \{L_i\}) = \mathrm{Hom}(N, \underset{I}{\otimes} L_i)$, and $\langle(\varphi_s), (\psi)_t\rangle^I_{S,T} = (\underset{S}{\otimes}\varphi_s)\circ(\underset{T}{\otimes}\psi_t)$ (note that $P^{*!}_{T,S}(\{N_t\}, \{M_s\}) = \mathrm{Hom}(\underset{T}{\otimes} N_t, \underset{S}{\otimes} M_s)$).

1.3.9. A *compound pseudo-tensor functor* $\tau^{*!}\colon \mathcal{N}^{*!} \to \mathcal{M}^{*!}$ between the compound pseudo-tensor categories is a pair of pseudo-tensor functors $\tau^*\colon \mathcal{N}^* \to \mathcal{M}^*$, $\tau^!\colon \mathcal{N}^{o!} \to \mathcal{M}^{o!}$ that give rise to the same usual functor $\tau\colon \mathcal{N} \to \mathcal{M}$, such that the diagrams

$$\begin{CD}
P^*_{I/S}(\{L_i\},\{M_s\})\times P^!_{I/T}(\{N_t\},\{L_i\}) @>{<\ \ >^I_{S,T}}>> P^{*!}_{T,S}(\{N_t\},\{M_s\})\\
@V{\tau^*\times\tau^!}VV @VV{\tau^{*!}}V\\
P^*_{I/S}(\{\tau L_i\},\{\tau M_s\})\times P^!_{I/T}(\{\tau N_t\},\{\tau L_i\}) @>{<\ \ >^I_{S,T}}>> P^{*!}_{T,S}(\{\tau N_t\},\{\tau M_s\})
\end{CD}$$

commute. Here the arrow $\tau^{*!}$ is defined in the obvious way using τ^* and $\tau^!$.

1.3.10. Let us explain what an augmented compound pseudo-tensor category is. Let $\mathcal{M}^{*!}$ be a compound tensor category and h an augmentation functor on $\mathcal{M}^*$. We say that h is *compatible* with the compound structure, or h is a $*$-*augmentation functor* on $\mathcal{M}^{*!}$ if it satisfies the following condition. Let $S, T \in Q(I)$ be a complementary pair and $i_0 \in I$

an element such that $\pi_T^{-1}(t_0) = \{i_0\}$, $|\pi_S^{-1}(s_0)| \geq 2$ where $s_0 = \pi_S(i_0)$, $t_0 = \pi_T(i_0)$. Set $I^0 := I \smallsetminus \{i_0\}$, $T^0 := T \smallsetminus \{t_0\}$. We may consider S and T^0 as a complementary pair in $Q(I^0)$. Our condition is commutativity of the diagram

$$\begin{array}{ccc} P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \times h(N_{t_0}) & \longrightarrow & P^*_{I^0/S}(\{L_i\}, M_s) \times P^!_{I^0/T^0}(N_t,\{L_i\}) \\ \big\downarrow{<\ \ >^I_{S,T}} & & \big\downarrow{<\ \ >^{I^0}_{S,T^0}} \\ P^{*!}_{T,S}(\{N_t\},\{M_s\}) \times h(N_{t_0}) & \longrightarrow & P^{*!}_{T^0,S}(\{N_t\},\{M_s\}) \end{array}$$

Here the upper horizontal arrow sends $((\varphi_s),(\psi_t),a)$ to $((\varphi_s, h_{I^0_{s_0,i_0}}(\varphi_{s_0},\psi_{t_0}(a))),\psi_t)$, and the lower one is the obvious extension of h_{T,t_0} to $P^{*!}_{T,\cdot}$.

One spells the compatibility property of a !-augmentation functor in a similar (dual) way. We call a compound pseudo-tensor category endowed with both $*$- and !-augmentation functors an *augmented* compound pseudo-tensor category.

The above definitions have obvious k-linear versions (replace product of sets by tensor product of k-modules, etc.).

1.3.11 Duality. The augmentation functors may be used to relate the pseudo-tensor categories $\mathcal{M}^*$ and $\mathcal{M}^{o!}$. For example, consider a !-augmentation functor $h^!$ on $\mathcal{M}^{*!}$. Assume that $h^!$ is representable by an object $\mathbf{1}$, so $h^!(M) = \mathrm{Hom}_{\mathcal{M}}(M,\mathbf{1})$, and for any $M \in \mathcal{M}$ the object $M' := \mathcal{H}om_{\mathcal{M}^*}(M,\mathbf{1})$ exists (here $\mathcal{H}om_{\mathcal{M}^*}$ means inner Hom with respect to $*$ structure, see 1.2.1). Then *the functor $\mathcal{M}^\circ \to \mathcal{M}$, $M \mapsto M'$, admits a canonical pseudo-tensor extension*

$$\mathcal{M}^{\circ !} \to \mathcal{M}^* \tag{1.3.11.1}$$

We need to define the maps $P^!_I(M,\{L_i\}) \to P^*_I(\{L'_i\}, M')$. Let I^1, I^2 be two copies of I. Set $J := I^1 \sqcup I^2$, $\widetilde{I} = I \sqcup \cdot$, and let $\pi_I\colon J \twoheadrightarrow I$, $\pi_{\widetilde{I}}\colon J \twoheadrightarrow \widetilde{I}$ be the projections defined by formulas $\pi_I(i^1) = \pi_I(i^2) = i$, $\pi_{\widetilde{I}}(i^1) = i$, $\pi_{\widetilde{I}}(i^2) = \cdot$; here $i \in I$, and $i^1 \in I^1$, $i^2 \in I^2$ are the copies of i. Then $I, \widetilde{I} \in Q(J)$ is a complementary pair (e.g., by 1.3.2 (ii)). Consider a J-family of objects $\{A_j\}$, $A_{i^1} = L'_i$, $A_{i^2} = L_i$, an I-family of objects $\{\mathbf{1}_i\}$ (copies of $\mathbf{1}$), and an $\widetilde{I}$-family of objects which are L_i for $i \in I \subset \widetilde{I}$ and the $\cdot$ one is M. We define the desired map is the composition

$$P_I^!(M,\{L_i\}) \to P^{*!}_{\tilde I, I}(\{L'_i, M\},\{\mathbf{1}_i\}) \to P^*_{\tilde I}(\{L'_i, M\},\mathbf{1}) \xrightarrow{\sim} P^*_I(\{L'_i\}, M')$$

Here the first arrow sends $\psi \in P^!_I(M,\{L_i\})$ to $\big\langle (\kappa_i),(\psi, id_{L'_i})\big\rangle^J_{I,\tilde I}$ (here $\kappa_i \in P^*_2(\{A_{i^1}, A_{i^2}\}, \mathbf{1}_i) = P^*_2(\{L'_i, L_i\},\mathbf{1})$ is the canonical pairing), the second arrow is $\lambda \mapsto h^!_I(\lambda)(id_{\mathbf{1}}^{\otimes I})$ (for $h^!_I$ see 1.2.7; we extend it to $P^{*!}$ in the obvious manner), and the last arrow is the canonical isomorphism (see 1.2.1). We leave it to the reader to check that the above construction is compatible with composition of operations.

1.3.12. Assume that the ! pseudo-tensor structure is representable (see 1.1.3). Then $P^*_{I,J}(\{L_i\},\{M_j\}) = P^*_I(\{L_i\}, \underset{J}{\otimes}{}^! M_j)$, and we may rewrite the $\langle\ \ \rangle^I_{S,T}$ data as a collection of canonical morphisms (*the compound tensor product maps*)

$$\otimes^I_{S,T}\colon P^*_{I/S}(\{L_i\},\{M_s\}) = \prod_S P^*_{I_s}(\{L_i\}, M_s) \to P^*_T(\{\underset{I_t}{\otimes}{}^! L_i\}, \underset{S}{\otimes}{}^! M_s), \tag{1.3.12.1}$$

$\otimes^I_{S,T} := \langle \cdot, (id_{\underset{I_t}{\otimes}{}^! L_i})\rangle^I_{S,T}$. The axioms (i), (ii) of 1.3.7 may be rewritten as commutativity of the following diagrams (here $I, S, T, \dots$ have the same meaning as in 1.3.2):

(i)
$$\begin{array}{ccc} P^*_{H/S}(\{K_h\},\{M_s\}) \times P^*_{I/H}(\{L_i\},\{K_h\}) & \longrightarrow & P^*_{I/S}(\{L_i\},\{M_s\}) \\ \big\downarrow {\scriptstyle \otimes^H_{S,V} \times \prod_V \otimes^{I_v}_{H_v,T_v}} & & \big\downarrow {\scriptstyle \otimes^I_{S,T}} \\ P^*_V(\{\underset{H_v}{\otimes}{}^! K_h\}, \underset{S}{\otimes}{}^! M_s) \times P^*_{T/V}(\{\underset{I_t}{\otimes}{}^! L_i\}, \underset{H_v}{\otimes}{}^! K_h) & \longrightarrow & P^*_T(\{\underset{I_t}{\otimes}{}^! L_i\}, \underset{S}{\otimes}{}^! M_s) \end{array}$$

(the horizontal arrows are composition maps)

(ii)
$$\begin{array}{ccc} P^*_{I/S}(\{L_i\},\{M_s\}) & \xrightarrow{\otimes^I_{S,T}} & P^*_T(\{\underset{I_t}{\otimes}{}^! L_i\}, \underset{S}{\otimes}{}^! M_s) \\ \big\downarrow {\scriptstyle \prod_U \otimes^{I_u}_{S_u,G_u}} & & \big\downarrow \\ P^*_{G/U}(\{\underset{I_g}{\otimes}{}^! L_i\}, \{\underset{S_u}{\otimes}{}^! M_s\}) & \longrightarrow & P^*_T(\{\underset{G_t}{\otimes}{}^!(\underset{I_g}{\otimes}{}^! L_i)\}, \underset{S}{\otimes}{}^! M_s) \end{array}$$

Here the right vertical arrow is the $*$ composition map with compatibility morphisms $\underset{G_t}{\otimes}{}^!(\underset{I_g}{\otimes}{}^! L_i) \to \underset{I_t}{\otimes}{}^! L_i$; the lower horizontal arrow is the composition of $\otimes^G_{U,T}$ and the map of P^*_T's arising from $\underset{U}{\otimes}{}^!(\underset{S_u}{\otimes}{}^! M_s) \to \underset{S}{\otimes}{}^! M_s$.

1.3.13. The simplest non-trivial $\otimes^I_{S,T}$ operations are those with $|S| = 2$ (the binary compound tensor products). They look as follows. If $S = \{1,2\}$ then the projection $I \twoheadrightarrow S$ amounts to a non-trivial decomposition $I = I_1 \sqcup I_2$. The complementarity of T, S means that the only non-trivial T-equivalence class I_t has two elements $i_1 \in I_1$, $i_2 \in I_2$, i.e., T coincides with $I_1 \vee I_2 =$ the union of I_1, I_2 with i_1, i_2 identified. We call the map

$$\underset{i_1,i_2}{\otimes} = \otimes^{I_1,I_2}_{i_1,i_2} := \otimes^I_{S,T}: P^*_{I_1}(\{L_i\}, M_1) \otimes P^*_{I_2}(\{L_i\}, M_2) \\ \to P^*_{I_1\vee I_2}(\{L_i, L_{i_1} \otimes^! L_{i_2}\}, M_1 \otimes^! M_2),$$

$\varphi_1, \varphi_2 \mapsto \varphi_1 \underset{i_1,i_2}{\otimes} \varphi_2$, the *binary tensor product at* i_1, i_2 (here the objects L_i under the $P^*_{I_1\vee I_2}$ sign are those for $i \in I \smallsetminus \{i_1, i_2\}$).

1.3.14. Assume that $\mathcal{M}^{\circ !}$ is actually a tensor category; we call such $\mathcal{M}^{*!}$ a *compound tensor category*. One may consider $\otimes^!$ as a tensor product on $\mathcal{M}$ itself; denote this tensor category as $\mathcal{M}^!$ (so $\mathcal{M}^!$ is the tensor category dual to $\mathcal{M}^{o!}$).

If we are in the k-linear situation, then we call $\mathcal{M}^{*!}$ *abelian* if $\mathcal{M}$ is an abelian k-category, P^*_I are left exact functors, and $\otimes^!$ is right exact.

A *compound tensor functor* $\tau^{*!}: \mathcal{N}^{*!} \to \mathcal{M}^{*!}$ between the compound tensor categories is a compound pseudo-tensor functor such that all the canonical morphisms $\nu_I: \tau(\underset{I}{\otimes}^! N_i) \to \underset{I}{\otimes} \tau(N_i)$ from 1.1.6 (ii) are isomorphisms, i.e., $\tau^!: \mathcal{N}^! \to \mathcal{M}^!$ is a tensor functor. Such $\tau^{*!}$ amounts to a pair $(\tau^*, \tau^!)$ where $\tau^*: \mathcal{N}^* \to \mathcal{M}^*$, $\tau^!: \mathcal{N}^! \to \mathcal{M}^!$ are, respectively, the pseudo-tensor and tensor extensions of the same functor $\tau: \mathcal{N} \to \mathcal{M}$ which commute with $\otimes^I_{S,T}$ maps.

Remark. If $\mathcal{M}$ is a compound tensor k-categrory then the category of superobjects in $\mathcal{M}$ is a compound tensor category in the obvious way, as well as the categories of graded super objects and DG superobjects (see 1.1.16 for the terminology).

1.3.15. If $\mathcal{M}^{*!}$ is a compound tensor category, then the right vertical arrow in the diagram 1.3.12 (ii) is an isomorphism. Therefore all the morphisms $\otimes^I_{S,T}$ are completely determined by binary ones (use this diagram to compute $\otimes^I_{S,T}$ inductively).

The binary tensor products are functorial, commutative and associative. Let us spell these properties explicitly. Functoriality means that the binary products are compatible with composition of $*$ operations in the obvious sense, and $\otimes^{I_1,I_2}_{i_1,i_2}$ coincides with the usual ! tensor product of morphisms if $I_1=\{i_1\}$, $I_2=\{i_2\}$. Commutativity means that $\otimes^{I_1,I_2}_{i_1,i_2}=\otimes^{I_2,I_1}_{i_2,i_1}$. To spell the associativity property assume that we have three sets I_1,I_2,I_3 and the elements $i_1\in I_1$, $i_2^1,i_2^3\in I_2$, $i_3\in I_3$; denote by $I_1\vee I_2\vee I_3$ the disjoint union of I_j's modulo the relation $i_1=i_2^1$, $i_2^3=i_3$. Take $\varphi_j\in P^*_{I_j}(\{L_i\},M_j)$. The associativity says that

$$(\varphi_1 \underset{i_1,i_2^1}{\otimes} \varphi_2) \underset{i_2^3,i_3}{\otimes} \varphi_3 = \varphi_1 \underset{i_1,i_2^1}{\otimes} (\varphi_2 \underset{i_2^3,i_3}{\otimes} \varphi_3) \\ \in P^*_{I_1\,\vee\, I_2\,\vee\, I_3}(\{L_i, L_{i_1}\otimes^! L_{i_2^1}, L_{i_2^3}\otimes^! L_{i_3}\}, M_1\otimes^! M_2\otimes^! M_3)$$

if $i_2^1\neq i_2^3$, and

$$(\varphi_1 \underset{i_1,i_2}{\otimes} \varphi_2) \underset{i_2,i_3}{\otimes} \varphi_3 = \varphi_1 \underset{i_1,i_2}{\otimes} (\varphi_2 \underset{i_2,i_3}{\otimes} \varphi_3) = (\varphi_1 \underset{i_1,i_3}{\otimes} \varphi_3) \underset{i_1,i_2}{\otimes} \varphi_2 \\ \in P^*_{I_1\vee I_2\vee I_3}(\{L_i, L_{i_1}\otimes^! L_{i_2}\otimes^! L_{i_3}\}, M_1\otimes^! M_2\otimes^! M_3)$$

if $i_2^1=i_2^3=i_2$.

It is easy to check that for a category $\mathcal{M}$ equipped with a tensor structure $\otimes^!$ and a pseudo-tensor structure P^* the data of functorial, commutative and associative binary tensor products amount to the whole data of compound tensor products, i.e., to a compound tensor structure.

Remark.[9] The obvious Action of the tensor category $\mathcal{M}^!$ on $\mathcal{M}$ extends canonically to an $\mathcal{M}^!$-Action on $\mathcal{M}^*$ defined by the maps $P^*_I(\{M_i\},N)\to P^*_I(\{A_i\otimes^! M_i\},(\otimes A_i)\otimes^! N)$, $\varphi\mapsto\varphi\otimes\{id_{A_i}\}$.

1.3.16. Let $\mathcal{M}^{*!}$ be a compound tensor category such that $\mathcal{M}^!$ has unit object $\mathbf{1}$ (as a tensor category). Then $h^!(M):=\operatorname{Hom}_{\mathcal{M}}(M,\mathbf{1})$ is a !-augmentation functor on $\mathcal{M}^{o!}$. We say that $\mathbf{1}$ is a *unit object* in $\mathcal{M}^{*!}$ if $h^!$ is compatible with the compound structure (see 1.3.10). Explicitly, this means that for any I-family of objects $\{L_i\}$ and an object M the binary tensor product at $i_0\in I$

$$\operatorname{End}\mathbf{1}\times P^*_I(\{L_i\},M)\to P^*_I(\{L_i,\mathbf{1}\otimes^! L_{i_0}\}_{i\neq i_0},\mathbf{1}\otimes^! M)=P^*_I(\{L_i\},M)$$

[9] We use terminology from 1.1.6(v).

sends $id_{\mathbf{1}} \times \varphi$ to φ for any $\varphi \in P_I^*(\{L_i\}, M)$.

Recall that in any tensor category $F = \operatorname{End} \mathbf{1}$ is an abelian semigroup (a commutative k-algebra in the k-linear situation) that acts canonically on any object M (the action $F \to \operatorname{End} M$ sends $f \in F$ to $f \otimes^! id_M \in \operatorname{End}(\mathbf{1} \otimes^! M) = \operatorname{End} M$). The above compatibility implies that the F-action on $P_I^*(\{L_i\}, M)$ through M coincides with the action through any L_i. If we are in the k-linear situation this means that P_I^* are canonically F-modules and $\mathcal{M}^{*!}$ is actually a compound tensor F-category.

We will call a compound tensor category $\mathcal{M}^{*!}$ which has a unit and is endowed with a $*$-augmentation functor h simply an *augmented compound tensor category.* We leave the definition of an *augmented compound tensor functor* between such categories to the reader.

1.3.17. Assume we are in the k-linear situation. We say that our unit object $\mathbf{1}$ is a *strong unit object* in $\mathcal{M}^{*!}$ if for any $I \in \mathcal{S}$, and $L_i, M \in \mathcal{M}$ one has $P_{\tilde{I}}^*(\{L_i, \mathbf{1}\}, M) = 0$.

1.3.18. A *compound operad* is the same as a compound pseudo-tensor category with single object. One may use compound operads to define compound tensor structures on some categories of modules (cf. 1.1.13). As in 1.1.13 we explain this in the k-linear setting.

Let R be an associative k-algebra, and $\mathcal{B}^{*!}$ a compound *strict* R-operad, i.e., a compound k-operad $\mathcal{B}^{*!}$ together with isomorphism of k-algebras $R \xrightarrow{\sim} \mathcal{B}_1$. Explicitly, such $\mathcal{B}^{*!}$ is a pair $(\mathcal{B}^*, \mathcal{B}^!)$, where $\mathcal{B}^*$ is a strict R-operad, and $\mathcal{B}^!$ is a strict R^o-operad (R^o is the opposite algebra, so $\mathcal{B}_I^*$ is an $(R - R^{\otimes I})$-bimodule and $\mathcal{B}_I^!$ is an $(R^{\otimes I} - R)$-bimodule), together with the pairings

$$\langle\ \ \rangle_{S,T}^I : (\underset{S}{\otimes} \mathcal{B}_{I_s}^*) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}_{I_t}^!) \to \mathcal{B}_S^! \underset{R}{\otimes} \mathcal{B}_T^* \tag{1.3.18.1}$$

defined for any complementary pair $S, T \in Q(I)$. These maps are morphisms of $(R^{\otimes S} - R^{\otimes T})$-bimodules, and the axioms 1.3.7 just say that the diagram (here I, S, T, H, V are as in 1.3.7(i))

$$\begin{array}{ccc}
(\underset{S}{\otimes}\mathcal{B}^*_{H_s}) \underset{R^{\otimes H}}{\otimes} (\underset{H}{\otimes}\mathcal{B}^*_{I_h}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes}\mathcal{B}^!_{I_t}) & \longrightarrow & (\underset{S}{\otimes}\mathcal{B}^*_{I_3}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes}\mathcal{B}^!_{I_t}) \\
\downarrow & & \\
(\underset{S}{\otimes}\mathcal{B}^*_{H_S}) \underset{R^{\otimes H}}{\otimes} (\underset{V}{\otimes}\mathcal{B}^!_{H_v}) \underset{R^{\otimes V}}{\otimes} (\underset{V}{\otimes}\mathcal{B}^*_{T_v}) & & \downarrow \\
\downarrow & & \\
\mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_V \underset{R^{\otimes V}}{\otimes} (\underset{V}{\otimes}\mathcal{B}^*_{T_v}) & \longrightarrow & \mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_T
\end{array}$$

commutes as well as the corresponding "dual" diagram, and $\langle\ \ \rangle^I_{S,T}$ coincides with $id_{\mathcal{B}^*_T}$, $id_{\mathcal{B}^!_S}$ if $\mid S\mid = 1$, resp. $\mid T \mid = 1$.

Now any such $\mathcal{B}^{*!}$ defines a compound pseudo-tensor structure on mod$R$. Namely, set (see 1.1.13)

$$P^*_I(\{L_i\}, M) = \mathrm{Hom}_{\mathrm{modR}^{\otimes I}}(\underset{I}{\otimes} L_i,\ M \underset{R}{\otimes} \mathcal{B}^*_I)$$

$$P^!_I(N, \{L_i\}) = \mathrm{Hom}_{\mathrm{modR}}\big(N, (\underset{I}{\otimes} L_i) \underset{R^{\otimes I}}{\otimes} \mathcal{B}^!_I\big)$$

The composition of operations comes from composition laws in $\mathcal{B}^*$ and $\mathcal{B}^!$. Note that the ! pseudo-tensor structure is representable (one has $\overset{!}{\underset{I}{\otimes}} L_i = (\underset{I}{\otimes} L_i) \underset{R^{\otimes I}}{\otimes} \mathcal{B}^!_I$), so we may use the format of 1.3.12.

The compound tensor product

$$\otimes^I_{S,T}\colon \underset{S}{\otimes} P^*_{I_s}(\{L_i\}, M_s) \to P^*_T(\{\overset{!}{\underset{I_t}{\otimes}} L_i\}, \underset{S}{\otimes}^! M_s)$$

sends $\otimes\varphi_s \in \underset{S}{\otimes} \mathrm{Hom}_{\mathrm{modR}^{\otimes I_s}}(\underset{I_s}{\otimes} L_i, M_s \underset{R}{\otimes} \mathcal{B}^*_{I_s})$ to the composition

$$\begin{aligned}
\underset{T}{\otimes}\big((\underset{I_t}{\otimes} L_i) \underset{R^{\otimes I_t}}{\otimes} \mathcal{B}^!_{I_t}\big) = (\underset{I}{\otimes} L_i) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes}\mathcal{B}^!_{I_t}) &\xrightarrow{\otimes\varphi_s} (\underset{S}{\otimes} M_s) \underset{R^{\otimes S}}{\otimes} (\underset{S}{\otimes}\mathcal{B}^*_{I_s}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes}\mathcal{B}^!_{I_t}) \\
&\xrightarrow{\langle\ \ \rangle^I_{S,T}} (\underset{S}{\otimes} M_s) \underset{R^{\otimes S}}{\otimes} \mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_T
\end{aligned}$$

We leave it to the reader to check commutativity of diagrams 1.3.12(i), (ii).

Note that $\otimes^!$ is a tensor product (i.e., we have a compound tensor structure) iff for any $J \twoheadrightarrow I$ the composition map $(\underset{I}{\otimes}\mathcal{B}^!_{J_i}) \underset{R^{\otimes I}}{\otimes} \mathcal{B}^!_I \to \mathcal{B}^!_J$ is an isomorphism.

1.3.19. Now assume that $\mathcal{B}^{*!}$ is a $*$-augmented compound operad (see 1.3.10). So we have a left R-module $\mathcal{B}^*_0$ and the maps $\mathcal{B}^*_I \underset{R}{\otimes} \mathcal{B}^*_0 \to \mathcal{B}_{I\setminus\{i_0\}}, i_0 \in I$, that are compatible with $\langle \quad \rangle$ in the sense explained in 1.3.10. According to 1.2.9, we get an augmentation functor h^* for the $*$ structure on mod$R$, $h^*(M) := M \underset{R}{\otimes} \mathcal{B}^*_0$. We leave it to the reader to check that h^* is a $*$-augmentation functor for our compound pseudo-tensor structure (see 1.3.10). Similarly, a !-augmentation for $\mathcal{B}^{*!}$ defines a !-augmentation $h^!$ on mod$R$ by formula $h^!(M) = \mathrm{Hom}_{\mathrm{mod}\, R}(M, \mathcal{B}^!_0)$.

1.3.20 Example. Assume we have a cocommutative Hopf algebra structure on R. It yields the similar structure on R^o (the coproduct map for R^o coincides with that for R). By 1.1.14 we have the strict R- and R^o operads $\mathcal{B}^* = \mathcal{B}^R$ and $\mathcal{B}^! = \mathcal{B}^{R^o}$ respectively. Let us show that $\mathcal{B}^*$ and $\mathcal{B}^!$ form a compound strict R-operad. We need to define the maps $\langle \quad \rangle^I_{S,T}$. For $S \in Q(I)$ denote by $\triangle^I_S: R^{\otimes S} \to R^{\otimes I}$ the morphism of k-algebras $\underset{S}{\otimes} \triangle^{(I_s)}$ (recall that $\triangle^{(I_s)}: R \to R^{\otimes I_s}$ is the I_s-multiple coproduct). Let us consider $R^{\otimes I}$ as $(R^{\otimes S} - R^{\otimes T})$-bimodule with respect to the action $(\otimes r_s)(\otimes r_i)(\otimes r_t) := \triangle^I_S(\otimes r_s) \cdot (\otimes r_i) \cdot \triangle^I_T(\otimes r_t)$. We have the obvious indentifications of $(R^{\otimes S} - R^{\otimes T})$-bimodules $(\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) = (\underset{S}{\otimes} R^{\otimes I_s}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} R^{\otimes I_t}) = (R^{\otimes I}) \underset{R^{\otimes I}}{\otimes} (R^{\otimes I}) = R^{\otimes I}$, $\mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_T = R^{\otimes S} \underset{R}{\otimes} R^{\otimes T}$. Consider the morphism of $(R^{\otimes S} - R^{\otimes T})$-bimodules

$$\nu\colon R^{\otimes S} \underset{R}{\otimes} R^{\otimes T} \to R^{\otimes I},$$

$\nu\big((\otimes r_s) \otimes (\otimes r_t)\big) := (\otimes r_s) 1 (\otimes r_t) = \triangle^I_S(\otimes r_s) \cdot \triangle^I_T(\otimes r_t)$. If S, T are complementary then ν is an isomorphism (use 1.3.2 (ii)), and our $\langle \quad \rangle^I_{S,T}$ is ν^{-1}.

We leave it to the reader to check that $\mathcal{B}^{*!} = (\mathcal{B}^*, \mathcal{B}^!, \langle \quad \rangle)$ is actually a compound operad (i.e., that the diagram from 1.3.17 commutes). If our Hopf algebra has counit then the operads $\mathcal{B}^*$ and $\mathcal{B}^!$ are augmented in a natural way (see 1.2.10); it is easy to see that $(\mathcal{B}^{*\wedge}, \mathcal{B}^{!\wedge}, \langle \quad \rangle)$ is an augmented compound operad.

§4 Rudiments of compound geometry

In this section $\mathcal{M}^{*!}$ is an abelian augmented compound tensor k-category (see 1.3.14, 1.3.16). We abbreviate $\otimes^!$ to $\otimes$. There are no assumptions about exactness of the $*$-augmentation functor h.

1.4.1. For $M \in \mathcal{M}$ set $\Gamma(M) := \mathrm{Hom}(\mathbf{1}, M)$. Then Γ is an augmentation functor on $\mathcal{M}^!$; in particular $\Gamma\colon \mathcal{M}^! \to k\mathrm{mod}$ is a pseudo-tensor functor (see 1.2.4, 1.2.7).

For a k-operad $\mathcal{B}$ we denote by $\mathcal{B}(\mathcal{M}^*)$ the category of $\mathcal{B}$ algebras in $\mathcal{M}^*$; we call them simply $\mathcal{B}^*$ *algebras*. In particular we have the category $\mathcal{L}ie(\mathcal{M}^*)$ of Lie* algebras. Similarly, if $\mathcal{B}^\wedge$ is an augmented k-operad then $\mathcal{B}^\wedge$ algebras in the augmented category $\mathcal{M}^*$ are called $\mathcal{B}^{\wedge *}$ algebras. In particular, we have the category of unital associative* algebras (see 1.2.8).

1.4.2 Matrix algebras. Assume we have $V, V' \in \mathcal{M}$ and $\langle\ \ \rangle \in P_2^*(\{V', V\}, \mathbf{1})$ a $*$ pairing. Then $E = V \otimes V'$ carries a canonical structure of associative* algebra which acts on V from the left and on V' from the right. Namely, the product operation $\cdot \in P_2^*(\{E, E\}, E)$ is the image of $id_V \otimes \langle\ \ \rangle \otimes id_{V'} \in \mathrm{E}nd(V) \otimes P_2^*(\{V', V\}, \mathbf{1}) \otimes \mathrm{E}nd(V')$ by the $\otimes^I_{S,T}$ map, where $I = \{1,2,3,4\}$, and S, T are equivalence relations $\{2 = 3\}$ and $\{1 = 2, 3 = 4\}$ respectively. The E-action on V is the image of $id_V \otimes \langle\ \ \rangle$ by the binary map $\otimes^{\{1\},\{2,3\}}_{1,2}\colon \mathrm{E}nd(V) \otimes P_2^*(\{V', V\}, \mathbf{1}) \to P_2^*(\{E, V\}, V)$ (see 1.3.13 for notations). We leave the definition of the right E-action on V', as well as verification of associativity, to the reader.

One often wants to know if the E-action on V identifies E with $\mathcal{E}ndV$. The following condition on $\langle\ \ \rangle$ assures this:

Definition. A $*$ pairing $\langle\ \ \rangle \in P_2^*(\{V', V\}, \mathbf{1})$ is *non-degenerate* if for every finite set I and $A_{i\in I}, B \in \mathcal{M}$ the canonical map

$$P_I^*(\{A_i\}, B \otimes V') \to P_{\tilde{I}}^*(\{A_i, V\}, B), \quad \varphi \mapsto (id_B \otimes \langle\ \ \rangle)(\varphi, id_V), \tag{1.4.2.1}$$

is an isomorphism.[10]

If $\langle\ \ \rangle$ is non-degenerate then for every $B \in \mathcal{M}$ one has $B \otimes V' \xrightarrow{\sim} \mathcal{H}om(V, B)$ (see 1.2.1, 1.2.2) and[11] $h(B \otimes V') \xrightarrow{\sim} \mathrm{Hom}(V, B)$. In particular,[12] $V' = \mathcal{H}om(V, \mathbf{1})$, i.e., V' is

[10] Here $id_B \otimes \langle\ \ \rangle \in P_2^*(\{B \otimes V', V\}, B)$.
[11] Take $I = \emptyset$ in (1.4.2.1).
[12] Take $B = \mathbf{1}$.

the $*$ dual to V, and $E = V \otimes V' \xrightarrow{\sim} \mathcal{E}nd V$, $h(E) \xrightarrow{\sim} \mathrm{End} V$. Our E is a unital associative* algebra: the unit element[13] is $id_V \in \mathrm{End} V = h(E)$.

1.4.3. We will often use the fact that $\otimes$ behaves naturally with respect to $*$ actions on multiples. Precisely, assume we have a map $I \to J$ of finite sets and families of objects $\{L_i\}_{i\in I}$, $\{M_j\}_{j\in J}$, $\{M'_j\}_{j\in J}$. Then comes the canonical map

$$\otimes^{I\sqcup J}_{J,\tilde{I}} \colon \underset{J}{\otimes} P^*_{\tilde{I}_j}(\{L_i, M_j\}, M'_j) \to P^*_{\tilde{I}}(\{L_i, \underset{J}{\otimes} M_j\}, \underset{J}{\otimes} M'_j) ,$$

$\otimes\varphi_j \mapsto \underset{J}{\otimes} \varphi_j$, compatible with composition of operations.

Example. Assume we have a family $\{\mathcal{B}_j\}_{j\in J}$ of k-operads together with the corresponding $\mathcal{B}_j$ module operads $\mathcal{C}_j$. Let L_j be a $\mathcal{B}^*_j$ algebra, M_j an L_j-module in $\mathcal{M}^*$ with respect to $\mathcal{C}_j$. Then $\underset{J}{\otimes} M_j$ carries canonical L_j-actions that mutually commute. Namely, for $j_0 \in J$ the j_0's action map $\mathcal{C}_{j_0 I} \to P^*_{\tilde{I}}(\{L_{j_0 I}, \underset{J}{\otimes} M_j\}, \underset{J}{\otimes} M_j)$ sends $c \in \mathcal{C}_{j_0 I}$ to $c_{M_{j_0}} \otimes (\underset{j\neq j_0}{\otimes} id_{M_j})$, where $c_{M_{j_0}} \in P^*_{\tilde{I}}(\{L_{j_0 I}, M_{j_0}\}, M_{j_0})$ is the c-action on M_{j_0}.

Recall that $h(L_j)$ is a $\mathcal{B}_j$ algebra, and M_j is automatically an $h(L_j)$-module (see 1.2.17). It is easy to see that the action of $h(L_{j_0})$ on $\underset{J}{\otimes} M_j$ that comes from the above L_{j_0}-action on $\underset{J}{\otimes} M_j$ coincides with the action $a \mapsto a_{M_{j_0}} \otimes (\otimes id_{M_j})_{j\neq j_0}$.

1.4.4 Lie* algebras and modules. Let L be a Lie* algebra. Denote by $\mathcal{M}(L)$ the category of L-modules in $\mathcal{M}^*$. Let us show that

$\mathcal{M}(L)$ is an *abelian augmented compound tensor k-category*

in a natural way. We will denote it as $\mathcal{M}(L)^{*!}$.

According to 1.2.18 $\mathcal{M}(L)$ carries a canonical abelian pseudo-tensor structure P^*_L. If $\{M_j\}_{j\in J}$ is a finite family of L-modules then L acts on $\underset{J}{\otimes} M_j$ via the sum of the above J commuting actions (i.e., by Leibnitz rule). This is the ! tensor product of L-modules. One easily checks that $\otimes^I_{S,T}$ maps send L-linear operations to L-linear ones; this provides the maps $\otimes^I_{S,T}$ for $\mathcal{M}(L)$. Compatibility axioms 1.3.12 are clear.

[13] See 1.2.8.

The object $\mathbf{1}$ with trivial L-action is a unit object in $\mathcal{M}_L^{*!}$. For an L-module M the L-action on M yields the morphism of k-modules $h(M) \to \mathrm{Hom}_{\mathcal{M}}(L, M)$. Denote by $h^L(M)$ its kernel. h^L is a $*$-augmentation functor on $\mathcal{M}(L)^{*!}$.

Remark. A morphism $L' \to L$ of Lie* algebras yields the "restriction of action" augmented compound tensor functor $\mathcal{M}(L)^{*!} \to \mathcal{M}(L')^{*!}$.

1.4.5 The Chevalley complex I. The constructions below (except Remark (ii)) use only $*$ pseudo-tensor structure on $\mathcal{M}$.

For a Lie* algebra L and an L-module M we have *the Chevalley complex* $C(L, M)$ wich is a complex with terms $C^n(L, M) := P_n^*(\{L, .., L\}, M)_{sgn}$ (the skew-coinvariants of the action of the symmetric group Σ_n on operations of n variables) and differential given by the standard formula (see below). Set $H^{\cdot}(L, M) = H^{\cdot}C(L, M)$; these are *cohomology of L with coefficients in M.*

As in the case of usual Lie algebras, the Chevalley complex is naturally defined in the more general "super" setting[14] (below we often drop prefix "super"). Namely, let L be a super Lie* algebra, M a super L-module. Consider the graded super k-module $\oplus P_n^*(\{L[1], .., L[1]\}, M)_{\Sigma_n}$ (coinvariants of the symmetric group action on super operations of n variables). It carries a canonical odd differential δ of degree 1 coming from the morphism $P_n^*(\{L[1], .., L[1]\}, M)[1] \to P_n^*(\{L[1], .., L[1]\}, M)$, $\varphi \mapsto \cdot_M(id_{L[1]}, \varphi) - \frac{1}{2}\varphi([\ \ ], id_{L[1]}, .., id_{L[1]})$ where $\cdot_M \in P_2(\{L, M\}, M)$ is the action of L on M. One has $\delta^2 = 0$, so we have a DG super k-module denoted by $C(L, M)$. Let us describe its natural symmetries.

Set $L_\dagger := \mathcal{C}one(id_L)$. This complex carries a canonical structure of DG Lie* algebra: the bracket is uniqely determined by property that $L \hookrightarrow L_\dagger$ is a morphism of DG Lie* algebras (then the complementary graded submodule $L[1] \subset L_\dagger$ is a commutative ideal, and the adjoint action of $L \subset L_\dagger$ on it equals the adjoint action of L).

The DG Lie algebra $h(L)_\dagger$ acts on $C(L, M)$ in a canonical way. Namely, $h(L) \subset h(L_\dagger)$ acts via the adjoint action on copies of $L[1]$ and the L-module structure on M, and the

[14] See 1.1.16 for notation.

action of its complement $h(L)[1]$ comes from the map $h(L[1]) \otimes P^*_n(\{L[1],..L[1]\},M) \to P^*_{n-1}(\{L,..,L\},M)$ which is minus sum of the canonical "convolution" maps for each of the n arguments.

Remarks. (i) If for every n the inner P object $\mathcal{P}^*_n(\{L,..,L\},M)$ is well-defined (see 1.2.1) then one has the inner Chevalley complex $\mathcal{C}(L,M)$ defined in the obvious way. It carries a canonical $*$ action of $L_\dagger$. There is a canonical morphism of DG $h(L_\dagger)$-modules $h(\mathcal{C}(L,M)) \to C(L,M)$ which is an isomorphism if our $\mathcal{P}$'s are P objects in the augmented sense (see 1.2.7).

(ii) Let us assume that the $*$ dual $L' := \mathcal{H}om^*(L,\mathbf{1})$ is well-defined and the canonical pairing $\langle\ \ \rangle \in P^*_2(\{L',L\},\mathbf{1})$ is non-degenerate (see 1.4.2). Then for every $n \geq 0$ and every $N \in \mathcal{M}$ one has $N \otimes L'^{\otimes n} = \mathcal{P}^*_n(\{L,..,L\},N)$ and $h(N \otimes L'^{\otimes n}) \xrightarrow{\sim} P^*_n(\{L,..,L\},N)$ (see 1.2.1, 1.2.7). Thus the inner Chevalley complex is well-defined (for an L-module M one has $\mathcal{C}(L,M)^n = M \otimes \mathrm{Sym}^n(L'[-1])$), and we have a canonical isomorphism of complexes

$$h(\mathcal{C}(L,M)) \xrightarrow{\sim} C(L,M). \tag{1.4.5.1}$$

(iii) The above constructions are functorial in the obvious manner. In particular, if both L and M are equipped with a DG structure (i.e., L is a DG Lie* superalgbera, M is a DG super L-module) then the super k-module $C(L,M)$ acquires an extra DG structure. *The Chevalley complex of DG Lie* algebra L with coefficients in DG L-module M* is this super k-module equipped with the "total" grading and differential. Same for the inner Chevalley complex.

1.4.6 Commutative$^!$ algebras and modules. We can do some commutative algebra in $\mathcal{M}^!$ as in any tensor k-category. Denote by $\mathcal{C}omu(\mathcal{M}^!)$ the category of commutative and associative algebras with unit in $\mathcal{M}^!$. An object $A \in \mathcal{C}omu(M^!)$ (we call it a *commutative$^!$ algebra*) is an object $A \in \mathcal{M}$ together with a commutative and associative product $\cdot = \cdot_A\colon A \otimes A \to A$ and a morphism $1 = 1_A\colon \mathbf{1} \to A$ which is a unit for $\cdot$. For a finite family $\{A_i\}$ of commutative$^!$ algebras the tensor product $\underset{I}{\otimes} A_i$ has an obvious structure of commutative$^!$ algebra; this is the coproduct of $\{A_i\}$ in $\mathcal{C}omu(\mathcal{M}^!)$. Therefore $\otimes$

defines a tensor category structure on $\mathcal{C}omu(\mathcal{M}^!)$; the object $\mathbf{1}$ endowed with an obvious product is a unit in $\mathcal{C}omu(\mathcal{M}^!)$.

For $A \in \mathcal{C}omu(\mathcal{M}^!)$ we denote by $\mathcal{M}(A)$ the category of unital A-modules in $\mathcal{M}^!$; this is an abelian k-category. The usual tensor product $\underset{A}{\otimes}$ of A-modules is right exact, so $\mathcal{M}(A)$ is an abelian tensor k-category with unit object A. The "forgetting of A-action" functor $\mathcal{M}(A) \to \mathcal{M}$ admits a left adjoint "induction" functor $\mathcal{M} \to \mathcal{M}(A)$, $M \mapsto A \otimes M$, which is a tensor functor. Below for an A-module N we denote by $\cdot_N \colon A \otimes N \to N$ action map.

Let us show that

$\mathcal{M}(A)$ *is an abelian augmented compound tensor k-category*

in a natural way. We denote it $\mathcal{M}(A)^{*!}$.

The tensor structure has been discussed already, so it remains to define the $*$ operations P^*_{AI}, the compound tensor product maps $\otimes^I_{S,T}$, and the $*$-augmentation functor.

Take $I \in \mathcal{S}$ and objects $\{L_i\}_{i\in I}$, M of $\mathcal{M}$. Assume that some L_{i_0}, $i_0 \in I$, and M are A-modules. We say that an operation $\varphi \in P^*_I(\{L_i\}, M)$ is *A-linear with respect to the i_0th variable* if the operations $\cdot_M(\varphi \otimes_{i_0} id_A), \varphi(id_{L_i}, \cdot_{L_{i_0}})_{i\neq i_0} \in P^*_I(\{L_i, A\otimes L_{i_0}\}_{i\neq i_0}, M)$ coincide. Here $\varphi \otimes_{i_0} id_A \in P^*_I(\{L_i, A \otimes L_{i_0}\}_{i\neq i_0}, A\otimes M)$. If all L_i are A-modules, then we denote by $P^*_{AI}(\{L_i\}, M) \subset P^*_I(\{L_i\}, M)$ the k-submodule of *A-polylinear* (i.e., A-linear with respect to each variable) operations. The composition of A-polylinear operations is again A-polylinear, so P^*_{AI} form a pseudo-tensor structure on $\mathcal{M}(A)$. The functors P^*_{AI} are left exact.

Let us define the structure maps $\otimes^I_{S,T} \colon \underset{S}{\otimes} P^*_{AI_s}(\{L_i\}, M_s) \to P^*_{AT}(\{\underset{AI_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s)$. The projections $\underset{I_t}{\otimes} L_i \to \underset{AI_t}{\otimes} L_i$ define the embedding $P^*_{AT}(\{\underset{AI_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s) \hookrightarrow P^*_T(\{\underset{I_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s)$, and it is easy to see that the composition

$$\underset{S}{\otimes} P^*_{AI}(\{L_{i_s}\}, M_s) \to \underset{S}{\otimes} P^*_{I_s}(\{L_i\}, M_s) \xrightarrow{\otimes^I_{S,T}} P^*_T(\{\underset{I_t}{\otimes} L_i\}, \underset{S}{\otimes} M_s) \to P^*_T(\{\underset{I_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s)$$

takes values in the image of this embedding. Therefore we may consider the composition as a morphism $\underset{S}{\otimes} P^*_{AI_s}(\{L_i\}, M_s) \to P^*_T(\{\underset{AI_t}{\otimes} L_{i_t}\}, \underset{AS}{\otimes} M_s)$; this is $\otimes^I_{S,T}$ map for $\mathcal{M}(A)^{*!}$.

Compatibility axioms 1.3.12 for $\mathcal{M}(A)^{*!}$ follow from those for $\mathcal{M}^{*!}$.

The object A, considered as an A-module, is a unit object in $\mathcal{M}(A)^{*!}$. Note that the morphism $1\colon \mathbf{1} \to A$ yields a canonical isomorphism $End_{\mathcal{M}(A)}(A) \xrightarrow{\sim} \Gamma(A)$ of commutative k-algebras. Therefore, by 1.3.16, $\mathcal{M}(A)^{*!}$ is actually a compound tensor $\Gamma(A)$-category.

We define the $*$-augmentation functor on $\mathcal{M}_A^{*!}$ simply as restriction of h to $\mathcal{M}(A)$ (note that h_{I,i_0} maps preserve A-polylinear operations).

Remarks. (i) The "induction" functor $\mathcal{M} \to \mathcal{M}(A)$ extends naturally to an augmented compound tensor functor $\mathcal{M}^{*!} \to \mathcal{M}(A)^{*!}$. Namely, we already mentioned that the induction is a ! tensor functor. The map

(1.4.6.1) $$P_I^*(\{F_i\}, G) \to P_{AI}^*(\{A \otimes F_i\}, A \otimes G)$$

is the composition $P_I^*(\{F_i\}, G) \to P_I^*(\{A \otimes F_i\}, A^{\otimes I} \otimes G) \to P_{AI}^*(\{A \otimes F_i\}, A \otimes G)$ where the first arrow sends φ to $\otimes_{I,I}^{I \sqcup I}(\varphi, \{id_A\}_{i \in I})$, the second one is the composition with $\cdot_I \otimes id_G$ where $\cdot_I\colon A^{\otimes I} \to A$ is the I-fold product. We leave it to the reader to check the compatibility with $\otimes_{S,T}^I$ maps (see 1.3.14). The arrow $h(F) \to h(A \otimes F)$ is $h(1_A \otimes id_F)$.

More generally, any morphism of commutative$^!$ algebras $f : B \to A$ yields a compound tensor functor (base change)

(1.4.6.2) $$f^* : \mathcal{M}(B)^{*!} \to \mathcal{M}(A)^{*!}$$

such that for $M \in \mathcal{M}(B)$ one has $f^*M = A \underset{B}{\otimes} M$. To define f^* you just replace in the above considerations $\mathcal{M}^{*!}$ by $\mathcal{M}(B)^{*!}$. The base change is compatible with composition of f's in the obvious way.

(ii) The induction pseudo-tensor functor $\mathcal{M}^* \to \mathcal{M}(A)^*$ is left adjoint[15] to the obvious forgetful pseudo-tensor functor $\mathcal{M}(A)^* \to \mathcal{M}^*$. To see this consider for $M \in \mathcal{M}(A)$ and $F_i, L_j \in \mathcal{M}$ the map $P_{I \sqcup J}^*(\{A \otimes F_i, L_j\}, M) \to P_{I \sqcup J}^*(\{F_i, L_j\}, M)$ sending an operation ψ to its composition with $1_A \otimes id_{F_i}\colon F_i \to A \otimes F_i$. This projection identifies the subspace of operations A-linear with respect to I variables with $P_{I \sqcup J}^*(\{F_i, L_j\}, M)$. The inverse map sends

[15] See 1.1.5.

$\varphi \in P^*_{I\sqcup J}(\{F_i, L_j\}, M)$ to the composition of $\varphi \otimes_I (\otimes id_A) \in P^*_{I\sqcup J}(\{A\otimes F_i, L_j\}, A^{\otimes I}\otimes M)$ with the product morphism $A^{\otimes I}\otimes M \to M$. If L_j are A-modules then our maps preserve operations A-linear with respect to J-variables, so $P^*_{A\ I\sqcup J}(\{A\otimes F_i, L_j\}, M)$ identifies with the subspace of $P^*_{I\sqcup J}(\{F_i, L_j\}, M)$ that consists of operations A-linear with respect to the J variables.

In the setting of (1.4.6.2) the pseudo-tensor functor $f^* : \mathcal{M}(B)^* \to \mathcal{M}(A)^*$ is left adjoint to the obvious forgetful pseudo-tensor functor $\mathcal{M}(A)^* \to \mathcal{M}(B)^*$.[16]

(iii) If a $*$ pairing $\langle\ \ \rangle \in P^*_2(\{L', L\}, \mathbf{1})$ in $\mathcal{M}$ is non-degenerate (see 1.4.2) then such is the corresponding pairing $\langle\ \ \rangle_A \in P^*_2(\{A\otimes L', A\otimes L\}, A)$ in $\mathcal{M}(A)$.

1.4.7 Differential $*$ operations. Many important $*$ operations between A-modules are not A-polylinear but belong to a larger class of *A-differential $*$ operations* we are going to define.

Let $\mathcal{J}\subset A\otimes A$ be the ideal of the diagonal, i.e., the kernel of the product map $A\otimes A\to A$. For an A-module L consider $A\otimes L$ as an $A\otimes A$-module and set $L^{(n)} := A\otimes L/\mathcal{J}^{n+1}(A\otimes L) = A^{(n)} \underset{A}{\otimes} L$. Now assume we have $M\in\mathcal{M}(A)$ and $L_i\in\mathcal{M}$. Pick some $i_0\in I$ and identify $P^*_I(\{L_i\}, M)$ with the subspace of operations in $P^*_I(\{L_i, A\otimes L_{i_0}\}, M)$ A-linear with respect to the i_0th variable (see Remark (ii) in 1.4.6). Assume that L_{i_0} is also an A-module. The operations vanishing on $\mathcal{J}^{n+1}(A\otimes L_{i_0})$ form a subspace of $P^*_I(\{L_i\}, M)$; its elements are *differential operations of order $\leq n$ with respect to the i_0th variable.* If $n=0$ these are the same as A-linear operations.

If all L_i's are A-modules then the $*$ operations which are differential operations of some order with respect to each variable are called *A-polydifferential $*$ operations of finite order.* The space of such operations is $\underset{n}{\cup} P^*_{A\,I}(\{L_i^{(n)}\}, M) \subset P^*_{A\,I}(\{A\otimes L_i\}, M) = P^*_I(\{L_i\}, M)$. They are stable with respect to composition, so A-polydifferential $*$ operations form another pseudo-tensor structure on $\mathcal{M}(A)$.

1.4.8 Action of Lie* algebras on commutative$^!$ algebras. If L is a Lie* algebra, then an action of L on a commutative$^!$ algebra A[17] is an L-module structure $\tau \in P^*_2(\{L, A\}, A)$

[16] To see this replace in the above argument $\mathcal{M}$ by $\mathcal{M}(B)$.

[17] Or an L-action by derivations.

on A (as on an object of $\mathcal{M}^*$) such that $\cdot_A\colon A\otimes A\to A$, $1_A\colon \mathbf{1}\to A$ are morphisms of L-modules.

Let τ be a fixed action of L on A. Then for $M\in\mathcal{M}(A)$ a *τ-action of L on M*[18] is an L-module structure on M (as on an object of $\mathcal{M}^*$) such that $\cdot_M\colon A\otimes M\to M$ is a morphism of L-modules. The A-modules equipped with τ-actions form an abelian category $\mathcal{M}(A,L,\tau)$. One may combine 1.4.4, 1.4.6 to show that

$\mathcal{M}(A,L,\tau)$ *is an abelian augmented compound tensor k-category*

in a natural way. We denote it $\mathcal{M}(A,L,\tau)^{*!}$.

Namely, for $M_i,N\in\mathcal{M}(A,L,\tau)$ set $P^*_{A,L,I}(\{M_i\},N) := P^*_{LI}(\{M_i\},N)\cap P^*_{AI}(\{M_i\},N)$ (the intersection is taken in $P^*_I(\{M_i\},N)$). This is the $*$ pseudo-tensor structure on $\mathcal{M}(A,L,\tau)$. Note that $\underset{AI}{\otimes}M_i$ is a quotient L-module of $\underset{I}{\otimes}M_i$, so we define the ! tensor product on $\mathcal{M}(A,L,\tau)$ as $\underset{A}{\otimes}$. The $\otimes^I_{S,T}$ operations all come from $\mathcal{M}(A)^{*!}$.

We may consider A as an object of $\mathcal{M}(A,L,\tau)$; this is the unit object.

The $*$-augmentation functor on $\mathcal{M}(A,L,\tau)^{*!}$ is $h^{\mathbf{L}}$ (see 1.4.4).

1.4.9 The Chevalley complex II. Let us assume that the $*$ dual $L' := \mathcal{H}om^*(L,\mathbf{1})$ of our Lie* algebra L exists and the canonical pairing $\langle\ \ \rangle\in P^*_2(\{L',L\},\mathbf{1})$ is non-degenerate (see 1.4.2). Below we give another interpretation of the inner Chevalley complexes from Remark (ii) in 1.4.5. As in 1.4.5, we deal with the general super setting (skipping the prefix "super").

Consider first the graded commutative$^!$ algebra $\mathrm{Sym}(L'[-1])$. It carries a canonical action of the graded Lie* algebra $L_\dagger$ from 1.4.5 (we forget about the differentials at the moment). Namely, $L\subset L_\dagger$ acts according to the coadjoint action, and its complement $L[1]\subset L_\dagger$ acts on the generators $L'[-1]$ by $-\langle\ \ \rangle\in P^*_2(\{L[1],L'[-1]\},\mathbf{1})$. Now the graded algebra $\mathrm{Sym}(L'[-1])$ admits a unique differential (odd of degree 1, see 1.1.16) such that the action of $L_\dagger$ are compatible with the differentials. Explicitly, on the generators $L'[-1]$ it is equal to the composition $L'[-2]\xrightarrow{\nu} L'[-1]\otimes L'[-1]\xrightarrow{-1/2\cdot}\mathrm{Sym}^2(L'[-1])$ where ν is the

[18] Or a τ-compatible action.

morphism dual to the Lie* bracket.[19] We denote this DG commutative$^!$ algebra equipped with the $L_\dagger$-action by $\mathcal{C}(L)$.

Let M be any L-module. Consider M as an $L_\dagger$-module[20] via the projection $L_\dagger \to L_\dagger/L[1] = L$. Then $L_\dagger$ acts on the graded $\mathrm{Sym}(L'[-1])$-module $M \otimes \mathrm{Sym}(L'[-1])$ via the tensor product of the above actions. As above, $M \otimes \mathrm{Sym}(L'[-1])$ admits a unique differential such that the actions of $L_\dagger$ and $\mathcal{C}(L)$ are compatible with the differentials. Explicitly, on the generators M the differential $M \to M \otimes L'$ is the morphism dual to the L-action. We denote this DG $\mathcal{C}(L)$-module equipped with $L_\dagger$-action by $\mathcal{C}(L, M)$. We leave it to the reader to check that it coincides with the inner Chevalley complex from Remark (ii) of 1.4.5.

If L acts on a commutative$^!$ algebra A by derivations, then $L_\dagger$ acts on $A \otimes \mathrm{Sym}(L'[-1])$ also by derivations, and the Chevalley differential is a derivation. Thus $\mathcal{C}(L, A)$ is a DG commutative$^!$ algebra equipped with a DG $L_\dagger$-action. If M is an A-module equipped with a compatible L-action (see 1.4.8) then $\mathcal{C}(L, M)$ is a DG $\mathcal{C}(L, A)$-module equipped with a compatible $L_\dagger$-action.

Remark. The morphism $\nu : L' \to L' \otimes L'$ is a Lie cobracket. The duality isomorphism $P_2^*(\{L, M\}, M) \xrightarrow{\sim} \mathrm{Hom}(M, L' \otimes M)$ identifies $*$ actions of L on M with ! coactions of L' on M. So the category of L-modules in $\mathcal{M}^*$ coincides with the category of L'-comodules in $\mathcal{M}^!$. If M is an L-module then $\mathcal{C}(L, M)$ is the usual Chevalley complex of the Lie coalgebra L' with coefficients in L'-comodule M in the tensor category $\mathcal{M}^!$.

1.4.10 Lie* algebroids. For $A \in \mathcal{C}omu(\mathcal{M}^!)$ a *Lie* A-algebroid*[21] is an A-module L, equipped with the following extra structures:

(a) a Lie* bracket $[\ \ ] \in P_2^*(\{L, L\}, L)$

(b) an action $\tau = \tau_L \in P_2^*(\{L, A\}, A)$ of L (considered as a Lie* algebra) on A

[19] So if $\langle\ \ \rangle^2 \in P_3^*(\{L' \otimes L', L, L\}, \mathbf{1})$ is the tensor product of 2 copies of $\langle\ \ \rangle$ then ν is uniquely determined by property $\langle\ \ \rangle^2(\nu, id_L, id_L) = \langle\ \ \rangle(id_{L'}, [\ \ ]) \in P_3^*(\{L', L, L\}, \mathbf{1})$.

[20] We forget about the differential on $L_\dagger$ at the moment.

[21] The definition of Lie algebroid in the usual (not compound) setting is recalled in 2.8.1 below; for more details see [M] or the first two sections of [BB].

that satisfy the following axioms:

(i) τ is A-linear with respect to the L-variable

(ii) The adjoint action ad_L is a τ-action of L (as a Lie* algebra) on L (as an A-module).

Notice that the structure $*$ operations (a), (b) are A-polydifferential (see 1.4.7).

Lie* A-algebroids form a category $\mathcal{LA}(A)$. Note that $\mathcal{L}ie(\mathcal{M}(A)^*)$ coincides with the full subcategory of $\mathcal{LA}(A)$ that consists of L with $\tau = 0$. The embedding $\mathcal{L}ie(\mathcal{M}(A)^*) \to \mathcal{LA}(A)$ admits a right adjoint functor $L \mapsto L^\circ := \operatorname{Ker}\tau$.

1.4.11. For $L \in \mathcal{LA}(A)$ an *L-module* M is an A-module M equipped with a τ-action $\tau_M \in P_2^*(\{L, M\}, M)$ of L (considered as a plain Lie* algebra) such that τ_M is A-linear with respect to L-variable. L-modules form a full subcategory[22] $\mathcal{M}(A, L) \subset \mathcal{M}(A, L, \tau)$ closed under subquotients and $\underset{A}{\otimes}$; clearly $A \in \mathcal{M}(A, L)$. Define the $*$ operations on $\mathcal{M}(A, L)$, as $P^*_{A,L}$, i.e., let us consider $\mathcal{M}(A, L)$ as a full augmented compound tensor subcategory $\mathcal{M}(A, L)^{*!}$ in $\mathcal{M}(A, L, \tau)^{*!}$. We see that

$\mathcal{M}(A, L)^{*!}$ *is an abelian augmented compound tensor k-category.*

Remark. The above notion of L-module is parallel to the one of *left* module for a usual Lie algebroid L_{us}. There seem to be no adequate compound analog of the notion of *right* L_{us}-module.

Let now E be any Lie* algebra, τ an action of E on A.

1.4.12 Lemma. (i) The induced A-module $A \otimes E$ has a unique structure of a Lie* A-algebroid such that the obvious morphism $1_A \otimes id_E : E \to A \otimes E$ is a morphism of Lie* algebras compatible with their actions on A. The functor $E \mapsto A \otimes E$ from the category of Lie* algebras acting on A to that of Lie* A-algebroids is left adjoint to the obvious "forgetting of A-action" functor.

(ii) The functor $\mathcal{M}(A, A \otimes E)^{*!} \to \mathcal{M}(A, E, \tau)^{*!}$ which assigns to an $A \otimes E$-module M the same M considered as an E-module is an equivalence of augmented compound tensor categories. □

[22] In the notation $\mathcal{M}(A, L, \tau)$ we consider L as a plain Lie* algebra acting on A.

1.4.13 The Chevalley complex III. As in 1.4.5, 1.4.9 we consider the super setting.

Let L be a Lie* A-algebroid. For $M \in \mathcal{M}(A, L)$ the A-polylinear operations form a subcomplex $C_A(L, M) \subset C(L, M)$ closed under the $h(L_\dagger)$-action. This is *the de Rham–Chevalley complex of L with coefficients in M*. Set $C_A(L) := C_A(L, A)$.

Assume now that the A-module L admits a dual A-module L' and the canonical $*$ pairing $\langle\ \ \rangle \in P^*_{A2}(\{L', L\}, A)$ is non-degenerate (in $\mathcal{M}(A)^{*!}$, see 1.4.2). There is a unique action ad'_L of the Lie* algebra L on L' (called *the coadjoint action*) such that $\langle\ \ \rangle \in P^*_2(\{L', L\}, A)$ is L-invariant (here L acts on L by the adjoint action, on A by τ).[23] It is easy to see[24] that ad'_L is a τ-action.

Consider the graded commutative$^!$ algebra $\mathrm{Sym}_A(L'[-1])$. The Lie* algebra L acts on it (via τ and ad'_L). This action extends to an action of the graded Lie* algebra $L_\dagger$ from 1.4.5.[25] Namely, $L[1] \subset L_\dagger$ kills A and acts on $L'[-1]$ by $\langle\ \ \rangle$.

We leave it to the reader to show that $\mathrm{Sym}_A(L'[-1])$ carries a unique differential d (odd of degree 1) such that the $L_\dagger$-action is compatible with the differentials. We denote the resulting DG commutative$^!$ algebra equipped with DG $L_\dagger$-action by $\mathcal{C}_A(L) = \mathcal{C}_A(L, A)$.

For $M \in \mathcal{M}(A, L)$ consider the graded $\mathrm{Sym}_A(L'[-1])$-module $M \underset{A}{\otimes} \mathrm{Sym}_A(L'[-1])$. It carries a natural action of the graded Lie* algebra $L_\dagger$ (namely, $L \subset L_\dagger$ acts according to the tensor product of the L-actions, and $L[1] \subset L_\dagger$ acts via the above action on $\mathrm{Sym}_A(L'[-1])$). Now $M \underset{A}{\otimes} \mathrm{Sym}_A(L'[-1])$ carries a unique differential such that the $L_\dagger$- and $\mathrm{Sym}_A(L'[-1])$-actions are compatible with the differentials. The resulting DG $\mathcal{C}_A(L)$-module equipped with DG $L_\dagger$-action is denoted by $\mathcal{C}_A(L, M)$. This is *the inner de Rham–Chevalley complex of L with coefficients in M*. We have $h(\mathcal{C}_A(L, M)) \xrightarrow{\sim} C_A(L, M)$ (cf. (1.4.5.1)).

It is easy to see that in situation of 1.4.12 the above construction amounts to that of 1.4.9: one has $\mathcal{C}_A(L) = \mathcal{C}(E, A)$ and $\mathcal{C}_A(L, M) = \mathcal{C}(E, M)$.

1.4.14 The tangent algebroid. For $M \in \mathcal{M}(A)$ we have the k-module of derivations

[23] Indeed, the operation $\langle\ \ \rangle(ad_L, id_{L'}) - \tau\langle\ \ \rangle \in P^*_3(\{L, L, L'\}, A)$ is A-linear with respect to the second argument L, so, since $\langle\ \ \rangle$ is non-degenerate (use Remark (ii) from 1.4.6) there is a unique $ad'_L \in P^*_2(\{L, L'\}, L')$ such that the above operation equals $\langle\ \ \rangle(id_L, ad'_L)$.

[24] Use universality property of $\langle\ \ \rangle$.

[25] We forget about the differential on $L_\dagger$ at the moment.

(with respect to ! tensor structure) $\mathrm{Der}(A, M) \subset \mathrm{Hom}_{\mathcal{M}}(A, M)$. The functor $\mathrm{Der}(A, \cdot)$ is representable: there is a universal derivation $d: A \to \Omega_A$.

More generally, for any finite collection of objects $\{N_i\}_{i\in I}$ of $\mathcal{M}$ we have a k-submodule $\mathrm{Der}(A, M)(\{N_i\})_I \subset P^*_{\tilde{I}}(\{N_i, A\}, M)$ of operations that are derivations with respect to A, i.e., those φ the $\varphi(id_{N_i}, \cdot_A) = \cdot_M(\varphi \otimes_1 id_A) + \cdot_M(\varphi \otimes_2 id_A) \in P^*_{\tilde{I}}(\{N_i, A \otimes A\}, M)$. Here $\varphi \otimes_1 id_A$, $\varphi \otimes_2 id_A \in P^*_{\tilde{I}}(\{N_i, A \otimes A\}, M)$ are the operations defined in 1.4.3; the labels 1,2 show which copy of A in $A \otimes A$ we insert in φ (the transposition of multiples symmetry of $A \otimes A$ interchanges them). Note that for any N_i-valued $*$ operations χ_i the composition $\varphi(\chi_i, id_A)$ is again a derivation with respect to A, and in case $I = \varnothing$ we have our old $\mathrm{Der}(A, M)$.

As follows from Remark (ii) in 1.4.6,[26] the map $P^*_{\tilde{I}}(\{N_i, \Omega_A\}, M) \to P^*_{\tilde{I}}(\{N_i, A\}, M)$, $\psi \mapsto \psi(id_{N_i}, d)$, identifies the subspace of A-linear with respect to Ω_A operations with $\mathrm{Der}(A, M)(\{N_i\})_I$. If N_i are also A-modules, then this map identifies $P^*_{A\tilde{I}}(\{N_i, \Omega_A\}, M)$ with the subspace of derivations A-linear with respect to N_i.

Assume that the object $\mathcal{D}er(A, M) := \mathcal{H}om_{\mathcal{M}(A)^*}(\Omega_A, M) \in \mathcal{M}(A)$ exists (see 1.3.12). By the above remark and Remark (ii) in 1.4.6 for any finite collection $\{G_i\}_{i\in I}$ one has a canonical identification $P^*_I\big(\{G_i\}, \mathcal{D}er(A, M)\big) = \mathrm{Der}(A, M)(\{G_i\})_I \subset P^*_{\tilde{I}}(\{G_i, A\}, M)$.

Let us show that $\Theta_A := \mathcal{D}er(A, A)$ (if it exists) is a Lie* A-algebroid in a natural way; this is *the tangent algebroid* of A. Namely, the action $\tau \in P^*_2(\{\Theta_A, A\}, A)$ is the universal derivation (that corresponds to id_{Θ_A} under the above identification), and $[\ ] \in P^*_2(\{\Theta_A, \Theta_A\}, \Theta_A)$ is uniquely defined by condition that τ is a Lie* action of Θ_A on A. Compatibilities 1.4.10 (i), (ii) are clear.

Remarks. (i) Θ_A is a universal Lie* A-algebroid: for any $L \in \mathcal{LA}(A)$ there is a unique morphism $L \to \Theta_A$.

(ii) Assume that the canonical $*$ pairing $\in P^*_{A2}(\{\Omega_A, \Theta_A\}, A)$ is non-degenerate (see 1.4.2). Then for an A-module M a structure of Θ_A-module on M amounts to that of an integrable connection $M \to \Omega_A \underset{A}{\otimes} M$.

[26]And left exactness of P^*.

Let M be an A-module. One may look for a universal Lie* A-algebroid $\mathcal{E}(M)$ acting on M. If $\mathcal{E}(M)$ exists then $\mathcal{E}(M)^\circ \subset \mathcal{E}(M)$ is the Lie* A-algebra that corresponds to the associative* A-algebra $\mathcal{E}nd\, M$; we denote it $\mathfrak{gl}(M)$.

1.4.15 Coisson algebras. A *compound Poisson*, or simply *coisson, bracket* on $A \in \mathcal{C}omu(\mathcal{M}^!)$ is a Lie* bracket $\{\ \} \in P_2^*(\{A,A\},A)$ such that the adjoint action $\tau_{\{\ \}}$ is an action of A (considered as a Lie* algebra) on A (as on a commutative$^!$ algebra). We call such $(A,\{\ \})$ a *coisson algebra.*

By 1.4.12 (i) a coisson bracket $\{\ \}$ yields a Lie* A-algebroid structure on $A\otimes A$. One checks easily that the kernel of the projection $A\otimes A \to \Omega_A$, $a\otimes b \mapsto adb$, is an ideal in $A\otimes A$, so Ω_A inherits the Lie* A-algebroid structure. In other words, Ω_A carries a unique Lie* A-algebroid structure such that $d\colon A\to\Omega_A$ is a morphism of Lie* algebras and $\tau_{\Omega_A}(d, id_A) = \{\ \}$.

Conversely, a structure of Lie* A-algebroid on Ω_A yields a binary $*$ operation $\{\ \}\colon= \tau_{\Omega_A}(d, id_A)$ on A. If $\{\ \}$ is skew-symmetric and $d\{\ \} = \{\ \}(d,d) \in P_2^*(\{A,A\},\ \Omega_A)$ then $\{\ \}$ is a coisson bracket. Therefore coisson brackets on A are the same as Lie* A-algebroid structures on Ω_A that satisfy the above conditions.

1.4.16. The category of coisson algebras $\mathcal{C}ois(\mathcal{M}^{*!})$ is a tensor category. Namely, for a finite family of coisson algebras $\{(A_i,\{\ \})\}_{i\in I}$, their tensor product is $(\underset{I}{\otimes} A_i, \{\ \})$ where $\{\ \}\colon= \sum\limits_I \{\ \}_i^\sim$ and $\{\ \}_{i_0}^\sim \in P_2^*(\{\underset{I}{\otimes} A_i, \underset{I}{\otimes} A_i\}, \underset{I}{\otimes} A_i)$ is the $\underset{I\smallsetminus\{i_0\}}{\otimes} A_i$-linear extension of $\{\ \}_{i_0}$ (see Remark (ii) in 1.4.6). The algebra $\mathbf{1}$ with zero coisson bracket is a unit object for this tensor structure.

Remark. There are obvious morphisms of coisson algebras $A_i \to \underset{I}{\otimes} A_i$ which mutually coisson commute in the obvious sense. This datum is universal: for any coisson algebra B the set $\mathrm{Hom}(\underset{I}{\otimes} A_i, B)$ is equal to the set of collections (φ_i) of mutually coisson commuting morphisms $\varphi_i :\ A_i \to B$.

1.4.17 Examples. (i) If L is a Lie* algebra then the symmetric$^!$ algebra $\mathrm{Sym}L$ carries a unique coisson bracket $\{\ \}$ such that $L = \mathrm{Sym}^1 L \hookrightarrow \mathrm{Sym}L$ is a morphism of Lie* algebras; this is the Kostant-Kirillov bracket. The functor $\mathcal{L}ie(\mathcal{M}^*) \to \mathcal{C}ois(\mathcal{M}^*), L \mapsto (\mathrm{Sym}L, \{\ \}),$

is left adjoint to the "forgetting of commutative$^!$ algebra structure" functor $\mathcal{C}ois(\mathcal{M}^{*!}) \to \mathcal{L}ie(\mathcal{M}^*)$.

(ii) More generally, if A is a commutative$^!$ algebra and L is a Lie* A-algebroid, then the symmetric$^!$ algebra $\mathrm{Sym}_A L$ of the A-module L carries a unique coisson bracket $\{\ \}$ such that $\{\mathrm{Sym}^p_A L, \mathrm{Sym}^q_A L\} \subset \mathrm{Sym}^{p+q-1}_A L$, and $\{\ \}$ coincides with $[\,]$ on $\mathrm{Sym}^1 \times \mathrm{Sym}^1$ and with τ on $\mathrm{Sym}^1 \times \mathrm{Sym}^0$. For example, $Sym\Theta_A$ is the algebra of functions on the cotangent bundle to $\mathrm{Spec}A$ and $\{\ \}$ is a canonical coisson structure on it.

(iii) The above constructions often come in twisted version. Let L be a Lie* algebra and $L^\flat$ a central extension of L by $\mathbf{1}$. The *twisted symmetric algebra* $\mathrm{Sym}^\flat L$ is the quotient of $\mathrm{Sym}(L^\flat)$ modulo the ideal generated by the image of the difference of embeddings $\mathbf{1} = \mathrm{Sym}^0(L^\flat) \hookrightarrow \mathrm{Sym}(L^\flat)$, $\mathbf{1} \subset L^\flat = \mathrm{Sym}^1(L^\flat) \hookrightarrow \mathrm{Sym}(L^\flat)$. This quotient algebra inherits the coisson bracket from $\mathrm{Sym}(L^\flat)$. Similarly, in the setting of (ii), if $L^\flat$ is a central extension of a Lie* algebroid L by A then the twisted symmetric algebra $\mathrm{Sym}^\flat_A L$ inherits a coisson bracket from $\mathrm{Sym}_A(L^\flat)$.

1.4.18 Hamiltonian reduction. The usual operations on Poisson algebras have obvious coisson versions. For example, if $(A, \{\ \})$ is a coisson algebra and $I \subset A$ is an ideal such that $\{I, I\} \subset I$, then I acts on A/I as a Lie* algebra. Set $B := (A/I)^I$ (:= the maximal I-invariant subobject of A/I; we tacitly assume that it exists). Then B is a ! subalgebra of A/I, and B inherits a coisson bracket $\{\ \}$ such that for $b_1, b_2 \in B$ and $\tilde{b}_i \in A$, $\tilde{b}_i \mathrm{mod} I = b_i$, one has $\{b_1, b_2\} = \{\tilde{b}_1, \tilde{b}_2\} \mathrm{mod} I$. In particular, for $(A, \{\ \}) \in \mathcal{C}ois(\mathcal{M}^{*!})$, $L \in \mathcal{L}ie(\mathcal{M}^*)$, and a morphism of Lie* algebras $\alpha\colon L \to A$, the commutative$^!$ algebra $B\colon = (A/\alpha(L)A)^L$ carries a canonical coisson bracket $\{\quad\}$ (here $\alpha(L)A$ is the ideal generated by $\alpha(L) \subset A$, so L acts on A via α and $\{\ \}$, $A/\alpha(L)A$ carries the quotient action, and $(\ )^L$ is the subobject of L-invariants). We call $(B, \{\ \})$ the *Hamiltonian reduction* of $(A, \{\ \})$.

1.4.19. Let $(A, \{\ \})$ be a coisson algebra. A *coisson* $(A, \{\ \})$*-module*, or simply $(A, \{\ \})$*-module*, is an A-module M (here A is considered as a commutative$^!$ algebra) together with a $\tau_{\{\ \}}$-action of A on M (here A is considered as a Lie* algebra) $\tau_M^{(A)} \in P_2^*(\{A, M\}, M)$ which is a derivation with respect to A-variable. The coisson modules form a category

$\mathcal{M}(A, \{\ \})$.

In fact $\mathcal{M}_{(A,\{\ \})}$ coincides with the category $\mathcal{M}(A, \Omega_A)$ where Ω_A is the Lie* A-algebroid from 1.4.15. The equivalence $\mathcal{M}(A, \Omega_A) \xrightarrow{\sim} \mathcal{M}(A, \{\ \})$ assigns to an Ω_A-module M same M with $\tau_M^{(A)}$ coming from the morphism of Lie* algebras $d: A \to \Omega_A$. Therefore $\mathcal{M}(A, \{\ \})$ carries a canonical augmented compound tensor structure; we denote it $\mathcal{M}(A, \{\ \})^{*!}$.

1.4.20 BRST (coisson version). This is a refined version of Hamiltonian reduction. Below we deal entirely with superobjects following conventions of 1.1.16; the prefix "super" is skipped. If M a graded superobject M we write $h(M)$ for the corresponding graded k-module, and $\hbar(M) := h(M)^{0even} = h(M^{0even})$. So $\hbar$ is the augmentation functor on the plain pseudo-tensor k-category $\mathcal{M}^{s\mathbb{Z}}$.[27] We assume that $k \supset \mathbb{Q}$.

Assume we have $L, L' \in \mathcal{M}^s$ and a $*$ pairing $\langle\ \ \rangle \in P_2^*(\{L', L\}, \mathbf{1})$. Consider $L[1] \oplus L'[-1]$ as a commutative $\mathbb{Z}$-graded Lie* algebra. Let $(L[1] \oplus L'[-1])^\flat = L[1] \oplus L'[-1] \oplus \mathbf{1}$ be its central extension by $\mathbf{1}$ (placed in degree 0) with commutator $\in P_2^*(\{L'[-1], L[1]\}, \mathbf{1}) = P_2^*(\{L', L\}, \mathbf{1})$ equal to $\langle\ \ \rangle$. Set $\mathcal{C}l_c := \mathrm{Sym}^\flat(L'[-1] \oplus L[1])$ (see 1.4.17(iii)).[28] This coisson $\mathbb{Z}$-graded algebra is called the *Clifford coisson algebra*. One has:

(i) There are obvious embeddings $\mathrm{Sym}(L'[-1]), \mathrm{Sym}(L[1]) \hookrightarrow \mathcal{C}l_c$; the product morphism $\mathrm{Sym}(L'[-1] \oplus L[1]) = \mathrm{Sym}(L'[-1]) \otimes \mathrm{Sym}(L[1]) \to \mathcal{C}l_c$ is an isomorphism of graded commutative$^!$ algebras.

(ii) The coisson bracket vanishes on both $L'[-1]$ and $L[1]$; the $\{\}$ pairing between $L'[-1]$ and $L[1]$ equals to $\langle\ \ \rangle$.

(iii) The subobject $L \otimes L' = L[1] \otimes L'[-1] \subset \mathcal{C}l_c^0$ is a Lie* subalgebra. The coisson bracket on it coincides with the bracket that comes from the associative* algebra structure as defined in 1.4.2. This subalgebra normalizes $L[1], L'[-1] \subset \mathcal{C}l_c^{\pm 1}$, and the corresponding adjoint action coincides with the action from 1.4.2.

Below we will need a simple lemma. For $A \in \mathcal{M}^s$ consider the coisson $\mathcal{C}l_c$-module $A \otimes \mathcal{C}l_c$. Let $F_c \subset h(A \otimes \mathcal{C}l_c)$ be the centralizer of $L[1] \subset \mathcal{C}l_c$. In other words, F_c is the

[27] With respect to even operations of degree 0, see 1.1.16.

[28] The lower index "c" is for "coisson" or "classical".

kernel of the morphism $\psi : h(A \otimes \mathcal{C}l_c) \to \mathrm{Hom}(L[1], A \otimes \mathcal{C}l_c)$ that comes from the adjoint action of $L[1]$ on $\mathcal{C}l_c$. It is clear that $F_c \supset h(A \otimes \mathrm{Sym}(L[1]))$.

1.4.21 Lemma. If $\langle\quad\rangle$ is non-degenerate (see 1.4.2) then $F_c = h(A \otimes \mathrm{Sym}(L[1]))$.

Proof. We have $h(A \otimes \mathcal{C}l_c^a) = \oplus\, h(A \otimes \mathrm{Sym}^n(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1]))$. Take any $\nu \in h(A \otimes \mathcal{C}l_c^a)$; let n be the largest number such that the nth component of ν is non-zero; denote it by $\bar{\nu} \in h(A \otimes \mathrm{Sym}^n(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1]))$. The morphism $\chi := \psi(\nu)$ sends $L[1]$ to the degree $a-1$ part of $A \otimes \mathrm{Sym}^{\leq n-1}(L'[-1]) \otimes \mathrm{Sym}(L[1])$; let $\bar{\chi}$ be its composition with the projection to $A \otimes \mathrm{Sym}^{n-1}(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1])$. Thus (see 1.4.2) $\bar{\chi} \in \mathrm{Hom}(L[1], A \otimes \mathrm{Sym}^{n-1}(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1])) = h(A \otimes (L'[-1] \otimes \mathrm{Sym}^{n-1}(L'[-1])) \otimes \mathrm{Sym}^{n-a}(L[1]))$. According to the Leibnitz formula the image of $\bar{\chi}$ by the product map $L'[-1] \otimes \mathrm{Sym}^{n-1}(L'[-1]) \to \mathrm{Sym}^n(L'[-1])$ equals $n\bar{\nu}$. If ν lies in F_c, i.e., $\chi = 0$, then $n\bar{\nu} = 0$, hence $n = 0$. $\square$

1.4.22. ¿From now on L is a Lie* algebra. We assume that the $*$ dual $L' := \mathcal{H}om(L, \mathbf{1})$ exists and the canonical pairing $\langle\quad\rangle \in P_2^*(\{L', L\}, \mathbf{1})$ is non-degenerate (see 1.4.2). Then the adjoint action of L on L, hence on $L[1]$, yields a morphism of Lie* algebras $\beta : L \to L \otimes L' = L[1] \otimes L'[-1] \subset \mathcal{C}l_c^0$. We may consider β as an element of $\bar{h}(\mathcal{C}l_c^0 \otimes L')$. Denote by $\tilde{\beta}_c \in \bar{h}(\mathcal{C}l_c[1]) = h(\mathcal{C}l_c^{1odd})$ its image by the product map $\mathcal{C}l_c \otimes L'[-1] \to \mathcal{C}l_c$.

Let A be a coisson algebra, and $\alpha : L \to A$ a morphism of Lie* algebras. Consider the graded coisson algebra $A \otimes \mathcal{C}l_c^{\cdot}$. Let $L_\dagger$ be the DG Lie* algebra from 1.4.5. We have a morphism of graded Lie* algebras

$$\ell : L_\dagger \to A \otimes \mathcal{C}l_c \tag{1.4.22.1}$$

where $\ell^0 = \alpha + \beta : L \to A \otimes \mathcal{C}l^0$, ℓ^{-1} is the composition $L[1] \hookrightarrow \mathcal{C}l^{-1} \to A \otimes \mathcal{C}l^{-1}$.

Let us rewrite α as $\tilde{\alpha} \in h(A \otimes L')^{even} \subset \bar{h}(A \otimes \mathcal{C}l_c[1])$. Let $\tilde{\beta} \in \bar{h}(A \otimes \mathcal{C}l_c[1])$ be the image of $\tilde{\beta}_c$ by the morphism of algebras $\mathcal{C}l_c \to A \otimes \mathcal{C}l_c$. Set

$$\mathfrak{d}_c := \tilde{\alpha} + 1/2\,\tilde{\beta} \in \bar{h}(A \otimes \mathcal{C}l_c[1]). \tag{1.4.22.2}$$

This is *the (classical) BRST current.* Its adjoint action is an odd derivation d of degree 1 of the graded coisson algebra $A \otimes \mathcal{C}l_c$ called *the BRST differential.*

1.4.23 Proposition. (i) One has $\{\mathfrak{d}_c, \mathfrak{d}_c\} = 0$, so $d^2 = 0$.

We denote the DG coisson algebra $(A \otimes \mathcal{C}l_c, d)$ by $\mathcal{C}_{\mathrm{BRST}}\ (L, A)_c$.

(ii) ℓ commutes with the differentials, so we have a morphism of DG Lie* algebras

$$\ell : L_\dagger \to \mathcal{C}_{\mathrm{BRST}}\ (L, A)_c. \tag{1.4.23.1}$$

We refer to this as *the BRST property* of $\mathfrak{d}$ or d. It determines $\mathfrak{d}$ and d uniquely:[29]

(iii) $\mathfrak{d}_c$ is a unique element of $h(A \otimes \mathcal{C}l_c[1])$ such that $[\mathfrak{d}_c, \ell^{-1}]$ equals ℓ^0, and d is a unique derivation of $A \otimes \mathcal{C}l$ of degree 1 such that $d\ell^{-1} = \ell^0$.

Proof. (i), (ii) are straightforward computations. (iii) follows from the next lemma. Let Der_c be the graded Lie algebra of derivations of the coisson algebra $A \otimes \mathcal{C}l_c$.

1.4.24 Lemma. The maps $h(A \otimes \mathcal{C}l_c^a) \to \mathrm{Der}_c^a \to \mathrm{Hom}(L^\Pi, A \otimes \mathcal{C}l_c^{a-1})$,[30] where the first arrow is the adjoint action, the second one is $\delta \mapsto \delta\ell^{-1}$, are injective for $a > 0$.

Proof. (a) Let δ be a coisson derivation of $A \otimes \mathcal{C}l_c$ of degree $a > 0$ such that $\delta\ell^{-1} = 0$, i.e., δ kills $L[1] \subset \mathcal{C}l_c$. Then it sends $L'[-1]$ and A to the centralizer of $L[1]$ in, respectively, $A \otimes \mathcal{C}l_c^{a+1}$ and $A \otimes \mathcal{C}l_c^a$, which is zero by 1.4.21. Thus $\delta = 0$.

(b) The kernel of the first arrow is contained in the centralizer of $L[1]$; by 1.4.21 it vanishes if $a > 0$. □

Considered as an object of an appropriate derived category $\mathcal{C}_{\mathrm{BRST}}(L, A)_c$ is called the *BRST reduction of* A with respect to α. Its cohomology form a graded coisson algebra $H^{\cdot}_{\mathrm{BRST}}(L, A)_c$. The reduction is *regular* if $H^{\neq 0}_{\mathrm{BRST}}(L, A)_c = 0$. Then the BRST reduction is a plain coisson algebra $H^0_{\mathrm{BRST}}(L, A)_c$.

1.4.25. Let us compare the BRST and Hamiltonian reductions (see 1.4.18). Set $\bar{A} := A/\alpha(L)A$. This is a commutative$^!$ algebra; the L-action on A (via α and the coisson bracket) yields an L-action on $\bar{A}$. Let I be the DG ideal of $\mathcal{C}_{\mathrm{BRST}}(L, A)_c$ generated by the image of $\ell^{-1} : L[1] \to A \otimes \mathcal{C}l_c^{-1}$. The morphism of graded algebras $A \otimes \mathrm{Sym}(L'[-1]) \to A \otimes \mathcal{C}l_c/I$ is surjective; by 1.4.23 its kernel is the ideal generated

[29] This is a version of a theorem of Akman [A].

[30] For a $\mathcal{D}$-module M M^Π is a copy of M with the opposite pairity. The Hom is a super k-module.

by $\alpha(L)$. Thus $\mathcal{C}_{\mathrm{BRST}}(L,A)_c/I = \bar{A}\otimes \mathrm{Sym}(L'[-1])$. One checks immediately that the BRST differential yields on $\bar{A}\otimes\mathrm{Sym}(L'[-1])$ the usual Chevalley differential (see 1.4.9), so we have a canonical surjective morphism of DG commutative$^!$ algebras

$$\mathcal{C}_{\mathrm{BRST}}(L,A)_c \twoheadrightarrow \mathcal{C}(L,\bar{A}). \tag{1.4.25.1}$$

Passing to zero cohomology we get a canonical morphism of commutative algebras

$$H^0_{\mathrm{BRST}}(L,A)_c \to \bar{A}^L. \tag{1.4.25.2}$$

It is easy to see that it is compatible with the coisson brackets.

1.4.26. The compound tensor structure on $\mathcal{M}$ can be encoded in a single pseudo-tensor structure $\mathcal{M}^c$. Its construction below looks pretty much ad hoc at the moment; in 3.2 we will see that it presents a "classical limit" of chiral structure.

So let us bring the sundry polylinear operations of a compound tensor category under one roof. For an I-family $L_i\in\mathcal{M}$, $M\in\mathcal{M}$, and $S\in Q(I)$ (see 1.3.1) set

$$P^c_I(\{L_i\},M)_S := P^*_S(\{\underset{i\in I_s}{\otimes} L_i\},M)\otimes(\underset{s\in S}{\otimes}\mathcal{L}ie_{I_s}). \tag{1.4.26.1}$$

The k-module of c *operations* is

$$P^c_I(\{L_i\},M) := \underset{S\in Q(I)}{\oplus} P^c_I(\{L_i\},M)_S. \tag{1.4.26.2}$$

The c operations compose in a natural manner. Namely, assume we have $J\twoheadrightarrow I$, and a J-family of objects $K_j\in\mathcal{M}$. Take $\varphi\in P^c_I(\{L_i\},M)_S$, $\psi_i\in P^c_{J_i}(\{K_j\},M)_{T_i}$ (where $T_i\in Q(J_i)$). For $R\in Q(J)$ the R-component $\varphi(\psi_i)_R\in P^c_J(\{K_j\},M)_R$ vanishes unless $R\leq T$,[31] I and R are transversal in $Q(T)$, and $\inf\{I,R\}=S$ (see 1.3.1). If these conditions are satisfied then $\varphi(\psi_i)_R=\varphi(\gamma_s)$ where for $s\in S$ one has[32] $\gamma_s := \otimes^{T_s}_{I_s,R_s}\psi_i \in P^*_{R_s}(\{\underset{j\in J_r}{\otimes}K_j\},\underset{i\in I_s}{\otimes}L_i)\otimes(\underset{t\in T_s}{\otimes}\mathcal{L}ie_{J_t})$ and the composition $\varphi(\gamma_s)$ is the tensor product of compositions of $*$ and Lie operations.[33] We leave it to the reader to check its associativity.

[31] Here $T:=\sqcup T_i\in Q(J)$.

[32] See 1.3.12 for notation.

[33] The composition of Lie operations is the tensor product of maps $\mathcal{L}ie_{I_s}\otimes(\underset{t\in T_r}{\otimes}\mathcal{L}ie_{J_t})\to\mathcal{L}ie_{J_r}$ labeled by $r\in R_s$. We use the fact that $T_r \xrightarrow{\sim} I_s$ (the transversality condition).

Therefore we defined an abelian[34] pseudo-tensor structure $\mathcal{M}^c$ on $\mathcal{M}$.

We define a natural $\mathbb{Z}$-grading on P^c as $P^c_I(\{L_i\},M)^n := \underset{S\in Q(I,|I|-n)}{\oplus} P^c_I(\{L_i\},M)_S$ where $Q(I,m)\subset Q(I)$ consists of quotients $I \twoheadrightarrow S$, $|S|=m$. Composition of c operations is compatible with the grading.

Notice that $P^*_I(\{L_i\},M) = P^c_I(\{L_i\},M)^0$ and $\mathrm{Hom}(\underset{I}{\otimes} L_i, M)\otimes \mathcal{L}ie_I = P^c_I(\{L_i\},M)^{|I|}$. The corresponding embedding and projection define the maps $P^! \otimes \mathcal{L}ie \to P^c \to P^*$ which are compatible with composition of operations, so we have the pseudo-tensor functors

$$\mathcal{M}^! \otimes \mathcal{L}ie \to \mathcal{M}^c \to \mathcal{M}^* \tag{1.4.26.3}$$

that extend the identity functor on $\mathcal{M}$.

Remark. It follows from 1.1.12 that the compound pseudo-tensor structure $\mathcal{M}^{*!}$ is uniquely determined by the pseudo-tensor structure $\mathcal{M}^c$ together with the $\mathbb{Z}$-grading on P^c.

1.4.27 Lemma. Lie algebras in $\mathcal{M}^c$ are the same as coisson algebras without unit in $\mathcal{M}^{*!}$.

Proof. A binary c operation $\mu \in P^c_2(M)$ is the same as a pair $(\cdot, \{\ \ \})$ where[35] $\cdot \in \mathrm{Hom}(M\otimes M, M)$ and $\{\ \ \}\in P^*_2(M)$. The skew-symmetry of μ amounts to symmetry of $\cdot$ and skew-symmetry of $\{\ \ \}$. The Jacobi identity for μ amounts to three identities (corresponding to the graded components):

– the Jacobi identity for $\cdot\otimes[\ \ ]$ which is the same as associativity of $\cdot$ (see 1.1.10),

– the Jacobi identity for $\{\ \ \}$,

– the identity which says that $\{\ \ \}$ acts by derivations of $\cdot$. □

[34] See 1.1.7.

[35] We use the identification $k \xrightarrow{\sim} \mathcal{L}ie_2$, $1\mapsto [\ \ ]$.

CHAPTER TWO: GEOMETRY OF $\mathcal{D}$-SCHEMES

Ломиться в двери пошлых аксиом,
Где лгут слова и красноречье храмлет?..

Б. Л. Пастернак, "Брюсову", 1926.[†]

In the following chapters the reader can understand "$\mathcal{D}$-module" as "super $\mathcal{D}$-module".[36]

§1 $\mathcal{D}$-modules: recollections and notation

Below k denotes our base field of characteristic zero. All schemes, algebras, etc., live over k.

2.1.1. The details for 2.1.1 - 2.1.6 may be found in, say, [B]. For a scheme X we denote by $\mathcal{M}_{\mathcal{O}}(X)$ the category of $\mathcal{O}$-modules on X ($:=$ quasicoherent sheaves of $\mathcal{O}_X$-modules). If X is smooth then $\mathcal{M}(X) = \mathcal{M}^r(X)$ (resp. $\mathcal{M}^\ell(X)$) denotes the category of right (resp. left) $\mathcal{D}$-modules on X ($:=$ sheaves of $\mathcal{D}_X$-modules quasicoherent as $\mathcal{O}_X$-modules). We usually deal with right $\mathcal{D}$-modules, and call them simply $\mathcal{D}$-modules. One has the obvious forgetting of $\mathcal{D}$-action functors $o^r\colon \mathcal{M}(X) \to \mathcal{M}_{\mathcal{O}}(X)$, $o^\ell\colon \mathcal{M}^\ell(X) \to \mathcal{M}_{\mathcal{O}}(X)$.

If G, L are left $\mathcal{D}$-modules, then $G \otimes L = G \underset{\mathcal{O}_X}{\otimes} L$ is a left $\mathcal{D}$-module in a natural way. Therefore $\mathcal{M}^\ell(X)$ is a tensor category with a unit object $\mathcal{O}_X$, and o^ℓ is a tensor functor.

[†] Breaking into the doors of vulgar axioms
Where words are lying and the eloquence is limping?..
B.L. Pasternak, "To Bryusov", 1926.
We owe this motto to A. Polishchuk. Unfortunetely, the English rendition is poor:
(a) Russian "poshlyi" is not the same as "vulgar"; see Nabokov, "Strong Opinions", pp 100–101.
(b) the absence of an English equivalent for the used impossible conjugation form of the verb "limp" making it a derivative of the noun "temple".

[36] See 1.1.16.

If M is a right $\mathcal{D}$-module, then $L \otimes M = L \underset{\mathcal{O}_X}{\otimes} M$ is a right $\mathcal{D}$-module (a field $\tau \in \Theta_X \subset \mathcal{D}_X$ acts on $L \otimes M$ as $(\ell \otimes m)\tau = \ell \otimes m\tau - \tau\ell \otimes m$). The sheaf $\omega_X = \Omega_X^{\dim X}$ has a canonical right $\mathcal{D}$-module structure (namely, $\nu\tau = -\mathcal{L}ie_\tau(\nu)$ for $\nu \in \omega_X$). The functor $\mathcal{M}^\ell(X) \to \mathcal{M}(X)$, $L \mapsto L^r := L\omega_X = L \otimes \omega_X$, is an equivalence of categories. The inverse equivalence is $M \mapsto M^\ell := M \otimes \omega_X^{-1}$.

We call the identification of derived categories

$$D\mathcal{M}^\ell(X) \xrightarrow{\sim} D\mathcal{M}^r(X), \qquad L \mapsto L\omega_X[\dim X], \tag{2.1.1.1}$$

the canonical equivalence. So $D^b\mathcal{M}(X)$ carries two standard t-structures, $\mathcal{M}^r$- and $\mathcal{M}^\ell$-ones, that differ by shift by $\dim X$. We denote the composition $D^b\mathcal{M}(X) = D^b\mathcal{M}^\ell(X) \xrightarrow{o^\ell} D^b\mathcal{M}_\mathcal{O}(X)$ simply o^ℓ.

2.1.2. A morphism $f: X \to Y$ of smooth varieties yields canonical functors

$$D\mathcal{M}(X) \underset{f^!}{\overset{f_*}{\rightleftarrows}} D\mathcal{M}(Y)$$

that satisfy standard compatibilities (the base change property, adjunction property for proper f, etc.).

If f is quasi-finite then f_* is left exact, and if f is affine then f_* is right exact with respect to the $\mathcal{M}^r$ t-structure.

The diagram of functors

$$\begin{array}{ccc} D\mathcal{M}_\mathcal{O}(X) & \xleftarrow{Rf^!} & D\mathcal{M}_\mathcal{O}(Y) \\ \uparrow o^r & & \uparrow o^r \\ D\mathcal{M}(X) & \xleftarrow{f^!} & D^b\mathcal{M}(Y) \\ \downarrow o^\ell & & \downarrow o^\ell \\ D\mathcal{M}_\mathcal{O}(X) & \xleftarrow{Lf^*} & D\mathcal{M}_\mathcal{O}(Y) \end{array} \tag{2.1.2.1}$$

commutes; here $Rf^!$, Lf^* are standard pull-back functors for $\mathcal{O}$-modules (recall that for any $F \in D\mathcal{M}_\mathcal{O}(Y)$ one has $(Lf^*F) \otimes \omega_X[\dim X] = Rf^!(F \otimes \omega_Y[\dim Y])$). In particular,

$f^!$ is right exact with respect to the $\mathcal{M}^\ell$ t-structure. We often denote the corresponding functor $\mathcal{M}^\ell(Y) \to \mathcal{M}^\ell(X)$ by f^*.

If f is a smooth morphism then $f^!$ is exact with respect to the $\mathcal{M}^\ell$ t-structure. The left $\mathcal{D}$-modules satisfy smooth descent property, so the categories $\mathcal{M}^\ell(U)$, U is a smooth X-scheme, form a sheaf of abelian categories on the smooth topology X_{sm}, which we denote $\mathcal{M}^\ell(X_{sm})$. One defines the category $\mathcal{M}^\ell(Z)$ for any smooth algebraic stack Z in a standard way.

2.1.3. If $i: X \hookrightarrow Y$ is a closed embedding then i_* is exact with respect to the $\mathcal{M}^r$ t-structure, and its right adjoint $i^!$ is left exact. They define mutually inverse equivalences between $\mathcal{M}(X)$ and the full subcategory $\mathcal{M}(Y)_X \subset \mathcal{M}(Y)$ that consists of $\mathcal{D}_Y$-modules that vanish on $Y \setminus X$ (Kashiwara's lemma).

The above functors can be described expilicitely as follows. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal of X. If $N \in \mathcal{M}(Y)$ then $i^! N$ coincides, as an $\mathcal{O}$-module, with the subsheaf of N that consists of sections killed by $\mathcal{I}$. For $M \in \mathcal{M}(X)$ one has $i_* M = M \underset{\mathcal{D}_X}{\otimes} (\mathcal{D}_Y / \mathcal{J}\mathcal{D}_Y)$. The identification $M \xrightarrow{\sim} i^! i_* M$ is $m \mapsto m \otimes 1$.

For $L \in \mathcal{M}^\ell(Y)$ and $M \in \mathcal{M}(X)$ there is a canonical isomorphism

$$i_*(i^* L \otimes M) \xrightarrow{\sim} L \otimes i_* M. \tag{2.1.3.1}$$

The corresponding isomorphism $i^* L \otimes M \xrightarrow{\sim} i^!(L \otimes i_* M)$ sends $l \otimes m$ to $l \otimes (m \otimes 1)$.

2.1.4. If X_i, $i \in I$, is a finite collection of smooth schemes then we have the exterior tensor product functors for both left and right $\mathcal{D}$-modules $\prod \mathcal{M}^r(X_i) \to \mathcal{M}^r(\prod X_i)$, $\prod \mathcal{M}^\ell(X_i) \to \mathcal{M}^\ell(\prod X_i)$, $(M_i) \mapsto \boxtimes M_i$. The canonical equivalence (2.1.1.1) identifies the corresponding derived functors.[37] The functors f_*, $f^!$ are compatible with $\boxtimes$: if $f_i: X_i \to Y_i$ are morphisms of smooth schemes then $(\prod f_i)_*(\boxtimes M_i) = \boxtimes(f_{i*} M_i)$, $(\prod f_i)^!(\boxtimes N_i) = \boxtimes(f_i^! N_i)$.

Assume that all X_i coincide with X; let $\Delta^{(I)}: X \to X^I$ be the diagonal embedding. For

[37] Notice that there is a canonical isomorphism $\omega_{\Pi X_i}[\dim \Pi X_i] \underset{\sim}{\longrightarrow} \boxtimes \omega_{X_i}[\dim X_i]$; in particular, it does not depend on the ordering of I.

$L_i \in D\mathcal{M}^\ell(X)$ one has a canonical isomorphism

$$\overset{L}{\otimes} L_i = \Delta^{(I)!}(\boxtimes L_i) \tag{2.1.4.1}$$

2.1.5. If X is affine then projectivity of a $\mathcal{D}_X$-module is a local property for Zariski or étale topology of X (see theorem 2.3.7 below where this fact is proven in a more general situation). For arbitrary X we call a locally projective $\mathcal{D}_X$-module of finite rank a $\mathcal{D}_X$-*vector bundle* or a $\mathcal{D}$-*vector bundle on* X. Notice that a $\mathcal{D}_X$-vector bundle need not be a locally free $\mathcal{D}_X$-module as shows the following example.[38]

Example. Assume that X is an affine curve, and let $V \subset \mathcal{D}_X$ be a $\mathcal{D}$-submodule such that V equals $\mathcal{D}_X$ outside of a point $x \in X$. Then V is a projective $\mathcal{D}_X$-module (since the category of $\mathcal{D}$-modules has homological dimension 1). According to 2.1.13 below, such V's are the same as open *k-vector subspaces* of the formal completion O_x of the local ring $\mathcal{O}_x$.

If V is locally free then (shrinking X if necessary) we can assume that it is generated by a single section ϕ, i.e., $V = \phi\mathcal{D}_X \subset \mathcal{D}_X$. Since $\phi\mathcal{D}_X = \mathcal{D}_X$ on $X \smallsetminus \{x\}$, our ϕ is a function, i.e., $V = t^n\mathcal{D}_X$ for some $n \geq 0$ (where t is a parameter at x). Therefore locally free V's correspond to open *ideals* in O_x.

2.1.6. For $M \in \mathcal{M}(X)$ set $h(M) := M \underset{\mathcal{D}_X}{\otimes} \mathcal{O}_X = M/M\Theta_X$; this is a sheaf on X_{Zar}. For a local section $m \in M$ we denote by $\overline{m}$ its class in $h(M)$. Note that $h(M)$ is a presheaf on étale topology in the obvious way. In fact, it is a sheaf on $X_{\text{ét}}$. Denote by $DR(M)$ the de Rham complex of M, so $DR(M)^i := M \underset{\mathcal{O}_X}{\otimes} \bigwedge^{-i} \Theta_X$. One has $h(M) = H^0DR(M)$; in fact, as an object of derived category of arbitrary sheaves on X_{Zar} or $X_{\text{ét}}$, $DR(M)$ equals to $M \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X$. Set $\Gamma_{DR}(X, M) := \Gamma(X, DR(M))$, $R\Gamma_{DR}(X, M) := R\Gamma(X, DR(M))$.

For a morphism $f: X \to Y$ of smooth schemes there is a canonical isomorphism $Rf_{\cdot}(DR(M)) \xrightarrow{\sim} DR(f_*M)$ in the derived category of sheaves on Y; here $f_{\cdot}$ is the sheaf-theoretical direct image, and $Rf_{\cdot}$ is its derived functor. For a more precise setting for this quasiisomorphism see 2.1.11. Two particular cases:

[38]We learned it from V.Ginzburg's talk about the work of Wilson on Calogero-Moser equation.

(a) The global de Rham cohomology coincide with the push-forward by projection $\pi: X \to$ (point): one has $H^{\cdot}_{DR}(X, M) := H^{\cdot}(X, DR(M)) = H^{\cdot}\pi_* M$.

(b) If $i: X \hookrightarrow Y$ is a closed embedding then the above quasiisomorphism arises as a canonical embedding of complexes $i.DR(M) \hookrightarrow DRi_*(M)$ defined by the obvious embedding of $\mathcal{O}_Y$-modules $i.M \hookrightarrow i_*M$. Thus $i.h(M) = h(i_*M)$.

Notice also that in situation (b) for $N \in D\mathcal{M}(Y)$ the obvious morphism $DR(Ri^! N) \to Ri^! DR(N)$ is a quasiisomorphism.[39]

Lemma. (i) If M is a non-zero *coherent* $\mathcal{D}_X$-module then $h(M) \neq 0$.

(ii) If M is a coherent $\mathcal{D}_X$-module with no O_X-torsion then for every $\mathcal{D}_X$-module L the map $\mathrm{Hom}(L, M) \to \mathrm{Hom}(h(L), h(M))$ is injective.

Proof. (i) (a) Using induction by the dimension of the support of M we can assume, by (b) above, that M is non-zero at the generic point of X. Replacing X by its open subset we can assume that M is a free $\mathcal{O}_X$-module.[40]

(b) Let us show that for any (non necessary coherent) non-zero $\mathcal{D}_X$-module M which is free as an $\mathcal{O}_X$-module one has $h(M) \neq 0$. We use induction by $\dim X$. Replacing X by its open subset we can find a smooth hypersurface $i : Y \hookrightarrow X$ such that the determinant of the normal bundle to Y is trivial. Let $j : U \hookrightarrow X$ be its complement. Set $M_Y := i^!(j_*j^*M/M) = (i * M^{\ell})^r$; this is a non-zero $\mathcal{D}$-module on Y which is free as an $\mathcal{O}_Y$-module by the condition on Y. Hence $h(M_Y) \neq 0$ by the induction hypothesis. Notice that $j.j^{\cdot}h(M) = h(j_*j^*M)$ and $h(i_*M_Y) = i.h(M_Y)$. Since h is right exact and i_*M_Y is a quotient of j_*j^*M we see that $h(M) \neq 0$.

(ii) (a) Since h is right exact it suffices to show that for every non-zero $L \subset M$ the map $h(L) \to h(M)$ is non-zero. The condition on M assures that we can replace X by any open subset, so we can assume that L, M, and M/L are free $\mathcal{O}_X$-modules.

(b) Let us show that for every (non-necessary coherent) non-zero $\mathcal{D}_X$-modules $L \subset M$ such that L, M, M/L are free $\mathcal{O}_X$-modules the map $h(L) \to h(M)$ is non-zero. We use

[39] Use the standard exact triangle $i_*Ri^!N \to N \to Rj_*j^*N \to i_*Ri^!N[1]$ where $j : Y \setminus i(X) \hookrightarrow Y$.

[40] Indeed, choose a good filtration on M, then by EGA IV 6.9.2 $\mathrm{gr}M$ is $\mathcal{O}_X$-free over a non-empty open $V \subset X$.

induction by $\dim X$ and follow notation from (i)(b) above. It suffices to show that the map $h(j_*j^*L) \to h(j_*j^*M)$ is non-zero. It admits as a quotient the map $h(L_Y) \to h(M_Y)$ which is non-zero by induction by $\dim X$. □

2.1.7. The details for 2.1.7 – 2.1.13 may be found in [S]. Let X be a smooth scheme. For an $\mathcal{O}$-module B on X we have the induced right $\mathcal{D}$-module $B_{\mathcal{D}} := B \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \in \mathcal{M}(X)$. The de Rham complex $DR(B_{\mathcal{D}})$ is acyclic in degrees $\neq 0$, and the projection $DR(B_{\mathcal{D}})^0 = B \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to B$, $b \otimes \partial \mapsto b\partial(1)$, induces the isomorphism $h(B_{\mathcal{D}}) \xrightarrow{\sim} B$. In other words, there is a canonical quasiisomorphism

$$\nu_B\colon DR(B_{\mathcal{D}}) \to B. \tag{2.1.7.1}$$

If C is another $\mathcal{O}$-module then for any $\varphi \in \operatorname{Hom}_{\mathcal{M}(X)}(B_{\mathcal{D}}, C_{\mathcal{D}})$ the corresponding map $h(\varphi)\colon h(B_{\mathcal{D}}) \to h(C_{\mathcal{D}})$ is a differential operator $B \to C$. In fact, the map

$$h\colon \operatorname{Hom}(B_{\mathcal{D}}, C_{\mathcal{D}}) \to \operatorname{Diff}(B, C), \tag{2.1.7.2}$$

where Diff is the vector space of all differential operators, is an isomorphism. The inverse map assigns to a differential operator $\psi\colon B \to C$ the morphism $\psi_{\mathcal{D}}\colon B_{\mathcal{D}} \to C_{\mathcal{D}}$, $b \otimes \partial \mapsto \psi_b \partial$, where $\psi_b \in C \otimes \mathcal{D}_X = \operatorname{Diff}(\mathcal{O}_X, C)$ is the differential operator $\psi_b(f) = \psi(bf)$.

Let $\mathcal{D}iff(X)$ be the category whose objects are $\mathcal{O}$-modules on X and morphisms are differential operators. We see that the induction $B \mapsto B_{\mathcal{D}}$ defines a fully faithful functor $\mathcal{D}iff(X) \hookrightarrow \mathcal{M}(X)$.

If N is a left $\mathcal{D}$-module then for any $\mathcal{O}$-module B one has a canonical isomorphism of $\mathcal{D}$-modules $(N \underset{\mathcal{O}_X}{\otimes} B)_{\mathcal{D}} \xrightarrow{\sim} N \otimes B_{\mathcal{D}}$, $(n \otimes b) \otimes \partial \mapsto (n \otimes b)\partial$. Therefore $N, B \mapsto N \otimes B := N \underset{\mathcal{O}_X}{\otimes} B$ is an action of the tensor category $\mathcal{M}^{\ell}(X)$ on $\mathcal{D}iff(X)$; the induction functor $\cdot_{\mathcal{D}}$ commutes with $\mathcal{M}^{\ell}(X)$-actions.[41]

2.1.8 Examples. (i) Let E be an $\mathcal{O}$-module on X^n, $n \geqslant 2$, that vanishes outside the diagonal $\Delta\colon X \hookrightarrow X^n$. We may consider E as a sheaf on X equipped with n $\mathcal{O}_X$-module structures. The identity map $id_E\colon E \to E$ is a differential operator if the two copies of E

[41] Here $\mathcal{M}^{\ell}(X)$ acts on $\mathcal{M}(X)$ by the usual tensor product.

are considered with any two of the above $\mathcal{O}_X$-module structures. So E is a well-defined object of $\mathcal{D}iff(X)$. The corresponding $\mathcal{D}$-module coincides with $\Delta^!(E \underset{\mathcal{O}_{X^n}}{\otimes} \mathcal{D}_{X^n})$.

(ii) For a right $\mathcal{D}$-module M its de Rham complex $DR(M)$ is a complex in $\mathcal{D}iff(X)$. If L is a left $\mathcal{D}$-module then $DR(L \otimes M) = L \otimes DR(M)$.

(iii) Let $i: X \hookrightarrow Y$ be a closed embedding. For B as above consider the canonical embeddings of $\mathcal{O}_Y$-modules $i.B \hookrightarrow i.(B \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X) \hookrightarrow i_*(B_{\mathcal{D}})$. The composition induces a canonical morphism of $\mathcal{D}_Y$-modules γ: $(i.B)_{\mathcal{D}} \to i_*(B_{\mathcal{D}})$ which is an isomorphism.

2.1.9. For any $M \in \mathcal{M}(X)$ the de Rham complex $DR(M)$ is a complex in $\mathcal{D}iff(X)$ (the de Rham differential is a first order differential operator), so we have the corresponding complex $DR(M)_{\mathcal{D}}$ of $\mathcal{D}$-modules. It is acyclic in degrees $\neq 0$, and the projection $DR(M)^0_{\mathcal{D}} = M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to M$, $m \otimes \partial \mapsto m\partial$, defines an isomorphism $H^0 DR(M)_{\mathcal{D}} \xrightarrow{\sim} M$. In other words we have a canonical quasiisomorphism

$$\mu_M\colon DR(M)_{\mathcal{D}} \to M. \tag{2.1.9.1}$$

2.1.10. One may replace modules by complexes of those to get the functors between the DG categories of complexes

$$DR\colon C\mathcal{M}(X) \to C\mathcal{D}iff(X), \qquad \cdot_{\mathcal{D}}\colon C\mathcal{D}iff(X) \to C\mathcal{M}(X) \tag{2.1.10.1}$$

together with natural morphisms ν, μ as in 2.1.7.1, 2.1.9.1.

The category $\mathcal{D}iff(X)$ is not abelian. Following Saito, a morphism ψ: $B \to C$ of complexes in $\mathcal{D}iff(X)$ is called a quasiisomorphism if such is $\psi_{\mathcal{D}}$. The derived category $D^b\mathcal{D}iff(X)$ is defined by inverting quasiisomorphisms.

Remark. A morphism $\psi : B \to C$ in $C\mathcal{D}iff(X)$ is called a *naive quasiisomorphism* if it is a quasiisomorphism of complexes of plain sheaves of vector spaces. It is clear that ψ is a quasiisomorphism if and only if for any $\mathcal{O}$-flat left $\mathcal{D}$-module N the morphism $id_N \otimes \psi : N \otimes B \to N \otimes C$ is a naive quasiisomorphism. If both B, C are bounded above complexes of *coherent* $\mathcal{O}$-modules then every naive quasiisomorphism ψ is necessarily a quasiisomorphism.[42] For general quasicoherent B, C this may be false (for example, take $B = DR(M)$ where M is a constant sheaf equal to the field of fractions of $\mathcal{D}_X$, $C = 0$).

Both μ and ν are quasiisomorphisms. They are compatible in the following sense: for any $B^{\cdot} \in K\mathcal{D}iff(X)$, $M^{\cdot} \in K\mathcal{M}(X)$ one has $DR(\mu_M) = \nu_{DR(M)}: DR(DR(M)_{\mathcal{D}}) \to DR(M)$, $(\nu_B)_{\mathcal{D}} = \mu_{B_{\mathcal{D}}}: DR(B_{\mathcal{D}})_{\mathcal{D}} \to B_{\mathcal{D}}$. Therefore DR and $\cdot_{\mathcal{D}}$ define mutually inverse equivalences between the derived categories:

$$DM(X) \overset{\sim}{\leftrightarrows} D\mathcal{D}iff(X). \tag{2.1.10.2}$$

For a morphism of schemes $f\colon X \to Y$ there is an obvious sheaf-theoretic push-forward functor $Rf.\colon D\mathcal{D}iff(X) \to D\mathcal{D}iff(Y)$. Equivalence (2.1.10.2) identifies it with f_*: there are canonical isomorphisms

$$DR(f_*M) = Rf.DR(M), \qquad f_*(B_{\mathcal{D}}) = (Rf.(B))_{\mathcal{D}}. \tag{2.1.10.3}$$

See particular cases in 2.1.8(iii), 2.1.6.

2.1.11 Remark. There seems to be no general way to find out whether a given D-module is induced or not. E.g. for an induced M its direct summand need not be induced (as shows the example from 2.1.5). A more handy category is that of *quasi-induced* $\mathcal{D}$-modules. One defines them as follows.

Let x be a (not-necessarily closed) point of X, $i_x : x \to X$ the embedding, and C an object of the derived category of $\mathcal{D}$-modules on X. Then $i_x^! C$ is defined to be the fiber of $i_U^! C$ over x where $U \subset X$ is a smooth locally closed subvariety containing x and i_U is the embedding $U \to X$. So $i_x^! C$ belongs to the derived category of $\mathcal{D}_x$-modules where $\mathcal{D}_x$ is the ring of differential operators on x. Let us say that C is *strongly non-negative* if for every $x \in X$ the Tor-dimension of $i_x^! C$ is non-negative. Equivalently, C is strongly non-negative if and only if for every $x \in X$ the Tor-dimension of the stalk C_x of C at x is $\leq \operatorname{codim} x$. It is easy to see that strong non-negativity implies the usual one (i.e., $H^i C = 0$

[42] Let M be the top non-zero cohomology of the cone of $\psi_{\mathcal{D}}$. Then M is a coherent $\mathcal{D}$-module such that $h(M) = 0$. We are done by the lemma from 2.1.6.

for $i < 0$) and that strong non-negativity is preserved under direct images with respect to arbitrary morphisms.

Remark. Assume that $k = \mathbb{C}$. A holonomic complex with regular singularities is strongly non-negative if and only if the complex of (usual, not perverse) sheaves corresponding to its the Verdier dual is non-positive.

We say that a $\mathcal{D}$-module on X is *quasi-induced* if it is strongly non-negative in the above sense. An induced $\mathcal{D}$-module is quasi-induced. The direct image of a quasi-induced $\mathcal{D}$-module under an affine morphism is quasi-induced.

2.1.12. Assume that $X \neq \emptyset$ is connected of dimension n.

We say that $M \in \mathcal{M}(X)$ is *constant* if M^ℓ is generated by global horizontal sections. The category of constant $\mathcal{D}$-modules is closed under subquotients. It identifies canonically with the category of vector spaces by means of the functors $M \mapsto \mathrm{Hom}(\mathcal{O}_X, M^\ell) = \Gamma^\nabla(X, M^\ell)$, $V \mapsto V \otimes \omega_X$.[43]

Assume that X is compact. Then every $M \in \mathcal{M}(X)$ admits the maximal constant quotient module $\pi : M \twoheadrightarrow M_{const}$.[44] The functor $M \mapsto M_{const}$ is right exact.

The adjunction formula for the projection from X to point (see 2.1.2) yields a canonical identification $M^\ell_{const} \xrightarrow{\sim} H^n_{DR}(X, M) \otimes \mathcal{O}_X$. For any point $s \in X$ the composition $H^n Ri^!_s M = M^\ell_s \xrightarrow{\pi_s} (M_{const})_s = H^n_{DR}(X, M)$ equals the n-th de Rham cohomology map for the canonical morphism $i_{s*}Ri^!_s M \to M$.

Let $S \subset X$ be any finite non-empty subset, $j_S : U \hookrightarrow X$ its complement. The standard exact triangle $\underset{S}{\oplus}\, i_{s*}Ri^!_s M \to M \to Rj_{S*}j^*_S M \to \underset{S}{\oplus}\, i_{s*}Ri^!_s M[1]$ shows that the kernel of the morphism $\pi_S = \oplus \pi_s : \underset{S}{\oplus}\, M^\ell_s \to H^n_{DR}(X, M)$ is equal to the image of the "residue" map $\mathrm{Res}_S = \oplus \mathrm{Res}_s : H^{n-1}_{DR}(U, M) \to \underset{S}{\oplus}\, M^\ell_s$.

2.1.13. Let $x \in X$ be a (closed) point; denote by i_x the embedding $\{x\} \hookrightarrow X$. For a $\mathcal{D}_X$-module M let $\Xi_x(M)$ be the set of all submodules $M_\xi \subset M$ such that M/M_ξ is supported

[43] To prove that these functors are mutually inverse notice that for a vector space V all sub-$\mathcal{D}$-modules of $V \otimes \mathcal{O}_X$ are of the form $W \otimes \mathcal{O}_X$, $W \subset V$; this follows from the irreducibility of the $\mathcal{D}$-module $\mathcal{O}_X$.

[44] A direct way to see the existence of M_{const}: reduce to the case where M^ℓ is coherent, then notice that $\dim \mathrm{Hom}_{\mathcal{D}}(M^\ell, \mathcal{O}_X) < \infty$.

at x. This is a subfilter in the ordered set of all submodules of M, so $\Xi_x(M)$ is a topology on M.

Consider the vector space $h(M)_x$. For $M_\xi \in \Xi_x(M)$ the sequence $0 \to h(M_\xi)_x \to h(M)_x \to h(M/M_\xi) \to 0$ is exact (since $H_{DR}^{-1}(M/M_\xi) = 0$). Therefore the map $N \mapsto h(N)_x \subset h(M)_x$ is a bijection between the set of $\mathcal{D}$-submodules $N \subset M$ containing M_ξ and that of vector subspaces of $h(M)_x$ containing $h(M_\xi)_x$ (since $\mathcal{D}$-submodules of M/M_ξ are the same as vector subspaces of $h(M/M_\xi)_x$). The subspaces $h(M_\xi)_x$, $M_\xi \in \Xi_x(M)$, form a topology on $h(M)_x$ which we call *the Ξ_x-topology.* We have seen that $M_\xi \mapsto h(M_\xi)_x$ is a bijection between $\Xi_x(M)$ and the set of open vector subspaces for the Ξ_x-topology.

Denote by $\hat{h}_x(M)$ the completion of $h(M)_x$ with respect to the Ξ_x-topology, i.e., the projective limit of the system of vector spaces $i_x^!(M/M_\xi) = h(M/M_\xi)_x$, $M_\xi \in \Xi_x(M)$. We consider $\hat{h}_x$ as a functor from $\mathcal{M}(X)$ to the category of complete separated topological vector spaces.[45] For any discrete vector space V one has[46] $\mathrm{Hom}_{cont}(\hat{h}_x(M), V) = \mathrm{Hom}(M, i_{x*}V)$, so the functor $\hat{h}_x$ is right exact.

Notice that if M is a coherent $\mathcal{D}$-module then $\hat{h}_x(M)$ is a profinite-dimensional vector space. Indeed, every M/M_ξ is a coherent $\mathcal{D}$-module supported at x, hence $i_x^!(M/M_\xi)$ is a finite-dimensional vector space.

We refer to any topology $\Xi_x^?(M)$ weaker than $\Xi_x(M)$ as a *topology at x on M.* As above, it yields a topology on $h(M)_x$. The corresponding completion $\hat{h}_x^?(M)$ equals $\varprojlim i_x^!(M/M_\xi)$, $M_\xi \in \Xi_x^?(M)$.

Examples. (i) Assume that M is an induced $\mathcal{D}$-module, $M = B_{\mathcal{D}}$ (see 2.1.7), so $h(M) = B$. Denote by $\Xi_x(B)$ the topology on B formed by all quasicoherent $\mathcal{O}$-submodules $P \subset B$ such that B/P is supported set-theoretically at x. Then $\Xi_x(M)$ is generated by submodules $P_{\mathcal{D}}$, $P \in \Xi_x(B)$.[47] Therefore $\hat{h}_x(M)$ is the completion of B with respect to the topology $\Xi_x(B)$. If B is $\mathcal{O}$-coherent then this is just the formal completion of B at x. We will

[45] In this work "topological vector space" always means vector space equipped with a *linear* topology (there is a base of neighbourhoods of 0 formed by linear subspaces).

[46] We would denote $\hat{h}_x(M)$ by i_x^*M but, unfortunately, the notation was already used at the end of 2.1.2.

[47] For $M_\xi \in \Xi_x(M)$ set $P := M_\xi \cap B$. Then $P \in \Xi_x(B)$ and $M_\xi \supset P_{\mathcal{D}}$.

prove in 2.1.16 that for arbitrary B the $\Xi_x(B)$-completion of B equals $B \otimes O_x$ where $O_x := \varprojlim \mathcal{O}_x/\mathrm{m}_x^n$ is the formal completion of the local ring at x.

(ii) Assume that M is any coherent $\mathcal{D}$-module. To compute $\hat{h}_x(M)$ one can represent M as a cokernel of a morphism $\varphi : \mathcal{D}_X^m \to \mathcal{D}_X^n$. We have seen that $\hat{h}_x(\mathcal{D}_X) = O_x$, so $\hat{h}_x(M)$ is the cokernel of the corresponding matrix of differential operators[48] $\varphi : O_x^m \to O_x^n$.

2.1.14 Lemma. (i) $\hat{h}_x(M)$ is the completion of the vector space $M \underset{\mathcal{D}_X}{\otimes} O_x$ with respect to the topology formed by the images of the maps $M_\xi \underset{\mathcal{D}_X}{\otimes} O_x \to M \underset{\mathcal{D}_X}{\otimes} O_x$, $M_\xi \in \Xi_x(M)$.

(ii) If the restriction of M to $X \smallsetminus \{x\}$ is a coherent $\mathcal{D}$-module then the latter topology is complete and separated, i.e.,

$$M \underset{\mathcal{D}_X}{\otimes} O_x \xrightarrow{\sim} \hat{h}_x(M). \tag{2.1.14.1}$$

Proof. (i) If M is supported at x then $M \underset{\mathcal{D}_X}{\otimes} O_x$ is equal to the coinvariants of the right $\mathcal{D}_X$-module action of Θ_X on $M \underset{\mathcal{O}_X}{\otimes} O_x = M$, hence $M \underset{\mathcal{D}_X}{\otimes} O_x = h(M)_x$.

If M is arbitrary then for $M_\xi \in \Xi_x(M)$ we have $\mathrm{Coker}(M_\xi \underset{\mathcal{D}_X}{\otimes} O_x \to M \underset{\mathcal{D}_X}{\otimes} O_x) = (M/M_\xi) \underset{\mathcal{D}_X}{\otimes} O_x = h(M/M_\xi)_x$, which implies (i).

(ii) We want to show that, under the condition of (ii), the morphism of vector spaces $M \underset{\mathcal{D}_X}{\otimes} O_x \to \hat{h}_x(M)$ just defined is an isomorphism. Our M is an extension of a $\mathcal{D}$-module M' supported at x by a coherent $\mathcal{D}$-module M''. We have aready checked the statement for M' (see the proof of (i)) and M'' (see Example (ii) of 2.1.13). Since the sequences $0 \to M'' \underset{\mathcal{D}_X}{\otimes} O_x \to M \underset{\mathcal{D}_X}{\otimes} O_x \to M' \underset{\mathcal{D}_X}{\otimes} O_x \to 0$ and $0 \to \hat{h}_x(M'') \to \hat{h}_x(M) \to \hat{h}_x(M') \to 0$ are exact, we are done. □

Remarks. (i) If M is a $\mathcal{D}$-module as in 2.1.14(ii) then $\hat{h}_x(M)$ admits an open profinite-dimensional subspace.[49] Such topological vector spaces are discussed in 2.6.9 under the name of *Tate vector spaces.*

(ii) If M is an arbitrary $\mathcal{D}$-module then the map $\phi_M : M \underset{\mathcal{D}_X}{\otimes} O_x \to \hat{h}_x(M)$ need not be an isomorphism. Here is an example. Assume that $X = \mathrm{Spec} R$ is an affine curve, so

[48] Here we use the fact that a morphism of profinite-dimensional vector spaces has closed image.

[49] Namely, the image of $\hat{h}_x(M_\xi)$ where $M_\xi \in \Xi_x(M)$ is a coherent $\mathcal{D}_X$-module (as explained in 2.1.13, $\hat{h}_x(M_\xi)$ is profinite-dimensional).

$R = H^0_{DR}(X, \mathcal{D}_X) = \mathrm{Ext}^1_{\mathcal{M}(X)}(\omega_X, \mathcal{D}_X)$. Let M be the universal extension of $R \otimes \omega_X$ by $\mathcal{D}_X$. Then $M \underset{\mathcal{D}_X}{\otimes} O_x = O_x/R \neq 0$ and $\hat{h}_x(M) = 0$ is the completion of O_x/R with respect to the quotient topology, i.e., $\hat{h}_x(M) = 0$.

However ϕ_M is always an isomorphism if M is either an induced $\mathcal{D}$-module (see Example (i) of 2.1.13 and 2.1.16) or isomorphic to a direct sum of coherent $\mathcal{D}$-modules. In fact, the argument used in the proof of statement (ii) of the lemma shows that it suffices to assume that the restriction of M to $X \setminus \{x\}$ satisfies either of these conditions.

(iii) Sometimes it is convemient to consider a finer structure than a plain topological vector space $\hat{h}_x(M)$. Namely, consider a directed set of all $\mathcal{D}$-coherent submodules $M_\alpha \subset M$ and set $\tilde{h}_x(M) := \varinjlim \hat{h}_x(M_\alpha)$. We consider $\tilde{h}_x(M)$ as an ind-object of the category of profinite-dimensional vector spaces. Recall that such an animal is the same as a left exact contravariant functor on the category of profinite-dimensional vector spaces with values in vector spaces. So $\tilde{h}_x(M)$ "represents" the functor $F \mapsto \varinjlim \mathrm{Hom}(F, \hat{h}_x(M_\alpha))$, F is a profinite-dimensional vector space. Notice that the dual object $\tilde{h}_x(M)^*$ – which is a pro-object of the category of vector spaces – equals $\varprojlim \mathrm{Hom}(M_\alpha, i_{x*}k)$.

2.1.15. We will also need a version of 2.1.13 in the case where x depends on parameters. So let Y be a scheme which we assume to be quasicompact and quasiseparated,[50] $x \in X(Y)$, $i_x : Y \hookrightarrow X \times Y$ the graph of x. Denote by $\mathcal{M}(X \times Y/Y)$ the category of right $\mathcal{D}_{X\times Y/Y} := \mathcal{D}_X \boxtimes \mathcal{O}_Y$-modules which are quasicoherent as $\mathcal{O}_{X\times Y}$-modules. Let $\mathcal{M}(X \times Y/Y)_x$ be the full subcategory of $\mathcal{D}_{X\times Y/Y}$-modules supported (set-theoretically) at the graph of x. A version of Kashiwara's lemma (see 2.1.3) says that the functor $i^!_x$ identifies $\mathcal{M}(X \times Y/Y)_x$ with the category of quasicoherent $\mathcal{O}_Y$-modules.

For $M \in \mathcal{M}(X \times Y/Y)$ we denote by $\Xi_x(M)$ the set of submodules $M_\xi \subset M$ such that $M/M_\xi \in \mathcal{M}(X \times Y/Y)_x$. This is a subfilter in the ordered set of all submodules of M. We have a $\Xi_x(M)$-projective system of quasicoherent $\mathcal{O}_Y$-modules $i^!_x(M/M_\xi)$ connected by surjective maps.

[50]The assumption is needed to assure that topological $\mathcal{O}_Y$-modules have local nature, i.e., proobjects in the category of quasicoherent $\mathcal{O}$-modules form a sheaf of categories on Y_{Zar}, and our constructions are compatible with localization of Y.

For $N \in \mathcal{M}(X \times Y/Y)$ set $h(N) := N \underset{\mathcal{D}_{X\times Y/Y}}{\otimes} \mathcal{O}_{X\times Y}$; this is a $p_Y^{\cdot}\mathcal{O}_Y$-module.[51] Since $i_x^!(M/M_\xi) = i_x^{\cdot}h(M/M_\xi)$ the projective limit of $i_x^!(M/M_\xi)$ equals to the completion of $i_x^{\cdot}h(M)$ with respect to the topology defined by the images of $i_x^{\cdot}h(M_\xi)$'s. We denote it by $\hat{h}_x(M)$.

If M is an induced module, $M = B \otimes \mathcal{D}_{X\times Y/Y}$ where B is an $\mathcal{O}_{X\times Y}$-module, then $h(M) = B$ and $\hat{h}_x(M)$ is the completion of B with respect to the topology formed by all quasicoherent $\mathcal{O}$-submodules $P \subset B$ which coincide with B outside of (the graph of) x. In particular, if B is $\mathcal{O}$-coherent then $\hat{h}_x(M)$ is the formal completion of B at x.

As in 2.1.14 for a coherent $\mathcal{D}_{X\times Y/Y}$-module M one identifies $\hat{h}_x(M)$ with the $\mathcal{D}_{X\times Y/Y}$-tensor product of M and the formal completion of $\mathcal{O}_{X\times Y}$ at x.

Remarks. (i) The above construction is compatible with base change by morphisms $f : Y' \to Y$: the obvious morphism of pro-$\mathcal{O}_{Y'}$-modules $\hat{h}_{xf}((f \times id_X)^* M) \to f^* \hat{h}_x(M)$ is an isomorphism.

(ii) If Y is a smooth variety and M is a right $\mathcal{D}_{X\times Y}$-module then differential operators on Y act on M in a continuous way with respect to Ξ_x-topology. Therefore $\hat{h}_x(M)$ is a sheaf of topological right $\mathcal{D}_Y$-modules. Similarly, if $\mathcal{D}_Y$ acts on M from the left then $\hat{h}_x(M)$ is a sheaf of topological left $\mathcal{D}_Y$-modules.

(iii) If F is a $\mathcal{D}$-module on Y and $\varphi : \hat{h}_x(M) \to F$ is a continuous $\mathcal{D}_Y$-morphism then it vanishes on the image of $\hat{h}_x(M_\xi)$ where $M_\xi \in \Xi_x(M)$ is a sub-$\mathcal{D}_{X\times Y}$-module of M. In particular, assume that $Y = X$, $x = id_X$, so $i_x = \Delta$, and $M = L \boxtimes \mathcal{O}_X$ for a $\mathcal{D}$-module L on X. Then every morphism φ as above is trivial (if $\dim X \neq 0$).

2.1.16. In this subsection we prove a result in commutative algebra we refered to in Example (i) of 2.1.13 and Remark (ii) in 2.1.14. It will not be used in the rest of this work.

Let A be a noetherian ring, $I \subset A$ an ideal such that A is complete in I-adic topology. Let C be the category of A-modules and $C_0 \subset C$ the full subcategory of A-modules M such that each element of M is annihilated by some power of I. The *I-topology* on an A-module M is the weakest topology such that every A-linear map from M to a module

[51] Here $p_Y : X \times Y \to Y$ is the projection.

from C_0 equipped with the discrete topology is continuous. The set B_M of all submodules $\tilde{M} \subset M$ such that $M/\tilde{M} \in C_0$ is a base of neighborhoods of 0 for the I-topology. If M is finitely generated then the I-topology coincides with the I-adic topology.

Theorem. (i) If N is a submodule of an A-module M then the I-topology on M induces the I-topology on N.

(iii) If A has finite Krull dimension then every A-module M is complete and separated for the I-topology.

Proof. To prove (i) we have to show that for every $\tilde{N} \in B_N$ there is an $\tilde{M} \in B_M$ such that $\tilde{M} \cap N = \tilde{N}$. Let P be the set of pairs $(L, \tilde{L})$ where L is a submodule of M and $\tilde{L} \in B_L$. Equip P with the following ordering: $(L, \tilde{L}) \leq (L', \tilde{L}')$ if $L \subset L'$ and $\tilde{L} = L \cap \tilde{L}'$. By Zorn's lemma there is a maximal $(L, \tilde{L}) \in P$ such that $(L, \tilde{L}) \geq (N, \tilde{N})$. It remains to show that $L = M$. Indeed, if $x \in M$, $x \notin L$ then one can construct $(L', \tilde{L}') > (L, \tilde{L})$ by putting $L' := L + Ax$, $\tilde{L}' := L' + U$ where $U \subset Ax$ is an open submodule of Ax such that $L \cap U = \tilde{L} \cap Ax$ (U exists because the I-topology on $L \cap Ax$ is induced by the I-topology on Ax).

Using (i) we reduce the proof of separatedness of M to the well known case where M is finitely generated (A need not have finite Krull dimension for this property).

Denote by $\hat{M}$ the completion of M with respect to the I-topology. Let $C^? \subset C$ be the full subcategory of A-modules M such that the map $M \to \hat{M}$ is bijective. We will show that $C^? = C$ if A has finite Krull dimension.

Lemma. $C^?$ has the following properties:

(i) Finitely generated modules belong to $C^?$.

(ii) Suppose that the sequence $0 \to M' \to M'' \to M \to 0$ is exact and $M \in C^?$. Then $M' \in C^? \Leftrightarrow M'' \in C^?$.

(iii) If $M_j \in C^?$, $j \in J$ then $\bigoplus_j M_j \in C^?$.

(iv) Suppose $M = \bigcup_i M_i$, $i = 1, 2, \dots$, $M_i \subset M_{i+1}$. If $M_i \in C^?$ for all i then $M \in C^?$.

Proof of Lemma. (i) is well known.

(ii) is a corollary of the left exactness of the functor $M \mapsto \hat{M}$, which is proved as follows. If the sequence $0 \to M_1 \to M_2 \to M_3$ is exact then $\mathrm{Ker}(\hat{M}_2 \to \hat{M}_3)$ is the closure of M_1 in $\hat{M}_2$, i.e., the completion of M_1 with respect to the topology induced from the I-topology on M_2. We have proved that the induced topology is the I-topology on M_1, so $\mathrm{Ker}(\hat{M}_2 \to \hat{M}_3) = \hat{M}_1$.

To prove (iii) it suffices to show that if $M = \bigoplus_j M_j$ then the image of the natural embedding $f : \hat{M} \to \prod_j \hat{M}_j$ equals $\bigoplus_j \hat{M}_j$. We only have to prove that $\mathrm{Im} f \subset \bigoplus_j \hat{M}_j$. Assume that $x = (x_j) \in \prod_j \hat{M}_j$ and $J' := \{j \in J | x_j \neq 0\}$ is infinite. Choose open submodules $U_j \subset M_j$ so that for every $j \in J'$ the image of x_j in M_j/U_j is nonzero. Then the image of x in $\prod_j M_j/U_j$ does not belong to $\oplus_j M_j/U_j$, so $x \notin \mathrm{Im} f$.

To prove (iv) it suffices to show that every $x \in \hat{M}$ belongs to the closure of $M_i \subset \hat{M}$ for some i (this closure equals $\hat{M}_i = M_i$). If this is not true then for every i there is an open submodule $U_i \subset M$ such that the image of x in $M/(M_i + U_i)$ is nonzero. We can assume that $U_1 \supset U_2 \supset \dots$. The submodule $U := \bigcup_i (M_i \cap U_i) \subset M$ is open and $U + M_i \subset U_i + M_i$. So for every i the image of x in $M/(M_i + U)$ is nonzero, i.e., the image of x in M/U does not belong to the image of M_i in M/U. This is impossible because $M = \bigcup_i M_i$. □

We assume now that A has finite Krull dimension. It remains to show that if a full subcategory $C^? \subset C$ satisfies properties (i)–(iv) from the lemma then $C^? = C$. Let $C_i \subset C$ be the full subcategory consisting of modules M such that $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p} \subset A$ with $\dim A/\mathfrak{p} > i$. We will prove by induction that $C_i \subset C^?$. Suppose we know that $C_{i-1} \subset C^?$. Let $M \in C_i$. Denote by P_i the set of all prime $\mathfrak{p} \subset A$ such that $\dim A/\mathfrak{p} = i$. The map $M \to \prod_{\mathfrak{p} \in P_i} M_{\mathfrak{p}}$ factors through $\sum_{\mathfrak{p} \in P_i} M_{\mathfrak{p}}$, and the kernel and cokernel of the morphism $M \to \sum_{\mathfrak{p} \in P_i} M_{\mathfrak{p}}$ belong to C_{i-1}. By (ii),(iii), and the induction assumption, to prove that $M \in C^?$ it suffices to show that $M_{\mathfrak{p}} \in C^?$ for each $\mathfrak{p} \in P_i$. Let $\mathfrak{p} \in P_i$. Our $M_{\mathfrak{p}}$ is an $A_{\mathfrak{p}}$-module such that each element of $M_{\mathfrak{p}}$ is annihilated by a power of $\mathfrak{p} A_{\mathfrak{p}}$. By (ii) and (iv) we are reduced to proving that vector spaces over $A_{\mathfrak{p}}/\mathfrak{p} A_{\mathfrak{p}}$ belong to $C^?$. By (iii) it suffices to show that $A_{\mathfrak{p}}/\mathfrak{p} A_{\mathfrak{p}} \in C^?$. The morphism $A/\mathfrak{p} \to A_{\mathfrak{p}}/\mathfrak{p} A_{\mathfrak{p}}$ is injective and its cokernel belongs to $C_{i-1} \subset C^?$. So $A_{\mathfrak{p}}/\mathfrak{p} A_{\mathfrak{p}} \in C^?$ by (i) and (ii). □

Remark. If A is a 1-dimensional local domain then the proof of statement (ii) of the above theorem is very simple. In this case every A-module M has a free submodule N such that $M/N \in C_0$, so we can assume that M is free. Then apply statement (iii) of the lemma.

§2 The compound tensor structure

In this section we describe a canonical augmented compound tensor structure on the category $\mathcal{M}(X)$, X is a smooth scheme. We have to define the tensor structure $\otimes^!$, the pseudo-tensor structure P^*, the compound tensor product maps $\otimes^I_{S,T}$ (see 1.3.13), and the augmentation functor h.

2.2.1 The ! tensor structure. The category $\mathcal{M}^\ell$ is a tensor category (see 2.1.1). We use the standard equivalence $\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}^\ell(X)$, $M \mapsto M^\ell := M\omega_X^{-1}$, to define the tensor product $\otimes^!$ on $\mathcal{M}(X)$, so we have $M_1 \otimes^! M_2 = M_1^\ell \underset{\mathcal{O}_X}{\otimes} M_2$. The object ω_X is a unit object for $\otimes^!$.

2.2.2. Consider the derived tensor product $\overset{L}{\otimes}{}^!$ on $D^b\mathcal{M}(X)$.[52] According to (2.1.4.1) for $M_i \in D^b\mathcal{M}(X)$ one has

$$\overset{L}{\otimes}{}^!_I M_i = \Delta^{(I)!}(\boxtimes(M_i[\dim X]))[-\dim X] = \Delta^{(I)!}(\boxtimes M_i) \otimes \lambda_I^{\otimes \dim X}[(|I|-1)\dim X]$$

where $\lambda_I := \det k^I$ (the last equality comes since $k[\dim X]^{\otimes I} = \lambda_I^{\otimes \dim X}[|I| \dim X]$). Therefore for $M_i \in \mathcal{M}(X)$ there is a canonical isomorphism

$$\otimes^!_I M_i = H^{(|I|-1)\dim X}\Delta^{(I)!}(\boxtimes M_i) \otimes \lambda_I^{\otimes \dim X}. \tag{2.2.2.1}$$

Example. Assume that $M_i = \omega_X$. One has $\omega_X^{\otimes^! I} = \omega_X$ and $\omega_X^{\boxtimes I} = \omega_{X^I} \otimes \lambda_I^{\otimes \dim X}$, so (2.2.2.1) comes from the identification $\omega_X = H^{(|I|-1)\dim X}\Delta^{(I)!}(\omega_{X^I})$.

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[52] Notice that $\overset{L}{\otimes}{}^!$ is identified with the tensor product on $D^b\mathcal{M}^\ell(X)$ by the above "standard" t-exact equivalence, and *not* by the canonical equivalence 2.1.1.1.

2.2.3 The $*$ pseudo-tensor structure. For an I-family L_i of $\mathcal{D}$-modules, $I \in \mathcal{S}$, and a $\mathcal{D}$-module M set

(2.2.3.1) $$P_I^*(\{L_i\}, M) := \mathrm{Hom}(\boxtimes L_i, \Delta_*^{(I)} M).$$

The elements of P_I^* are called $*$ *I-operations.* They have étale local nature: we have the sheaf $\underline{P}_I^*(\{L_i\}, M)$, $U \mapsto P_I^*(\{L_i|_U\}, M|_U)$, on the étale topology of X. For a surjective map $\pi\colon J \twoheadrightarrow I$ the composition map

$$P_I^*(\{L_i\}, M) \otimes (\otimes_I P_{J_i}^*(\{K_j\}, L_i)) \longrightarrow P_J^*(\{K_j\}, M)$$

sends $\varphi \otimes (\otimes \psi_i)$ to $\varphi(\psi_i)$ defined as the composition

$$\boxtimes_J K_j \xrightarrow{\boxtimes \psi_i} \boxtimes_I \Delta_*^{(J_i)} L_i = \Delta_*^{(\pi)} (\boxtimes_I L_i) \xrightarrow{\Delta_*^{(\pi)}(\varphi)} \Delta_*^{(\pi)} \Delta_*^{(I)} M = \Delta_*^{(J)} M$$

where $\Delta^{(\pi)} = \prod_I \Delta^{(J_i)}\colon X^I \hookrightarrow X^J$. The composition is associative, so P_I^* define on $\mathcal{M}(X)$ an abelian pseudo-tensor structure. We denote this pseudo-tensor category by $\mathcal{M}(X)^*$. The pseudo-tensor categories $\mathcal{M}(U)^*$, $U \in X_{\text{ét}}$, form a sheaf of pseudo-tensor categories on the étale topology of X.

Remark. One has $P_I^*(\{L_i\}, M) = \mathrm{Hom}(\otimes^* L_i, M)$ where $\otimes^* L_i := \Delta^{(I)*}(\boxtimes L_i)$ is the projective system of all the $\mathcal{D}_{X^I}$-module quotients of $\boxtimes L_i$ supported on the diagonal, considered as a projective system of $\mathcal{D}_X$-modules via the $\Delta^{(I)!}$-equivalence.

2.2.4 Examples. (i) Let us describe the $*$ pseudo-tensor structure on the subcategory of induced modules (see 2.1.7). For $\mathcal{O}$-modules $\{F_i\}_{i \in I}, G$ denote by $\mathrm{Diff}_I(\{F_i\}, G) \subset \mathrm{Hom}_k\Big(\underset{k}{\otimes} F_i, G\Big)$ the subspace of differential I-operations (those φ's for which the maps $F_i \to G$, $f_i \mapsto \varphi\Big(f_i \otimes \Big(\underset{j \neq i}{\otimes} f_j\Big)\Big)$, are differential operators for any i and any fixed local sections $f_j \in F_j$, $j \neq i$). Equivalently, $\mathrm{Diff}_I(\{F_i\}, G) = \mathrm{Diff}(\boxtimes F_i, \Delta_\cdot^{(I)} G)$.

The differential operations are stable with respect to composition so they define a pseudo-tensor structure on the category $\mathcal{D}iff(X)$; denote this pseudo-tensor category by $\mathcal{D}iff(X)^*$. One has $P_I^*(\{F_{i\mathcal{D}}\}, G_{\mathcal{D}}) = \mathrm{Hom}((\boxtimes F_i)_{\mathcal{D}}, (\Delta_\cdot G)_{\mathcal{D}}) = \mathrm{Diff}_I(\{F_i\}, G)$ (see 2.1.7, 2.1.8(iii)), i.e., $\cdot_{\mathcal{D}}$ extends to a fully faithful pseudo-tensor functor

(2.2.4.1) $$\cdot_{\mathcal{D}}\colon \mathcal{D}iff(X)^* \hookrightarrow \mathcal{M}(X)^*.$$

(ii) Assume that $|I| \geq 2$ and one of the $\mathcal{D}$-modules L_i is lisse (e.g., equal to ω_X). Then $P_I^*(\{L_i\}, M) = 0$.

2.2.5. Let $i: X \hookrightarrow Y$ be an embedding of smooth schemes. The left exact functors $H^0 i_*$, $H^0 i^!$ between $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ act on P_I^* in the obvious way, i.e., they are pseudo-tensor functors. If Y is a closed subscheme then $H^0 i^!$ is right adjoint to $i_* = H^0 i_*$ as pseudo-tensor functors (which means that $P_I^*(\{i_* N_i\}, M) = P_I^*(\{N_i\}, H^0 i^! M)$ for any $N_i \in \mathcal{M}(X)$, $M \in \mathcal{M}(Y)$), so i_* is a fully faithful embedding of pseudo-tensor categories.

2.2.6 The tensor product maps. We have to define the canonical maps (see 1.3.12)

$$\otimes_{S,T}^I\colon \otimes_S P_{I_s}^*(\{L_i\}, M_s) \to P_T^*(\{\otimes_{I_t}^! L_i\}, \otimes_S^! M_s)$$

According to the first formula in 2.2.2, one has canonical isomorphisms in $D^b\mathcal{M}(X^T)$ (here $\alpha := k[-\dim X]$, so $\alpha^{-1} = k[\dim X]$):

$$\begin{aligned}
\Delta^{(\pi_T)!}(\boxtimes_I L_i) &= \boxtimes_T(\Delta^{(I_t)!}(\boxtimes_{I_t} L_i)) = \boxtimes_T((\overset{L}{\otimes}{}^!_{I_t} L_i) \otimes \alpha^{-1} \otimes \alpha^{\otimes I_t}) \\
&= (\boxtimes_T(\overset{L}{\otimes}{}^!_{I_t} L_i)) \otimes (\alpha^{-1})^{\otimes T} \otimes \alpha^{\otimes I}, \\
\Delta^{(\pi_T)!}(\boxtimes_S(\Delta_*^{(I_s)} M_s)) &= \Delta^{(\pi_T)!}\Delta_*^{(\pi_S)}(\boxtimes_S M_s) = \Delta_*^{(T)}\Delta^{(S)!}(\boxtimes_S M_s) \\
&= \Delta_*^{(T)}\left(\overset{L}{\otimes}{}^!_S M_s\right) \otimes \alpha^{-1} \otimes \alpha^{\otimes S}.
\end{aligned}$$

In the second line we used the base change canonical isomorphism $\Delta^{(\pi_T)!}\Delta_*^{(\pi_S)} = \Delta_*^{(T)}\Delta^{(S)!}$ (recall that $X^S, X^T \subset X^I$ are transveral and $X^S \cap X^T = X$, see 1.3.2 (i)).

Consider the diagonal embeddings $\Delta^{(S)}\colon k \hookrightarrow k^S$, $\Delta^{(\pi_T)}\colon k^T \hookrightarrow k^I$. The embedding $\Delta^{(\pi_S)}\colon k^S \to k^I$ induces an isomorphism between the quotients $k^S/k \xrightarrow{\sim} k^I/k^T$. Since $\alpha^{-1} \otimes \alpha^{\otimes S} = \det((k^S/k)[-\dim X])$, $(\alpha^{-1})^{\otimes T} \otimes \alpha^{\otimes I} = \det((k^I/k)^T[-\dim X])$ we get the identification $\alpha^{-1} \otimes \alpha^{\otimes S} = (\alpha^{-1})^{\otimes T} \otimes \alpha^{\otimes I}$. Now $\otimes_{S,T}^I$ is the composition

$$\otimes_S P_{I_s}^*(\{L_i\}, M_s) \to \mathrm{Hom}(\boxtimes_I L_i, \boxtimes_S \Delta_*^{(I_s)} M_s) \to \mathrm{Hom}(\boxtimes_T(\otimes_{I_t}^! L_i), \Delta_*^{(T)}(\otimes_S^! M_s))$$

where the first arrow is $\boxtimes_S$ and the second one is $H^{(|I|-|T|)\dim X}\Delta^{(\pi_T)!}$.

The commutativity of diagrams 1.3.12 (i), (ii) is clear. Therefore we have defined the compound tensor category structure on $\mathcal{M}(X)$. This compound tensor category $\mathcal{M}(X)^{*!}$ is abelian (see 1.3.14) with a strong unit object ω_X (see 1.3.16, 1.3.17, 2.2.4 (ii)).

The above constructions are compatible with étale localization therefore $\mathcal{M}(U)^{*!}$, $U \in X_{\text{ét}}$, form a sheaf of compound tensor categories $\mathcal{M}(X_{\text{ét}})^{*!}$ over $X_{\text{ét}}$.

2.2.7 The augmentation functor. Let us show that the de Rham functor $h = h_X$ (see 2.1.6) is a non-degenerate $*$-augmentation functor on $\mathcal{M}(X_{\text{ét}})^{*!}$ (see 1.2.5, 1.3.10). We have to define for a finite set I, $|I| \geq 2$, and $i_0 \in I$ a canonical morphism of sheaves

$$h_{I,i_0}\colon P^*_I(\{L_i\}, M) \otimes h(L_{i_0}) \to P^*_{I\smallsetminus\{i_0\}}(\{L_i\}, M) \tag{2.2.7.1}$$

Namely, for $\varphi \in P^*_I(\{L_i\}, M)$ and $\ell_i \in L_i$ one has

$$h_{I,i_0}(\varphi \otimes \bar{\ell}_{i_0})\Big(\underset{I\smallsetminus\{i_0\}}{\boxtimes} \ell_i\Big) = Tr_{i_0}\varphi\left(\underset{I}{\boxtimes}\ell_i\right). \tag{2.2.7.2}$$

Here $Tr_{i_0}\colon pr_{I,i_0\cdot}\, \Delta^{(I)}_* M \to \Delta^{(I\smallsetminus\{i_0\})}_* M$ is the trace map for the projection $pr_{I,i_0}\colon X^I \to X^{I\smallsetminus\{i_0\}}$. It is easy to see that h_{I,i_0} is well-defined by (2.2.7.2). Compatibilities 1.2.5(i),(ii) and 1.3.10 are straightforward, so h makes $\mathcal{M}(X)^{*!}$ an augmented compound tensor category.

We leave it to the reader to check that h is non-degenerate, i.e., (see 1.2.5) that the morphisms $P^*_I(\{L_i\}, M) \to \mathrm{Hom}(h(L_{i_0}), P^*_{I\smallsetminus\{i_0\}}(\{L_i\}, M)$ coming from (2.2.7.1) are injective.

Let $Sh(X)^{\otimes}$ be the tensor category of sheaves of k vector spaces. According to 1.2.7, h extends to a pseudo-tensor functor

$$h:\, \mathcal{M}(X)^* \to Sh(X)^{\otimes}. \tag{2.2.7.3}$$

The maps $h_I\colon P^*_I(\{L_i\}, M) \to \mathrm{Hom}(\otimes_I h(L_i), h(M))$ are $h_I(\varphi)(\otimes\bar{\ell}_i) = \overline{\varphi(\boxtimes\ell_i)}$; here we consider $\overline{\varphi(\boxtimes\ell_i)}$ as a section of $\Delta^{(I)\cdot}h(\Delta^{(I)}_* M) = h(M)$ (see 2.1.6).

If $i\colon X \hookrightarrow Y$ is a closed embedding, then $i_*\colon \mathcal{M}(X)^* \hookrightarrow \mathcal{M}(Y)^*$ is compatible with the augmentation functors, i.e., it is an augmented pseudo-tensor functor (see 1.2.8).

Remark. The map $h_I: P_I^*(\{L_i\}, M) \to \mathrm{Hom}(\otimes_I h(L_i), h(M))$ can be not injective. However it is injective if M is either an induced $\mathcal{D}$-module (e.g. if M is supported at a single point) or a coherent $\mathcal{D}_X$-module. Indeed, in the induced case by left exactness of h we can assume that L_i's are also induced (replace L_i by $L_{i\mathcal{D}} \twoheadrightarrow L_i$ if necessary), and use then 2.2.4(i). In the coherent case use statement (ii) of Lemma from 2.1.6.

The rest of this section contains an auxiliary technical material; we suggest the reader to skip it returning when necessary.

2.2.8. Let us show that our augmented compound tensor structure comes from certain augmented compound strict $\mathcal{D}_X$-operad $\mathcal{B}^{*!}$ in the way explained in 1.3.18, 1.3.19.

Let us define $\mathcal{B}^{*!} = (\mathcal{B}^*, \mathcal{B}^!, <\ \ >^I_{S,T})$. For a finite set I let $\mathcal{B}^*_I$ be the tensor product of I copies of $\mathcal{D}_X$ considered as *left* $\mathcal{D}$-modules. This is a left $\mathcal{D}$-module on X, and the right $\mathcal{D}_X$-action on each copy of $\mathcal{D}_X$ defines the right $\mathcal{D}_X^{\otimes I}$-action on $\mathcal{B}^*_I$ (here $\mathcal{D}_X^{\otimes I}$ is the I^{th} tensor power of $\mathcal{D}_X$ considered as sheaf of k-algebras). So $\mathcal{B}^*_I$ is a $(\mathcal{D}_X - \mathcal{D}_X^{\otimes I})$-bimodule. Similarly, let $\mathcal{B}^!_I$ be the $\otimes^!$-product of I copies of $\mathcal{D}_X$ considered as *right* $\mathcal{D}_X$-modules; this is a $(\mathcal{D}_X^{\otimes I} - \mathcal{D}_X)$-bimodule.

Note that $\mathcal{B}^*_I$ is generated, as right $\mathcal{D}_X^{\otimes I}$-module, by the subsheaf $\mathcal{O}_X = \mathcal{O}_X 1^{\otimes I}$ on which $\mathcal{O}_X^{\otimes I} \subset \mathcal{D}_X^{\otimes I}$ acts via the product map $\mathcal{O}_X^{\otimes I} \to \mathcal{O}_X$. In fact $\mathcal{B}^*_I = \mathcal{O}_X \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \mathcal{D}_X^{\otimes I}$. Similarly, $\mathcal{B}^!_I = \mathcal{D}_X^{\otimes I} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} (\omega_X^{(1-|I|)})$ as left $\mathcal{D}_X^{\otimes I}$-modules where $\omega_X^{(1-|I|)}$ is the $\mathcal{O}_X$-tensor power.

For a map $J \to I$ the composition morphisms for our augmented operads $\mathcal{B}^*_I \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\mathcal{B}^*_{J_i}) \to \mathcal{B}^*_J$, $(\otimes \mathcal{B}^!_{J_i}) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} \mathcal{B}^!_I \to \mathcal{B}^!_J$ are the isomorphisms of $(\mathcal{D}_X - \mathcal{D}_X^{\otimes J})$-, resp. $(\mathcal{D}_X^{\otimes J} - \mathcal{D}_X)$-bimodules, uniquely determined by the property that $f_I \otimes (\otimes f_{J_i}) \mapsto f_I \Pi f_{J_i} \in \mathcal{O}_X \subset \mathcal{B}^*_J$, $(\otimes \nu_{J_i}) \otimes \nu_I \mapsto (\Pi \nu_{J_i})\nu_I \in \omega_X^{(1-|J|)} \subset \mathcal{B}^!_J$ for $f_I \in \mathcal{O}_X \subset \mathcal{B}^*_I$, $\nu_I \in \omega_X^{(1-|I|)} \subset \mathcal{B}^!_I$, etc.

It remains to define $<\ \ >^I_{S,T}: \left(\underset{S}{\otimes} \mathcal{B}^*_{I_s}\right) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} \left(\underset{T}{\otimes} \mathcal{B}^!_{I_t}\right) \to \mathcal{B}^!_S \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_T$. One has $\left(\underset{S}{\otimes} \mathcal{B}^*_{I_s}\right) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} \left(\underset{T}{\otimes} \mathcal{B}^!_{I_t}\right) = \mathcal{O}_X^{\otimes S} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} (\mathcal{D}_X \omega_X^{-1})^{\otimes I} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \omega_X^{\otimes T}$ where the $\mathcal{O}_X^{\otimes I}$-module structure on $\mathcal{O}_X^{\otimes S}$, $\omega_X^{\otimes T}$ comes from the π_S and π_T product morphisms $\mathcal{O}_X^{\otimes I} \to \mathcal{O}_X^{\otimes S}$, $\mathcal{O}_X^{\otimes I} \to \mathcal{O}_X^{\otimes T}$. To see this $(\mathcal{D}_X^{\otimes S}, \mathcal{D}_X^{\otimes T})$-bimodule more clearly note that the left $\mathcal{D}_X^{\otimes I} \otimes \mathcal{D}_X^{\otimes I}$-module $(\mathcal{D}_X \omega_X^{-1})^{\otimes I}$

is the (sheaf-theoretic) restriction to the diagonal $X \subset X^I$ of the irreducible holonomic left $\mathcal{D}$-module $\mathcal{F}_I$ on $X^I \times X^I$ supported on $X^I \subset X^I \times X^I$ and generated by the $\mathcal{O}_{X^I}$-module $(\omega_X^{-1})^{\boxtimes I}$. Therefore, the left $\mathcal{D}_X^{\otimes S} \otimes \mathcal{D}_X^{\otimes T}$-module $\mathcal{O}_X^{\otimes S} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} (\mathcal{D}_X \omega_X^{-1})^{\otimes I} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \mathcal{O}_X^{\otimes T}$ is the restriction to $X \subset X^S \times X^T$ of the pull-back $\mathcal{F}_{S,T}$ of $\mathcal{F}_I$ to $X^S \times X^T \subset X^I \times X^I$. This $\mathcal{F}_{S,T}$ is the irreducible holonomic left $\mathcal{D}$-module on $X^S \times X^T$ supported on $X^S \times X^T \cap X^I = X$ and generated by $\omega_X^{-|I|}$. Hence

$$\Big(\underset{S}{\otimes} \mathcal{B}^*_{I_s}\Big) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) = ((\Delta^{(S)}, \Delta^{(T)})^{\cdot} \mathcal{F}_{S,T}) \underset{\mathcal{O}_X^{\otimes T}}{\otimes} \omega_X^{\otimes T}$$

On the other hand, it is easy to see that the left $\mathcal{D}^{\otimes S} \otimes \mathcal{D}^{\otimes T}$-module $\mathcal{B}^!_S \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_T \underset{\mathcal{O}_X^{\otimes T}}{\otimes} (\omega_X^{-1})^{\otimes T}$ is the restriction to $X \subset X^S \times X^T$ of the same $\mathcal{F}_{S,T}$ (the generator is $\omega_X^{-|I|} = \omega_X^{1-|S|} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X \underset{\mathcal{O}_X^{\otimes T}}{\otimes} (\omega_X^{-1})^{\otimes T} \subset \mathcal{B}^!_S \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_T \underset{\mathcal{O}_X^{\otimes T}}{\otimes} (\omega_X^{-1})^{\otimes T}$). The resulting isomorphism $(\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) \xrightarrow{\sim} \mathcal{B}^!_S \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_T$ is our $< \ >^I_{S,T}$.

The axioms of augmented compound operad are immediate.

2.2.9. Let us check that the augmented compound tensor structure defined on $\mathcal{M}(X)$ by $\mathcal{B}^{*!}$ (see 1.3.18, 1.3.19) coincides with that defined in 2.2.1 – 2.2.7. It is clear that $\otimes^!_I L_i = (\otimes L_i) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} \mathcal{B}^!_I$. The sheaf-theoretic restriction of the $\mathcal{D}_{X^I}$-module $\Delta^{(I)}_* M$ to the diagonal is a $\mathcal{D}_X^{\otimes I}$-module. There is a unique isomorphism of $\mathcal{D}_X^{\otimes I}$-modules

$$M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I \xrightarrow{\sim} \Delta^{(I)\cdot} \Delta^{(I)}_* M \tag{2.2.9.1}$$

which induces the identity isomorphism between the subsheaves $M = M \otimes 1^{\otimes I}$ and $M = \Delta^{(I)!} \Delta^{(I)}_* M$. This yields the isomorphism

$$\mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}(\otimes L_i, M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I) \xrightarrow{\sim} P^*_I(\{L_i\}, M). \tag{2.2.9.2}$$

We leave it to the reader to identify the maps $< \ >^I_{S,T}$.

2.2.10. We finish this section with several remarks about the $*$ pseudo-tensor structure. The operad $\mathcal{B}^*_I$ satisfies conditions (i), (ii) of 1.2.3. therefore, for any *coherent* $\mathcal{D}$-modules

L_i and an arbitrary $\mathcal{D}$-module M, the object $\mathcal{P}_I(\{L_i\}, M) \in \mathcal{M}(X)$ exists. Explicitly, one has

$$\mathcal{P}_I(\{L_i\}, M) = \underline{\mathrm{Hom}}_{\mathcal{D}_X^{\otimes I}}(\underset{k}{\otimes} L_i, M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{\tilde{I}}). \tag{2.2.10.1}$$

Consider the morphism of $(\mathcal{O}_X, \mathcal{D}_X^{\otimes I})$-bimodules $\mathcal{B}^*_I \to \mathcal{B}^*_{\tilde{I}}$, $b \mapsto 1. \otimes b$ (recall that $\tilde{I} = \{\,\cdot\,\} \sqcup I$). The corresponding morphism of $(\mathcal{D}_X, \mathcal{D}_X^{\otimes I})$-bimodules $\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_I \to \mathcal{B}^*_{\tilde{I}}$ is isomorphism. It yields the identification $M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_I = M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{\tilde{I}}$, hence also

$$\mathcal{P}_I(\{L_i\}, M) = \underline{\mathrm{Hom}}_{\mathcal{D}_X^{\otimes I}}(\underset{k}{\otimes} L_i, M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_I). \tag{2.2.10.2}$$

Note that the right $\mathcal{D}_X$-action on $M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_{\tilde{I}}$ (used to define the $\mathcal{D}$-module structure on the $\underline{\mathrm{Hom}}$ sheaf) identifies with the right $\mathcal{D}$-module structure on $M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_I$ that comes from the right $\mathcal{D}_X$-module structure on M and the left one on $\mathcal{B}^*_I$ (see 2.1.1).

Remark. The functor $\mathcal{P}^*_I$ obviously commutes with inverse images for étale morphisms, i.e., it is defined on the sheaf of categories $\mathcal{M}(X_{\text{ét}})$. It also commutes with direct images for closed embeddings (as follows immediately from 2.2.5).

2.2.11. If we deal with induced $\mathcal{D}_X$-modules, i.e., $L_i = F_{i\mathcal{D}}$, $M = G_{\mathcal{D}}$, where F_i are coherent $\mathcal{O}$-modules, then the corresponding $\mathcal{D}$-module $\mathcal{P}^*_I$ can be described as follows. Consider the sheaf of differential I-operations $\mathrm{Diff}_I(\{F_i\}, G)_X$.[53] This is an object of $\mathcal{D}iff(X)$ by 2.1.8(i), so we have the corresponding $\mathcal{D}$-module $\mathrm{Diff}_I(\{F_i\}, G)_{\mathcal{D}} = \mathrm{Diff}_I(\{F_i\}, G)_{X\mathcal{D}}$. Now the canonical $\tilde{I}$-operation $\in \mathrm{Diff}_{\tilde{I}}(\{\mathrm{Diff}_I(\{F_i\}, G)_X, F_i\}, G) = P^*_{\tilde{I}}(\{\mathrm{Diff}_I(\{F_i\}, G)_{\mathcal{D}}, F_{i\mathcal{D}}\}, G_{\mathcal{D}})$ yields an isomorphism[54]

$$\mathrm{Diff}_{IX}(\{F_i\}, G)_{\mathcal{D}} \xrightarrow{\sim} \mathcal{P}^*_I(\{F_{i\mathcal{D}}\}, G_{\mathcal{D}}). \tag{2.2.11.1}$$

2.2.12. So for a $\mathcal{D}_X$-coherent L and any $M \in \mathcal{M}(X)$ we have the $\mathcal{H}om$ $\mathcal{D}_X$-module $\mathcal{H}om^*(L, M) = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X) \in \mathcal{M}(X)$ equipped with a universal "evaluation"

[53] Which is the X-sheafified version of the vector space of differential I-operations from 2.2.4(i).

[54] To check that our arrow is an isomorphism one reduces (using left exactness of our functors) to the case $F_i = \mathcal{O}_X$ where the statement is obvious.

$*$ pairing $\varepsilon \in P_2^*\ (\{\mathcal{H}om^*(L,M),L\},M)$. If K is another coherent $\mathcal{D}$-module, then the composition $*$ pairing $c \in P_2^*(\{\mathcal{H}om^*(L,M),\mathcal{H}om^*(K,L)\},\mathcal{H}om^*(K,M))$ is defined; the composition is associative and compatible with evaluation (see 1.2.2).

Remark. ε is just the usual evaluation morphism $\underline{\mathrm{Hom}}\Big(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big) \underset{k}{\otimes} L \to M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$. To describe c explicitly, it is convenient to understand $\underline{\mathrm{Hom}}_{\mathcal{D}_X}\Big(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big)$ in the way different from what we used before: let us assume now that the $\mathcal{D}_X$-module structure on $M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ comes from the right $\mathcal{D}_X$-module structure on M and the left one on $\mathcal{D}_X$ while the action of $\mathcal{D}_X$ on $\underline{\mathrm{Hom}}_{\mathcal{D}_X}\Big(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big)$ comes from the right $\mathcal{D}_X$-action on $\mathcal{D}_X$. Since $M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X = M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{\{1,2\}}$ has a canonical automorphism interchanging the two $\mathcal{D}$-module structures on $M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ we can still identify $\mathcal{H}om^*(L,M)$ with $\underline{\mathrm{Hom}}_{\mathcal{D}_X}\Big(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big)$. After this identification the $*$ pairing c becmes the usual composition

$$\underline{\mathrm{Hom}}\Big(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big) \underset{k}{\otimes} \underline{\mathrm{Hom}}\Big(K, L \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big) \to \underline{\mathrm{Hom}}\Big(K, \Big(M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big) \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big)$$
$$= \underline{\mathrm{Hom}}\Big(K, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X\Big) \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X.$$

2.2.13. We see that for any coherent $M \in \mathcal{M}(X)$ the $\mathcal{D}$-module $\mathcal{E}nd^*(M) = \mathcal{H}om^*(M,M)$ is an associative algebra in $*$ sense acting on M. This action is universal: for any associative aglebra A in $\mathcal{M}(X)^*$ an A-module structure on M is the same as a morphism of associative algebras $A \to \mathcal{E}nd^* M$.

Set $M' := \mathcal{H}om^*(M,\omega_X)$; this is again a coherent module. We have a canonical pairing $\langle\ \ \rangle \in P_2^*(\{M',M\},\omega_X)$.

If M is a $\mathcal{D}_X$-vector bundle (see 2.1.5) then $\langle\ \ \rangle$ is non-degenerate (see 1.4.2); in particular, $M \otimes M' \xrightarrow{\sim} \mathcal{E}nd^* M$.

2.2.14. According to 1.3.12, the duality functor $'\colon \mathcal{M}(X)^{\circ}_{coh} \to \mathcal{M}(X)$ extends to the pseudo-tensor functor $'\colon \mathcal{M}(X)^{\circ !}_{coh} \to \mathcal{M}(X)^*$. It induces an equivalence between the full pseudo-tensor subcategories of $\mathcal{D}_X$-vector bundles.

Example. Let F be a vector bundle on X, F^* the dual bundle, $F^\circ := F^* \otimes \omega_X$. Then $F_{\mathcal{D}}$ is a $\mathcal{D}_X$-vector bundle and there is a canonical identification $F'_{\mathcal{D}} := (F_{\mathcal{D}})' \xrightarrow{\sim} F^\circ_{\mathcal{D}}$

(defined by the image of the canonical pairing $\in \mathrm{Hom}(F\otimes F^\circ,\omega_X)\subset \mathrm{Diff}_2(\{F,F^\circ\},\omega_X)=P_2^*(\{F_{\mathcal{D}},F^\circ_{\mathcal{D}}\},\omega_{\mathcal{D}})$ by the canonical morphism $\omega_{\mathcal{D}}:=\omega_{X\mathcal{D}}\to\omega_X$). Denote by ${}_{\mathcal{D}}F$ the left $\mathcal{D}$-module $\mathcal{D}_X\underset{\mathcal{O}_X}{\otimes}F$; we see that $(F'_{\mathcal{D}})^\ell = {}_{\mathcal{D}}F^*$.

If F_i, G are vector bundles on X then the duality isomorphism $'$: $\mathrm{Hom}(G'_{\mathcal{D}},\otimes^! F'_{i\mathcal{D}})\xrightarrow{\sim} P_I^*(\{F_{i\mathcal{D}}\},G_{\mathcal{D}})$ is the composition of the following identifications (see 2.2.8 for notation) $\mathrm{Hom}({}_{\mathcal{D}}G^*,\otimes({}_{\mathcal{D}}F_i^*)) = \mathrm{Hom}_{\mathcal{O}_X}(G^*,\mathcal{B}_I^*\underset{\mathcal{O}_X^{\otimes I}}{\otimes}(\otimes F_i^*)) = \mathrm{Hom}_{\mathcal{O}_{X^I}}(\boxtimes F_i, G\underset{\mathcal{O}_X}{\otimes}\mathcal{B}_I^*) = \mathrm{Hom}(\boxtimes F_{i\mathcal{D}},(\Delta^{(I)}_{\cdot}G)_{\mathcal{D}}) = \mathrm{Hom}(\boxtimes F_{i\mathcal{D}},\Delta^{(I)}_{*}G_{\mathcal{D}})$.

2.2.15. A remark about the augmentation functor. Take $I,J\in\mathcal{S}$, the families of objects $\{L_i\}$, $\{A_j\}$, M and consider the composition morphism for $I\hookrightarrow I\sqcup J$ in our augmented pseudo-tensor structure $\underline{P}^*_{I\sqcup J}(\{L_i,A_j\},M)\otimes\underset{J}{\otimes}h(A_j)\to\underline{P}^*_I(\{L_i\},M)$. We can rewrite it as a morphism

$$h_{IJ}\colon P^*_{I\sqcup J}(\{L_i,A_j\},M)\to\mathrm{Hom}\,(\underset{J}{\otimes}h(A_j),\underline{P}^*_I(\{L_i\},M))\,. \tag{2.2.15.1}$$

2.2.16 Lemma. The map $h_{I,J}$ is injective. Its image is the subspace of those $\varphi\colon\underset{J}{\otimes}h(A_j)\to P_I^*(\{L_i\},M)$ that the map $\widetilde{\varphi}\colon(\otimes L_i)\otimes(\otimes A_j)\to M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*$, $(\otimes\ell_i)\otimes(\otimes a_j)\mapsto\varphi(\otimes\bar a_j)(\otimes\ell_i)$, is a differential operator with respect to either of variables A_j (see 2.1.8(iii)).

In other words, φ satisfies the following condition (here $\mathcal{I}$ is the kernel of $\mathcal{O}_X^{\otimes I\sqcup J}\to\mathcal{O}_X$):

($*$) For any local section $s\in(\otimes L_i)\otimes(\otimes A_j)$ one has $\widetilde{\varphi}(\mathcal{I}^N s)=0$ for $N\gg 0$.

Proof. Let $Tr\colon M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}^*_{I\sqcup J}\to M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}^*_{I\sqcup J}\underset{\mathcal{D}_X^{\otimes J}}{\otimes}\mathcal{O}_X^{\otimes J}=M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*$ be the trace map along J-variables. The corresponding morphism of $\mathcal{D}_X^{\otimes I\sqcup J}$-modules $M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}^*_{I\sqcup J}\to\mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}\Big(\mathcal{D}_X^{\otimes I}\otimes\mathcal{O}_X^{\otimes J},M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*\Big)$, which assigns to $b\in M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}^*_{I\sqcup J}$ the map $\alpha\mapsto Tr(b\alpha)$, is injective. Its image is the subspace of those $\psi\colon\mathcal{D}_X^{\otimes I}\otimes\mathcal{O}_X^{\otimes J}\to M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*$ that kill some $\mathcal{I}^N\subset\mathcal{O}_X^{\otimes I\sqcup J}\subset\mathcal{D}_X^{\otimes I}\otimes\mathcal{O}_X^{\otimes J}$. This morphism maps $P^*_{I\sqcup J}(\{L_i,A_j\},M)$ injectively to

$$\mathrm{Hom}_{\mathcal{D}_X^{\otimes I\sqcup J}}\Big((\otimes L_i)\otimes(\otimes A_j),\mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}\Big(\mathcal{D}_X^{\otimes I}\otimes\mathcal{O}_X^{\otimes J},M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*\Big)$$
$$=\mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}\Big((\otimes L_i)\otimes(\otimes h(A_j)),M\underset{\mathcal{D}_X}{\otimes}\mathcal{B}_I^*\Big)=\mathrm{Hom}(\otimes h(A_j),\underline{P}_I^*(\{L_i\},M))$$

The image coincides with the subspace defined by condition $(*)$. Since this map coincides with $h_{I,J}$, we are done. $\square$

The following technical lemma will be of use. Let L_i, M be $\mathcal{D}$-modules on X, $\varphi \in P_I^*(\{L_i\}, M)$, $x \in X$ a closed point. Set $U_x := X \smallsetminus \{x\}$. We use the terminology of 2.1.13.

2.2.17 Lemma. (i) The polylinear map $h(\varphi) : \otimes h(L_i) \to h(M)$ is Ξ_x-conitnuous with respect to each variable.

(ii) Assume that the restriction to U_x of each L_i to U_x is a countably generated $\mathcal{D}$-module. The $h(\varphi)$ is Ξ_x-continuous.

(iii) Assume that for certain $i_0 \in I$ every L_i but, possibly, L_{i_0} is a coherent $\mathcal{D}_X$-module. Then for each $M_\xi \in \Xi_x(M)$ there exists $L_{i_0\xi} \in \Xi_x(L_{i_0})$ such that

$$\varphi(L_{i_0\xi} \boxtimes (\underset{i \neq i_0}{\boxtimes} L_i)) \subset \Delta_*^{(I)} M_\xi.$$

Proof. (i) Pick $i_0 \in I$ and $\bar{l}_i \in h(L_i)_x$ for $i \neq i_0$. We have to show that the map $h(L_{i_0}) \to h(M)$, $\bar{l}_{i_0} \mapsto h(\varphi)(\bar{l}_{i_0} \otimes (\otimes \bar{l}_i))$, is Ξ_x-continuous. This is clear since this map comes from a morphism of $\mathcal{D}$-modules $L_{i_0} \to M$ that comes from φ and $\bar{l}_i$'s by the augmentation functor structure on h.

(ii) An induction by $|I|$ shows (use the augmentation functor structure on h) that to prove continuity of $h(\varphi)$ it suffices to find for every $M_\xi \in \Xi_x(M)$ some $L_{i\xi} \in \Xi_x(L_i)$ such that $h(\varphi)(\otimes L_{i\xi}) \subset M_\xi$. Shrinking X if necessary we can assume that X is affine and we have functions t_a, $a = 1, .., n$, on it which are local coordinates at x and have no other common zero. Let $l_{i1}, l_{i2}, ..$ be sections of L_i that generate the restriction of L_i to U_x as a $\mathcal{D}_{U_x}$-module. The quotient $\Delta_*^{(I)} M / \Delta_*^{(I)} M_\xi = \Delta_*(M/M_\xi)$ is a $\mathcal{D}_{X^I}$-module supported at point $\Delta^{(I)}(x)$, so every its section is killed by sufficiently high power of each t_a lifted from either copy of X. Therefore, using induction by $b = 1, 2, ..$, we can choose integers $n(a, b, i) \geq 0$ such that $h(\varphi)(\underset{I}{\boxtimes} t_a^{n(a,b_i,i)} l_{ib_i}) \in \Delta_*^{(I)} M_\xi \subset \Delta_*^{(I)} M$ for every $1 \leq b_i \leq b$. Our $L_{i\xi}$ is the $\mathcal{D}$-submodule of L_i generated by $\{t_a^{n(a,b,i)} l_{ib}\}$.

(iii) Choose for $i \neq i_0$ a finite set $\{l_{ij}\}$ of local sections of L_i at x that generates L_i. Now take for $L_{i_0\xi}$ the intersection of the preimages of $\Delta_*^{(I)} M_\xi$ by all maps $L_{i_0} \to \Delta_*^{(I)} M$, $l \mapsto \varphi(l \boxtimes (\boxtimes l_{ij}))$. $\square$

We see that in situation (ii) the morphism $h_I(\varphi)\colon \otimes h(L_i)_x \to h(M)_x$ extends by continuity to the map $\hat h_I(\varphi)_x\colon \widehat{\otimes}\, \hat h_x(L_i) \to \hat h_x(M)$. Here $\widehat{\otimes}\, \hat h_x(L_i) := \varprojlim \otimes h_x(L_i/L_{i\,\xi_i})$.

If we decide to play with better objects $\tilde h_x$ from Remark (iii) of 2.1.14 instead of plain topological vector spaces then the countability condition of (ii) becomes irrelevant.

§3 $\mathcal{D}_X$-SCHEMES

2.3.1. A *commutative $\mathcal{D}_X$-algebra* is a commutative unital $\mathcal{O}_X$-algebra equipped with an integrable connection along X. Unless it is stated explicitly otherwise our $\mathcal{D}_X$-algebras are assumed to be $\mathcal{O}_X$-quasicoherent. A *$\mathcal{D}_X$-scheme* is an X-scheme equipped with an integrable connection along X. Denote by $\mathcal{C}omu_{\mathcal{D}}(X)$, $\mathcal{S}ch_{\mathcal{D}}(X)$ the corresponding categories. We have a fully faithful functor $\mathrm{Spec}\colon \mathcal{C}omu_{\mathcal{D}}(X)^\circ \hookrightarrow \mathcal{S}ch_{\mathcal{D}}(X)$; its essential image is the category of $\mathcal{D}_X$-schemes affine over X.

Remark. Let x_a be a coordinate system on X. Assume we have a system of non-linear partial differential equations $P_\alpha(u_i, \partial_a u_i, ...) = 0$ where P_α are polynomial functions with coefficients in functions on X. It yields a $\mathcal{D}_X$-algebra $R = R(\{P_\alpha\})$ defined as the quotient of the free $\mathcal{D}_X$-algebra $\mathrm{Sym}(\oplus \mathcal{D}_X \cdot u_i)$ modulo the relations P_α. A solution of our system with values in a (non necessary quasi-coherent) $\mathcal{D}_X$-algebra Φ is a collection of $\phi_i \in \Phi$ such that $P_\alpha(\phi_i, \partial_a \phi_i, ..) = 0$. This is the same as a morphism of $\mathcal{D}_X$-algebras $R \to \Phi$, $u_i \mapsto \phi_i$. Therefore $\mathcal{D}_X$-algebras provide a "coordinate free" languge for non-linear pde's.

Set $\mathcal{C}omu^!(X) := \mathcal{C}omu(\mathcal{M}^!(X))$ (we use the notation of 1.4.6). The equivalence of tensor categories $\mathcal{M}^!(X) \xrightarrow{\sim} \mathcal{M}^\ell(X)$, $A \mapsto A^\ell = A\omega_X^{-1}$, yields the equivalence

$$\mathcal{C}omu^!(X) \xrightarrow{\sim} \mathcal{C}omu_{\mathcal{D}}(X). \tag{2.3.1.1}$$

2.3.2. Denote by $\mathcal{S}ch(X)$, $\mathcal{C}omu(X)$ the category of X-schemes, resp. that of commutative $\mathcal{O}_X$-algebras (quasicoherent as $\mathcal{O}_X$-modules). We have the obvious forgetting of connection functors

$$o\colon \mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(X), \quad \mathcal{S}ch_{\mathcal{D}}(X) \to \mathcal{S}ch(X). \tag{2.3.2.1}$$

They admit the left, resp. right, adjoint functors

$$\mathcal{C}omu(X) \to \mathcal{C}omu_{\mathcal{D}}(X), \quad \mathcal{S}ch(X) \to \mathcal{S}ch_{\mathcal{D}}(X). \tag{2.3.2.2}$$

We denote them both by $\mathcal{J}$. For $R \in \mathcal{C}omu(X)$ $\mathcal{J}R$ is generated by R. Explicitly, it is the $\mathcal{D}_X$-algebra quotient of the symmetric algebra $\mathrm{Sym}^{\cdot}\left(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} R\right)$ modulo the ideal generated by elements $\partial(1 \otimes r_1 \cdot 1 \otimes r_2 - 1 \otimes r_2 r_2) \in \mathrm{Sym}^2\left(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} R\right) + \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} R$ and $\partial(1 \otimes 1_R - 1) \in \mathcal{D}_X \otimes R + \mathcal{O}_X$ where $r_i \in R$, $\partial \in \mathcal{D}_X$, 1_R is the unit of R. This construction is compatible with localization. The functor $\mathcal{J}\colon \mathcal{S}ch(X) \to \mathcal{S}ch_{\mathcal{D}}(X)$ is $\operatorname{Spec} R \mapsto \operatorname{Spec} \mathcal{J}R$ for affine schemes; for arbitrary schemes use glueing. Note that for any $Z \in \mathcal{S}ch(X)$ the canonical projection $\mathcal{J}Z \to Z$ induces a bijection $(\mathcal{J}Z)_{\mathcal{D}}(X) \xrightarrow{\sim} Z(X)$ where $(\mathcal{J}Z)_{\mathcal{D}}(X)$ is the set of all horizontal sections $X \to Z$.

2.3.3. Here is a different description of $\mathcal{J}R$. Denote by $\mathcal{I}$ the kernel of the product map $\mathcal{O}_X^{\otimes 2} = \mathcal{O}_X \underset{k}{\otimes} \mathcal{O}_X \to \mathcal{O}_X$; for $n \geq 1$ set $\mathcal{O}_{X(n)} := \mathcal{O}_X^{\otimes 2}/\mathcal{I}^n$. These $\mathcal{O}_{X(n)}$ form a projective system of algebras; there are two obvious morphisms $\mathcal{O}_X \rightrightarrows \mathcal{O}_{X(n)}$. Therefore any $\mathcal{O}_X$-algebra B yields two $\mathcal{O}_{X(n)}$-algebras: $B \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X(n)}$ and $\mathcal{O}_{X(n)} \underset{\mathcal{O}_X}{\otimes} B$.

2.3.4 Lemma. For any $R, B \in \mathcal{C}omu(X)$ one has

$$\operatorname{Hom}(\mathcal{J}R, B) = \varprojlim \operatorname{Hom}\left(\mathcal{O}_{X(n)} \underset{\mathcal{O}_X}{\otimes} R, B \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X(n)}\right) \tag{2.3.4.1}$$

where the left and right Hom mean, respectively, the morphisms of $\mathcal{O}_X$- and $\mathcal{O}_{X(n)}$-algebras.

Proof. Our statement is X-local, so we may assume that X is affine, $X = \operatorname{Spec} C$. Thus $\mathcal{O}_X$-algebras are the same as C-algebras. For a C-algebra B denote by J_B the kernel of the map $B \underset{k}{\otimes} C \to B$, $b \otimes c \mapsto bc$. Set $\mathcal{T}B := \varprojlim B \otimes C/J_B^n$. Let us consider $\mathcal{T}B$ as a C-algebra via the morphism $C \to \mathcal{T}B$, $c \mapsto 1 \otimes c$. The projection $\mathcal{T}B \to C$, $b \otimes c \mapsto bc$, is a morphism of C-algebras. Note that $\mathcal{T}B$ is a $\mathcal{D}_X$-algebra in the obvious way (the elments $b \otimes 1 \in \mathcal{T}B$ are horizontal). It is easy to see that the functor $\mathcal{C}omu(X) \to \mathcal{C}omu_{\mathcal{D}}(X)$, $B \mapsto \mathcal{T}B$, is *right* adjoint to the functor o: $\mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(X)$. Therefore we have $\operatorname{Hom}(o\mathcal{J}R, B) = \operatorname{Hom}(\mathcal{J}R, \mathcal{T}B) = \operatorname{Hom}(R, o\mathcal{T}B)$. The latter set coincides with $\varprojlim \operatorname{Hom}$ in the statement of 2.3.4; we are done. $\square$

So for any X-scheme Y the k-points of $\mathcal{J}Y$ are the same as pairs (x,γ), $x\in X$, γ is a section of Y on the formal neighborhood of x. We call $\mathcal{J}Y$ the *jet scheme of* Y.

Remark. The above constructions are compatible with étale localization, so we may replace schemes by algebraic spaces.

2.3.5. For $A^\ell\in \mathcal{C}omu_{\mathcal{D}}(X)$ we denote by $A^\ell[\mathcal{D}_X]$ the ring of differential operators with coefficients in A^ℓ. By definiton, this is a sheaf of associative algebras equipped with a morphism of algebras $A^\ell\to A^\ell[\mathcal{D}_X]$ and that of Lie algebras $\Theta_X\to A^\ell[\mathcal{D}_X]$ which satisfy the relations $\tau\cdot a-a\cdot\tau=\tau(a)$, $f\cdot\tau=f\tau$ for $a\in A^\ell$, $f\in\mathcal{O}_X\subset A^\ell$, $\tau\in\Theta_X$, and universal with respect to these data in the obvious sense. The embedding of Θ_X extends to the morphism of associative algebras $\mathcal{D}_X\to A^\ell[\mathcal{D}_X]$. One checks easily that the morphism of sheaves $A^\ell\underset{\mathcal{O}_X}{\otimes}\mathcal{D}_X\to A^\ell[\mathcal{D}_X]$, $a\otimes\partial\mapsto a\cdot\partial$ is an isomorphism.

For any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the corresponding sheaf $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$ on $\mathcal{Y}_{\text{ét}}$.

2.3.6. An A^ℓ-module in the tensor category $\mathcal{M}^\ell(X)$ is the same as a left $A^\ell[\mathcal{D}_X]$-module quasicoherent as $\mathcal{O}_X$-module. Similarly, an A-module in $\mathcal{M}^!(X)$ is the same as right $A^\ell[\mathcal{D}_X]$-module quasicoherent as an $\mathcal{O}_X$-module. We denote by $\mathcal{M}^\ell(X,A^\ell)$, $\mathcal{M}(X,A^\ell)=\mathcal{M}^r(X,A^\ell)$ the categories of left, resp. right, $A^\ell[\mathcal{D}_X]$-modules. These are tensor categories (canonically identified) with unit object equal to A^ℓ, resp. A.

Every morphism of commutative $\mathcal{D}_X$-algebras $f:A^\ell\to B^\ell$ yields an adjoint pair of functors $f^*:\mathcal{M}(X,A^\ell)\to\mathcal{M}(X,B^\ell)$, $f_\cdot:\mathcal{M}(X,B^\ell)\to\mathcal{M}(X,A^\ell)$ where $f^*M:=B^\ell\underset{A^\ell}{\otimes}M$, $f_\cdot N$ is N considered as an $A^\ell[\mathcal{D}_X]$-module. Notice that f^* is a tensor functor.

The faithfully flat descent works in any abelian tensor category, so $A^\ell[\mathcal{D}_X]$-modules are local objects with respect to flat topology.[55] So for any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the corresponding categories $\mathcal{M}^\ell(\mathcal{Y})$, $\mathcal{M}^r(\mathcal{Y})=\mathcal{M}(\mathcal{Y})$. There objects are sheaves of left, resp. right, $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules which are quasicoherent as $\mathcal{O}_Y$-modules. The categories $\mathcal{M}^\ell(\mathcal{U})$, $\mathcal{M}(\mathcal{U})$ for $\mathcal{U}\in\mathcal{Y}_{\text{ét}}$ form sheaves of categories $\mathcal{M}^\ell(\mathcal{Y}_{\text{ét}})$ on $\mathcal{Y}_{\text{ét}}$. One has $\mathcal{M}^\ell(\operatorname{Spec}A^\ell)=\mathcal{M}^\ell(X,A)$, $\mathcal{M}(\operatorname{Spec}A^\ell)=\mathcal{M}(X,A^\ell)$.

[55] f is faithfully flat as a morphism of commutative algebras in the tensor category $\mathcal{M}^\ell(X)$ if and only if it is faithfully flat as a morphism of plain $\mathcal{O}_X$-algebras.

Most of the facts mentioned in §2.1 render easily to the $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules setting. E.g., if G, L are left $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules and M is right one, then $G \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} L$ is left $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module in a natural way, and $L \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} M$ is right one. We have the standard equivalence $\mathcal{M}^{\ell}(\mathcal{Y}) \to \mathcal{M}^r(\mathcal{Y})$, $L \mapsto L\omega_X$. Any $N \in \mathcal{M}(\mathcal{Y})$ admits a canonical finite left resolution $(DR(N))_{\mathcal{D}}$ whose terms are induced $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules.

2.3.7 Theorem. Assume that $\mathcal{Y}$ is a $\mathcal{D}_X$-scheme which is affine (as a k-scheme). Then projectivity of an $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module is a local property for the Zariski or étale topology.

Proof. It occupies 2.3.8–2.3.10.

2.3.8. Let us first prove the theorem for a finitely generated $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module P. Suppose that P is projective over some étale covering of $\mathcal{Y}$. We need to show that P is projective. Take $N \in \mathcal{M}(\mathcal{Y})$. Consider the sheaf $\underline{\mathrm{Hom}}(P,N)$ on $\mathcal{Y}_{\text{ét}}$. It depends on N in exact way, so projectivity of P follows if we prove that $H^i(\mathcal{Y}_{\text{ét}}, \underline{\mathrm{Hom}}(P,N)) = 0$ for $i > 0$.

This is true if N is an induced module, $N = F \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} \mathcal{D}_X$. Indeed, in this situation $\underline{\mathrm{Hom}}(P,N)$ admits a structure of quasicoherent $\mathcal{O}_{\mathcal{Y}}$-module. Namely, the $\mathcal{O}_{\mathcal{Y}}$-module structure on $\underline{\mathrm{Hom}}(P,N)$ comes from an $\mathcal{O}_{\mathcal{Y}}$-action on N inherited from F. Quasicoherence is étale local property, so it suffices to check it over opens where P is projective. There $\underline{\mathrm{Hom}}(P,N)$ is isomorphic to a direct summand of a sum of several copies of N, which is obviously quasicoherent.

To prove acyclicity of $\underline{\mathrm{Hom}}(P,N)$ for arbitrary N use the fact that N admits a finite left resolution by induced modules (see 2.1.9, 2.3.6). We are done.

2.3.9. Without the finite generatedness assumption Theorem 2.3.7 is nontrivial even if X is a point. In this case it was conjectured by Grothendieck (see Remark 9.5.8 from [Gr]) and proved by Raynaud and Gruson (even for the fpqc topology); see [RG], p.82, 3.1.4. We will use the method of [RG]. It is based on the following fact: a flat module M over a ring R is projective if and only if M has the following properties:

(i) M is a direct sum of countably generated modules;

(ii) M is a Mittag-Leffler module

(the implication projectivity$\Rightarrow$(i) is due to Kaplansky [Ka]; the fact that (i) and (ii) imply projectivity is proved in [RG], p.74). A flat module M is said to be Mittag-Leffler if for some (or for every) representation of M as a direct limit of finitely generated projective modules P_i (i belongs to a directed ordered set I) the projective system (P_i^*) satisfies the Mittag-Leffler condition (i.e., for every $i \in I$ there exists $j \geq i$ such that $\mathrm{Im}(P_j^* \to P_i^*) = \mathrm{Im}(P_k^* \to P_i^*)$ for all $k \geq j$). Here $P_i^* := \mathrm{Hom}_R(P_i, R)$.

Now Theorem 2.3.7 is easily reduced to the following lemma.

2.3.10 Lemma. Assume that X is affine and $\mathcal{Y}$ is an affine $\mathcal{D}_X$-scheme. Then a flat $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module M is globally flat, i.e., $\Gamma(\mathcal{Y}, M)$ is flat over $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X])$.

Proof. M has a finite left resolution consisting of globally flat $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules (indeed, M is $\mathcal{O}_{\mathcal{Y}}$-flat, so the canonical resolution $(DR(M))_{\mathcal{D}}$ has the required property). Therefore it suffices to show that if there is an exact sequence $0 \to F_1 \to F_0 \to M \to 0$ with F_0, F_1 globally flat and M flat then M is globally flat, i.e., for every left $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module L the morphism $\Gamma(\mathcal{Y}, F_1) \otimes_{\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X])} \Gamma(\mathcal{Y}, L) \to \Gamma(\mathcal{Y}, F_0) \otimes_{\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X])} \Gamma(\mathcal{Y}, L)$ is injective. The map $\Gamma(\mathcal{Y}, F_i) \otimes_{\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X])} \Gamma(\mathcal{Y}, L) \to \Gamma(\mathcal{Y}, F_i \otimes_{\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]} L)$ is an isomorphism (represent F_i as a direct limit of finitely generated free $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules). Finally, the morphism $F_1 \otimes_{\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]} L \to F_0 \otimes_{\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]} L$ is injective because M is flat. $\square$

2.3.11. We see that for any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ there is a notion of $\mathcal{Y}$-locally projective $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module. Notice that for $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules on an affine $\mathcal{Y}$ the property of being finitely generated is obviously étale local. So we know what are $\mathcal{Y}$-locally finitely generated $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. We call a $\mathcal{Y}$-locally finitely generated and projective $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module a *$\mathcal{D}_X$-vector bundle on $\mathcal{Y}$*.

2.3.12. According to 1.4.6 (and §2.2) for a commutative $\mathcal{D}_X$-algebra A^ℓ the category $\mathcal{M}(X, A^\ell)$ carries a canonical augmented compound tensor structure. As in 1.4.6 for $L_i, M \in \mathcal{M}(X, A^\ell)$ we denote the corresponding space of A^ℓ-polylinear $*$ operations by $P^*_{A\,I}(\{L_i\}, M)$.

In fact, this structure comes from an augmented compound strict $A^\ell[\mathcal{D}_X]$-operad $\mathcal{B}^{*!}_A$. We leave the explicit construction of $\mathcal{B}^{*!}_A$ to the reader (repeat 2.2.8, replacing everywhere

$\mathcal{D}_X$ by $A^\ell[\mathcal{D}_X]$). The discussion 2.2.10 – 2.2.14 remains literally valid for $A^\ell[\mathcal{D}_X]$-modules if we replace the words "coherent $\mathcal{D}_X$-module" by "finitely presented $A^\ell[\mathcal{D}_X]$-module". The lemma 2.2.16 also remains valid.

According to Remark (i) of 1.4.6 for every morphism of commutative $\mathcal{D}_X$-algebras $f : B^\ell \to A^\ell$ the corresponding tensor functor f^* from 2.3.6 extends to a compound tensor functor (base change)

$$f^* : \mathcal{M}(X, A^\ell)^{*!} \to \mathcal{M}(X, B^\ell)^{*!}. \tag{2.3.12.1}$$

We also have a forgetful paseudo-tensor functor

$$f_\cdot : \mathcal{M}(X, B^\ell)^{*} \to \mathcal{M}(X, A^\ell)^{*} \tag{2.3.12.2}$$

right adjoint to f^* considered as a $*$ pseudo-tensor functor (see Remark (ii) of 1.4.6).

Lemma. $\mathcal{M}(X, A^\ell)^{*!}$ has local nature with respect to flat topology of SpecA^ℓ.

Proof. We already discussed the flat descent of $A^\ell[\mathcal{D}_X]$-modules in 2.3.6, so it remains to check that the A-polylinear $*$ operations satisfy the flat descent property. For $L_i, M \in \mathcal{M}(X, A^\ell)$ the above adjunction identifies the descent data with operations $\psi \in P^*_{A\,I}(\{L_i\}, B^\ell \underset{A^\ell}{\otimes} M)$ whose composition with the two arrows $B^\ell \underset{A^\ell}{\otimes} M \rightrightarrows B^\ell \underset{A^\ell}{\otimes} B^\ell \underset{A^\ell}{\otimes} M$, $b\otimes m \mapsto b\otimes 1\otimes m, 1\otimes b\otimes m$, coincide. Now the exact sequence $0 \to M \to B^\ell \underset{A^\ell}{\otimes} M \to B^\ell \underset{A^\ell}{\otimes} B^\ell \underset{A^\ell}{\otimes} M$ (the latter arrow is the difference of the above two standard ones) together with the left exactness of P^*_A implies the desired descent property. □

For L_i, M as above an *A-polydifferential $*$ operation* is an operation $\varphi \in P^*_I(\{L_i\}, M)$ such that for every finitely generated $A^\ell[\mathcal{D}_X]$-submodules $L'_i \subset L_i$ the corresponding restriction $\in P^*_I(\{L'_i\}, M)$ is an A-polydifferential $*$ operation of finite order (see 1.4.7). The composition of A-polydifferential operations is A-polydifferential, so they define another pseudo-tensor structure on $\mathcal{M}(X, A^\ell)$.

Lemma. A-polydifferential $*$ operations have local nature with respect to the étale topology of SpecA^ℓ.

Proof. The case of arbitrary A-polydifferential operations reduces immediately to that of A-polydifferential operations of finite order, and then to that of A-polylinear ones (replace

L_i by $L_i^{(n)}$ as in 1.4.7 and notice that $L_i^{(n)}$ have étale local nature). Then use the previous lemma. $\square$

Therefore for any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the compound tensor category $\mathcal{M}(\mathcal{Y})^{*!}$ and the sheaf of compound tensor categories $\mathcal{M}(\mathcal{Y}_{\text{ét}})^{*!}$ on $\mathcal{Y}_{\text{ét}}$, the pull-back functors f^* are compound tensor functors, etc. The polydifferential $*$ operations also make sense in this setting.

Remarks. (i) One has the following immediate analog of 2.2.10. Let $\{L_i\}$ be a finite collection of finitely presented $A^\ell[\mathcal{D}_X]$-modules, M any $A^\ell[\mathcal{D}_X]$-module. Then the $A^\ell[\mathcal{D}_X]$-module $\mathcal{P}^*_{A\,I}(\{L_i\}, M)$ is well-defined. The functor $\mathcal{P}^*_{A\,I}$ is compatible with flat pull-backs. So the functor $\mathcal{P}^*_I$ makes sense on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$.

(ii) Thus for every finitely presented $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module L its dual $L' := \mathcal{H}om^*(L, \mathcal{O}_\mathcal{Y}\omega_X)$ is well-defined. In particular, if the $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module of differentials $\Omega_\mathcal{Y} = \Omega_{\mathcal{Y}/X}$ is finitely presented[56] then $\Theta_\mathcal{Y} := \Omega'_\mathcal{Y}$ is well-defined.

(iii) If L is a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$ then L' is also a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$, and the canonical pairing is non-degenerate (see 1.4.2).

2.3.13. We are going to discuss the notion of smoothness for algebraic $\mathcal{D}_X$-spaces. First let us recall the definition of formally smooth algebras.

Let $\mathcal{M}$ be any abelian tensor k-category with unit object (so $\otimes$ is right exact). A commutative algebra A in $\mathcal{M}$ is *formally smooth* if for any (commutative) algebra R and an ideal $I \subset R$ such that $I^2 = 0$, any morphism $f: A \to R/I$ may be lifted to a morphism $\tilde{f}: A \to R$. It suffices to check this in the case when f is an isomorphism, i.e., $R/I = A$, $f = id$ (replace arbitrary R, I, f by $\widetilde{R} := A \underset{R/I}{\times} R$, $\widetilde{I} := \mathrm{Ker}(\widetilde{R} \to A) = I$, $f = id$). Note that the symmetric algebra $\mathrm{Sym}V$ is formally smooth if V is a projective object of $\mathcal{M}$.

Assume that $\mathcal{M}$ has many projective objects. We may represent A as a quotient B/J, $B = \mathrm{Sym}\,V$ for a projective V. Then A is formally smooth iff the projection $B/J^2 \to B/J = A$ has a section $\sigma: A \to B/J^2$ (if σ exists, then for any R, I, f as above f lifts to $\hat{f}: B/J^2 \to R$ and one may set $\tilde{f} = f\sigma$). Note that the sections σ correspond

[56] This happens if $\mathcal{Y}$ is (locally) finitely $\mathcal{D}_X$-presented.

bijectively to derivations $\partial: B \to J/J^2$ such that $\partial(x) = x \bmod J^2$ for $x \in J$ (namely, $\sigma(b \bmod J) = b - \partial(b)$). We may consider instead of ∂ the corresponding morphism of B-modules $\varphi: \Omega_B/J \cdot \Omega_B \to J/J^2$; the condition on ∂ means that $\varphi\nu = id_{J/J^2}$ where $\nu: J/J^2 \to \Omega_B/J \cdot \Omega_B$ is the usual map $\nu(x \bmod J^2) = dx \bmod J \cdot \Omega_B$. Therefore $A = B/J$ is formally smooth iff ν is a split injection. Set $\Omega_A^{(-1)} := \mathrm{Ker}\nu$. Since $\mathrm{Coker}\,\nu = \Omega_A$ and $\Omega_B/J \cdot \Omega_B$ is a projective A-module, we see that

A is formally smooth iff $\Omega_A^{(-1)} = 0$ and Ω_A is a projective A-module.

Remark. $\Omega_A^{(-1)}$ is independent of the presentation $A = B/J$; in fact, $\Omega_A^{(-1)}$ is the -1^{th} cohomology of the cotangent complex of A (see [Gr], [Il]).

Let us return to our tensor category $\mathcal{M}^\ell(X)$. Assume for the moment that X is affine; let A^ℓ be a commutative $\mathcal{D}_X$-algebra.

2.3.14 Proposition. (i) A^ℓ is formally smooth iff it is formally smooth as $\mathcal{O}_X$-algebra and Ω_{A^ℓ} is a projective $A^\ell[\mathcal{D}_X]$-module.

(ii) A^ℓ is formally smooth as $\mathcal{O}_X$-algebra iff it is formally smooth as k-algebra.

Proof. The forgetting of $\mathcal{D}_X$-action tensor functor $o^\ell: \mathcal{M}^\ell(X) \to \mathcal{M}_{\mathcal{O}}(X)$ sends projective objects to projective ones, as well as the corresponding functor $o^\ell_A: \mathcal{M}^\ell(X, A^\ell) \to A^\ell\mathrm{mod}$. Therefore $o^\ell\Omega_{A^\ell} = \Omega_{o^\ell A^\ell}$, $o^\ell\Omega^{(-1)}_{A^\ell} = \Omega^{(-1)}_{o^\ell A^\ell}$, and (i) follows from the above criterion. The statement (ii) follows from the next lemma (the morphism $\Omega_{A^\ell} \to \Omega_X \underset{\mathcal{O}_X}{\otimes} A^\ell$ left inverse to μ corresponds to the canonical derivation $\nabla: A^\ell \to \Omega_X \underset{\mathcal{O}_X}{\otimes} A^\ell$).

2.3.15 Lemma. If A is an $\mathcal{O}_X$-algebra then A is a formally smooth $\mathcal{O}_X$-algebra iff A is formally smooth as k-algebra and the morphism of A-modules $\mu: \Omega_X \underset{\mathcal{O}_X}{\otimes} A \to \Omega_A$ is a split injection.

Proof. Since X is smooth over k, the standard long exact sequence reduces to $0 \to \Omega_A^{(-1)} \to \Omega^{(-1)}_{A/\mathcal{O}_X} \to \Omega_X \underset{\mathcal{O}_X}{\otimes} A \to \Omega_A \to \Omega_{A/\mathcal{O}_X} \to 0$; this implies the lemma. Here is another ad hoc proof. Let us represent A as B/J, $B = \mathrm{Sym}\,V$, V is a free $\mathcal{O}_X$-module. The formal smoothness of A over $\mathcal{O}_X$ means that the morphism of A-modules $\nu: J/J^2 \to \Omega_{B/\mathcal{O}}/J{\cdot}\Omega_{B/\mathcal{O}}$ admits a left inverse. The formal smoothness over k means that the morphism $\tilde{\nu}: J/J^2 \to$

$\Omega_{B/k}/J\cdot\Omega_{B/k}$, admits a left inverse. It remains to apply to $P = J/J^2$, $Q = \Omega_{B/k}/J\cdot\Omega_{B/k}$, $R = \Omega_X \underset{\mathcal{O}_X}{\otimes} A$ the following

Easy Lemma. Assume we have morphisms $P \xrightarrow{f} Q \xleftarrow{g} R$ where g is a split injection. Then $P \to Q/\mathrm{Im} g$ is a split injection iff such are both $f\colon P \to Q$ and $R \to Q/\mathrm{Im} f$.

Proof. Let us represent Q as $R \oplus S$, so $g(r) = (r, 0)$, $f(p) = (\varphi(p), \psi(p))$. The morphism $R \to Q/f(P)$ is a split injection iff there is $\gamma\colon S \to R$ such that $\varphi = \gamma\psi$, i.e., when $\varphi \in \mathrm{Hom}(S, R)\psi$; the morphism f is a split injection iff $id_P \in \mathrm{Hom}(R, P)\varphi + \mathrm{Hom}(S, P)\psi$. We need to show that $id_P \in \mathrm{Hom}(S, P)\psi$ iff $id_P \in \mathrm{Hom}(R, P)\varphi + \mathrm{Hom}(S, P)\psi$ and $\varphi \in \mathrm{Hom}(S, R)\psi$, which is clear. $\square$

2.3.16. ¿From now on we do *not* assume that X is affine. For a $\mathcal{D}_X$-algebra A^ℓ the $A^\ell[\mathcal{D}_X]$-modules $\Omega_{A^\ell}, \Omega^{(-1)}_{A^\ell}$ are compatible with étale localization of A^ℓ. Therefore for any $\mathcal{D}_X$-scheme, or algebraic $\mathcal{D}_X$-space, $\mathcal{Y}$ we have $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules $\Omega_{\mathcal{Y}}$, $\Omega^{(-1)}_{\mathcal{Y}}$. We say that $\mathcal{Y}$ is *formally smooth* if $\Omega^{(-1)}_{\mathcal{Y}} = 0$ and $\Omega_{\mathcal{Y}}$ is a locally projective $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module. In fact, $\mathcal{Y}$ is formally smooth if and only if for every affine scheme S equipped with a $\mathcal{D}_X$-scheme structure and every closed $\mathcal{D}_X$-subscheme $S_0 \subset S$ defined by an ideal I with $I^2 = 0$ the map $\mathrm{Mor}_{\mathcal{D}_X}(S, \mathcal{Y}) \to \mathrm{Mor}_{\mathcal{D}_X}(S_0, \mathcal{Y})$ is surjective. To prove the "if" statement notice that locally a $\mathcal{D}_X$-morphism $f : S_0 \to \mathcal{Y}$ can be extended to a morphism $S \to \mathcal{Y}$ and these extensions form a torsor over $\mathcal{H}om(f^*\Omega_{\mathcal{Y}}, I)$. Such a torsor is the same as an extension of $f^*\Omega_{\mathcal{Y}}$ by I. Now $f^*\Omega_{\mathcal{Y}}$ is locally projective and therefore projective (see 2.3.7), so $\mathrm{Ext}(f^*\Omega_{\mathcal{Y}}, I) = 0$ and our torsor is trivial.

We say that $\mathcal{Y}$ is *smooth* if it is formally smooth and $\Omega_{\mathcal{Y}}$ is a finitely generated $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module. Then $\Omega_{\mathcal{Y}}$ is a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$.[57]

Remark. One may argue that the term "smooth" should be reserved for formally smooth $\mathcal{D}_X$-schemes which have locally $\mathcal{D}_X$-finite type (i.e., have a covering by $\mathrm{Spec}\, A^\ell$ where A^ℓ is a finitely generated $\mathcal{D}_X$-algebra), or $\mathcal{D}_X$-finite presentation. These conditions are a priori stronger than the one we consider above. We do not know if a smooth $\mathcal{D}_X$-scheme of $\mathcal{D}_X$-finite type is necessarily $\mathcal{D}_X$-finitely presented.

[57] In fact, the latter property is the only property of smoothness that will be actually used in this book.

2.3.17. A morphism $\varphi : \mathcal{Y} \to \mathcal{Z}$ of algebraic $\mathcal{D}_X$-spaces is *formally smooth* if for every (S, S_0) as above and any pair of φ-compatible morphisms $f : S_0 \to \mathcal{Y}$, $g : S \to \mathcal{Z}$ lifts to a morphism $\tilde{f} : S \to \mathcal{Y}$. The reformulation of this property in terms of relative differentials is left to the reader. One says that φ is *formaly étale* if, in addition, $\tilde{f}$ is unique.

2.3.18. For a smooth $\mathcal{D}_X$-scheme $\mathcal{Y}$ a *coordinate system* on $\mathcal{Y}$ is a morphism $\nu\colon \mathcal{Y} \to \operatorname{Spec}\operatorname{Sym}^{\cdot}(\mathcal{D}_X^n)$, $n \geq 0$, such that $d\nu\colon \nu^*\Omega_{\operatorname{Sym}^{\cdot}(\mathcal{D}_X^n)} \to \Omega_{\mathcal{Y}}$ is an isomorphism. The jet schemes $\mathcal{J}Z$, Z/X is smooth, have local coordinate systems (for any étale morphism $\varphi\colon Z \to \mathbb{A}_X^n$ the morphism $\mathcal{J}\varphi\colon \mathcal{J}Z \to \mathcal{J}\mathbb{A}_X^n = \operatorname{Spec}\operatorname{Sym}^{\cdot}(\mathcal{D}_X^n)$ is a coordinate system on $\mathcal{J}Z$). This is **not** true (even at a generic point!) for arbitrary smooth $\mathcal{D}_X$-schemes.

Here is the reason. Assume that our $\mathcal{Y} = \operatorname{Spec} A^\ell$ is a smooth reduced[58] $\mathcal{D}_X$-scheme such that $\Omega_{A^\ell} \simeq A^\ell[\mathcal{D}_X]$. Note that any invertible element of $A^\ell[\mathcal{D}_X]$ lies in $A^\ell \subset A^\ell[\mathcal{D}_X]$. Denote by $\mathcal{L}$ the image of A^ℓ by any isomorhism $A^\ell[\mathcal{D}_X] \xrightarrow{\sim} \Omega_{A^\ell}$. This is a canonically defined line subbundle of Ω_A. If $\mathcal{Y}$ admits locally a coordinate system, then $\mathcal{L}$ is an integrable subbundle, i.e., $d(\mathcal{L}) \subset \mathcal{L} \wedge \Omega_{A^\ell} \subset \Omega^2_{A^\ell}$.

2.3.19 Example. Let us construct a smooth A^ℓ with non-integrable $\mathcal{L}$. Take $X = \operatorname{Spec} k[t]$, $B^\ell = k[t, u, v, u', v', \ldots, v'^{-1}]$ (here $u' := \partial_t u$, etc., so B^ℓ is the localization of $\operatorname{Sym}^{\cdot}(\mathcal{D}_X^2) = k[t, u, v, u', \ldots]$ by v'). Let $I \subset B^\ell$ be the $\mathcal{D}_X$-ideal generated by $\varphi = v + u'v'$; set $A^\ell := B^\ell / I$. Consider the morphism of $B^\ell[\mathcal{D}_X]$-modules

$$\chi\colon B^\ell[\mathcal{D}_X] \oplus B^\ell[\mathcal{D}_X] \to \Omega_{B^\ell}, \quad (\partial_1, \partial_2) \mapsto \partial_1 d\varphi + \partial_2 \xi,$$

where $\xi := du + (u'/v')dv \in \Omega_{B^\ell}$. Localizing B^ℓ (and A^ℓ) further by $v' - u''v' + u'v''$, we see that χ becomes an isomorphism. Now A^ℓ is smooth, and the map $A^\ell[\mathcal{D}_X] \to \Omega_{A^\ell}$, $\partial \mapsto \partial\xi_{A^\ell}$, is an isomorphism. Here ξ_{A^ℓ} is restriction of ξ to $\operatorname{Spec} A^\ell$; this is a generator of $\mathcal{L}$. One has $\xi_{A^\ell} \wedge d\xi_{A^\ell} = \xi_{A^\ell} \wedge d(u'/v') \wedge dv = -\xi_{A^\ell} \wedge d(v/v'^2) \wedge dv = \alpha\xi_{A^\ell} \wedge dv \wedge dv'$, $\alpha \neq 0$, where u and v are considered as elements of A^ℓ. It is easy to check that $dv = \beta\partial_t\xi_{A^\ell}$, $\beta \neq 0$, so $\xi_{A^\ell} \wedge d\xi_{A^\ell} \neq 0$ and $\mathcal{L}$ is non-integrable.

[58] We do not know if a smooth $\mathcal{D}_X$-scheme is necessarily reduced.

§4 The spaces of horizontal sections

In this section we assume that $X \neq \emptyset$ is connected of dimension n.

2.4.1. Denote by $\mathcal{C}omu(k)$ the category of commutative unital k-algebras. For any $F \in \mathcal{C}omu(k)$ the constant left $\mathcal{D}$-module $F \otimes \mathcal{O}_X$ is a $\mathcal{D}_X$-algebra in the obvious way. This functor $\mathcal{C}omu(k) \to \mathcal{C}omu_{\mathcal{D}}(X)$ identifies $\mathcal{C}omu(k)$ with the full subcategory of constant $\mathcal{D}_X$-algebras.[59] It has a right adjoint functor $\Gamma^\nabla : \mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(k)$, $\Gamma^\nabla(A^\ell) = \Gamma^\nabla(X, A^\ell) = \mathrm{Hom}(\mathcal{O}_X, A^\ell) = H^{-n}_{DR}(X, A)$.

We would like to find out if our embedding admits a left adjoint functor

$$\mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(k), \quad A^\ell \mapsto \langle A\rangle = \langle A\rangle(X). \tag{2.4.1.1}$$

So we want to see if for $A^\ell \in \mathcal{C}omu_{\mathcal{D}}(X)$ the functor $\langle A\rangle$: $F \mapsto \mathrm{Hom}(A^\ell, F \otimes \mathcal{O}_X)$ on $\mathcal{C}omu(k)$ is representable. Notice that $\operatorname{Spec}\langle A\rangle$ is the space of horizontal sections $X \to \operatorname{Spec} A^\ell$.

It is clear that $\langle A\rangle$ is a projective limit of commutative algebras. Namely, consider the set $\mathcal{C}$ of ideals $I \subset A^\ell$ such that A^ℓ/I is a constant $\mathcal{D}$-algebra. The intersection of two ideals from $\mathcal{C}$ belongs to $\mathcal{C}$, so it is a subfilter in the ordered set of all ideals in A^ℓ. For any $I \in \mathcal{C}$ we have a commutative algebra $\Gamma^\nabla(X, A^\ell/I)$, and for $I' \supset I$ the surjective map $\Gamma^\nabla(X, A^\ell/I) \twoheadrightarrow \Gamma^\nabla(X, A^\ell/I')$. Our $\langle A\rangle$ is the $\mathcal{C}$-projective limit of these algebras.

Notice that $\langle A\rangle(X)$ is covariantly functorial with respect to open embeddings of X's. Precisely, for any open $X' \subset X$ the algebra $\langle A\rangle(X)$ is a quotient algebra of $\langle A\rangle(X')$ in the obvious way.

For a point $x \in X$ the fiber[60] $A^\ell_x := H^0 i_x^* A^\ell = A^\ell \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_x/\mathrm{m}_x$ of A^ℓ at x is a commutative algebra. A $\mathcal{D}_X$-algebra epimorphism $A^\ell \to F \otimes \mathcal{O}_X$, $F \in \mathcal{C}omu(k)$, induces an epimorphism $A^\ell_x \to (F \otimes \mathcal{O}_X)_x = F$, so we get a canonical morphism $A^\ell_x \to \langle A\rangle$ with dense image. Therefore $\operatorname{Spec}\langle A\rangle$ is naturally an ind-subscheme of any fiber $\operatorname{Spec} A^\ell_x$.

2.4.2 Proposition. If X is compact, then $\langle A\rangle$ is a true commutative algebra.

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[59] i.e., those $\mathcal{D}_X$-algebras which are constant as a $\mathcal{D}_X$-module, see 2.1.12.

[60] Here i_x is the embedding $\{x\} \hookrightarrow X$.

Proof. We want to show that A admits the maximal constant $\mathcal{D}$-algebra quotient. Consider the maximal constant $\mathcal{D}$-module quotient $\sigma_A : A \twoheadrightarrow A_{const}$ (see 2.1.12). Now take the quotient of A modulo the ideal generated by the kernel of σ_A. $\square$

We saw in 2.1.12 that $A^\ell_{const} = H^n_{DR}(X, A) \otimes \mathcal{O}_X$, so $\langle A\rangle$ is a quotient of $H^n_{DR}(X, A)$. In 2.4.3–2.4.5 below we give some "explicit" descriptions of $\langle A\rangle$. We assume that X is compact.

Let $\mathcal{J}^\ell \subset A^\ell \otimes A^\ell$ be the kernel of the multiplication map $A^\ell \otimes A^\ell \to A^\ell$. Consider the morphism $\sigma_A^{\otimes 2} : A^\ell \otimes A^\ell \to H^n_{DR}(X, A)^{\otimes 2} \otimes \mathcal{O}_X$. Restricting it to $\mathcal{J}$ and passing to de Rham cohomology we get a morphism

$$\nu : H^n_{DR}(X, \mathcal{J}) \to H^n_{DR}(X, A)^{\otimes 2}. \tag{2.4.2.1}$$

The projection $H^n_{DR}(X, A) \twoheadrightarrow \langle A\rangle$ together with the product on $\langle A\rangle$ identifies $\langle A\rangle$ with a quotient of $H^n_{DR}(X, A)^{\otimes 2}$.

2.4.3 Lemma. One has $\langle A\rangle = \mathrm{Coker}\nu$.

Proof. Look at the commutative diagram

$$\begin{array}{ccc} A^\ell & \longrightarrow & \langle A\rangle \otimes \mathcal{O}_X \\ \uparrow & & \uparrow \\ A^\ell \otimes A^\ell & \longrightarrow & H^n_{DR}(X, A)^{\otimes 2} \otimes \mathcal{O}_X \end{array}$$

According to the proof of 2.4.2, $\langle A\rangle \otimes \mathcal{O}_X$ is the quotient of A^ℓ modulo the ideal $A^\ell \cdot \mathrm{Ker}\sigma_A$, or, equivalently, the quotient of $A^\ell \otimes A^\ell$ modulo $\mathcal{J} + \mathrm{Ker}\sigma_A \otimes A^\ell + A^\ell \otimes \mathrm{Ker}\sigma_A$. This is $\mathrm{Coker}\nu \otimes \mathcal{O}_X$, q.e.d. $\square$

Here is another description of $\langle A\rangle$. Let Δ: $X \hookrightarrow X \times X$ be the diagonal embedding, j: $U := X \times X \smallsetminus \Delta(X) \hookrightarrow X \times X$ its complement. Consider the morphism $j_* j^* A \boxtimes A \to \Delta_* A[-n+1]$ in $D^b\mathcal{M}(X \times X)$ defined as the composition of the canonical "residue" morphism $j_*j^*(A \boxtimes A) \to \Delta_*\Delta^!(A \boxtimes A)[1] \xrightarrow{\sim} \Delta_*(A \overset{L}{\otimes}{}^! A)[-n+1]$ and the product $A \overset{L}{\otimes}{}^! A \to A \otimes^! A \to A$. Passing to de Rham cohomology[61] we get a morphism of vector spaces

$$\mu : H^{2n-1}_{DR}(U, A \boxtimes A) \to H^n_{DR}(X, A). \tag{2.4.3.1}$$

[61] One has $H^\cdot_{DR}(X \times X, j_*j^*B) = H^\cdot_{DR}(U, B)$, $H^\cdot_{DR}(X \times X, \Delta_*C) = H^\cdot_{DR}(X, C)$.

2.4.4 Lemma. One has $\langle A\rangle = \mathrm{Coker}\mu$.

Proof. Denote by C the cone of our morphism $j_*j^*A\boxtimes A[-1]\to \Delta_*A[-n]$; this is an object of $D^b\mathcal{M}(X\times X)$.[62] One has[63] $\mathrm{Coker}\mu = H^{2n}_{DR}(X\times X, C)$.

Consider the obvious morphism $A\boxtimes A\to C$. Denote by D its cone, so we have the exact triangle $A\boxtimes A\to C\to D$. The complex D is supported on the diagonal; its top non-zero cohomology is $H^{n-1}D=\Delta_*\mathcal{J}$. Therefore $H^{2n-1}_{DR}(X\times X, D) = H^n_{DR}(X,\mathcal{J})$ and all the higher de Rham cohomology vanish. So the long exact sequence of de Rham cohomology identifies $H^{2n}_{DR}(X\times X, C)$ with the cokernel of the arrow $H^{2n-1}_{DR}(X\times X, D)\to H^{2n}_{DR}(X\times X, A\boxtimes A)$ which may be rewritten as $\nu' : H^n_{DR}(X,\mathcal{J})\to H^n_{DR}(X,A)^{\otimes 2}$.

So we see that $\mathrm{Coker}\mu = \mathrm{Coker}\nu'$. We leave it to the reader to check that our ν' equals the map ν from (2.4.2.1). Now use 2.4.3. $\square$

Now let $S\subset X$ be a finite non-empty subset. We have the canonical residue map $\mathrm{Res}_S = \Sigma\mathrm{Res}_s$: $H^{n-1}_{DR}(X\smallsetminus S, A)\to \bigoplus_{s\in S} A^\ell_s$ (see 2.1.12); denote its image by V_S. Consider the product $\bigotimes_S A^\ell_s \twoheadrightarrow \langle A\rangle$ of our surjections $A^\ell_s\twoheadrightarrow\langle A\rangle$ (see 2.4.1, 2.4.2). As follows from the definition of $\langle A\rangle$ (and 2.1.12), its kernel contains $f(V_S)$ where $f:\bigoplus_S A^\ell_s\to\bigotimes_S A^\ell_s$ maps $a_s\in A^\ell_s$ to $a_s\otimes 1^{\otimes S\smallsetminus\{s\}}$. Hence it contains the ideal I_S generated by $f(V_S)$. So we have a ring epimorphism $\alpha_S : (\bigotimes_S A^\ell_s)/I_S\to\langle A\rangle$.

2.4.5 Lemma. One has $\alpha_S : (\bigotimes_S A^\ell_s)/I_S \xrightarrow{\sim} \langle A\rangle$.

Proof. Consider the canonical projections $\pi_s : A^\ell_s\twoheadrightarrow (A^\ell_{const})_s = H^n_{DR}(X,A)$. According to 2.1.12 the kernel of $\pi_S=\Sigma\pi_s : \bigoplus_S A^\ell_s\twoheadrightarrow H^n_{DR}(X,A)$ equals to V_S.

(a) Assume that S consists of a single point $s\in X$. As follows from the proof of 2.4.2 $\langle A\rangle\otimes\mathcal{O}_X$ is the quotient of A^ℓ modulo the ideal generated by the kernel of the projection $\pi : A^\ell\twoheadrightarrow A^\ell_{const}$. Therefore, passing to the s-fibers, we see that $\langle A\rangle$ is the quotient of A^ℓ_s modulo the ideal generated by $\mathrm{Ker}\pi_s$, i.e., by V_s, and we are done.

(b) Assume that $|S|\geq 2$. Take any $s_0\in S$. Then $A^\ell_{s_0}\cap V_S = V_{s_0}$ (indeed, both subspaces are equal to the kernel of π_{s_0}). Therefore the morphism $A^\ell_{s_0}\to\bigotimes_S A^\ell_s$ yields a morphism

[62] In the most important case $n=1$ our C is actually a $\mathcal{D}$-module, not merely a complex.
[63] Since U is non-compact $H^{2n}_{DR}(U,B)=0$ for every $B\in\mathcal{M}(U)$.