$\zeta : A^\ell_{s_0}/I_{s_0} \to (\bigotimes_S A^\ell_s)/I_S$. Since $A^\ell_{s_0}+V_S = \bigoplus_S A^\ell_s$ (which follows from the surjectivity of π_{s_0}) we see that ζ is surjective. The composition of ζ with α_S is the map $\alpha_{s_0} : A^\ell_{s_0}/I_{s_0} \to \langle A\rangle$ for the one point set $\{s_0\}$ which we know to be an isomorphism by (a). Thus α_S is an isomorphism. □

Remark. One may also deduce 2.4.5 in the case $S = \{s\}$ (i.e., (a) above) from 2.4.4. Namely, set $U_s := X \smallsetminus \{s\}$; denote by i_s, $\tilde{i}_s$ the embeddings $\{s\} \hookrightarrow X$, $X = \{s\} \times X \hookrightarrow X \times X$. Then $\tilde{i}^!_s(A \boxtimes A) = (i^!_s A) \otimes A$, so we have a commutative diagram

$$\begin{array}{ccc} H^{2n-1}_{DR}(U, A \boxtimes A) & \xrightarrow{\ \mu\ } & H^n_{DR}(X, A) \\ \uparrow & & \uparrow \\ A^\ell_s \otimes H^{n-1}_{DR}(U_s, A) & \xrightarrow{\ \mu_s\ } & A^\ell_s \end{array}$$

the vertical arrows come from canonical morphisms $\tilde{i}_{s*}\tilde{i}^!_s(A \boxtimes A) \to A \boxtimes A$, $i_{s*}i^!_s A \to A$, and μ_s sends $a \otimes b$ to $a\,\mathrm{Res}_s b$. The standard long exact sequences show that the vertical arrows are surjective; the kernel of the right one is $\mathrm{Im}\,(\mathrm{Res}_s)$. Since $\mathrm{Res}_s(b) = \mu_s(1 \otimes b)$, one has $\mathrm{Im}\,(\mathrm{Res}_s) \subset \mathrm{Im}(\mu_s)$. Therefore, $\mathrm{Coker}\mu = A^\ell_s/\mathrm{Im}(\mu_s) = A^\ell_s/I_s$, so, by 2.4.4, we have $A^\ell_s/I_s \xrightarrow{\sim} \langle A\rangle$. One checks immediately that this is the morphism of 2.4.5.

2.4.6. Let $j : U \hookrightarrow X$ be a non-empty Zariski open subset (we assume that X is compact). For $A^\ell \in \mathcal{C}omu_{\mathcal{D}}(U)$ the left $\mathcal{D}$-module $j.A^\ell := H^0 j_* A^\ell$ is a $\mathcal{D}_X$-algebra in the obvious way. Let $\Xi^c(A)$ denotes the set of $\mathcal{D}_X$-subalgebras $A^\ell_\xi \subset j.A^\ell$ such that $j^* A^\ell_\xi = A^\ell$. The intersection of two subalgebras from $\Xi^c(A)$ belongs to $\Xi^c(A)$, so this is a subfilter in the ordered set of all $\mathcal{D}_X$-subalgebras of $j.A^\ell$. We have a $\Xi^c(A)$-projective system of algebras $\langle A_\xi\rangle(X)$ connected by epimorphisms. The canonical maps $\langle A\rangle(U) \to \langle A_\xi\rangle(X)$ yield a morphism (the limit is taken over $\Xi^c(A)$)

$$\langle A\rangle(U) \to \varprojlim \langle A_\xi\rangle(X). \tag{2.4.6.1}$$

2.4.7 Lemma. This is an isomorphism.

Proof. It suffices to show that any morphism[64] $\varphi: A^\ell \to F \otimes \mathcal{O}_U$ is the restriction to U of some morphism $\varphi_X: A^\ell_\xi \to F \otimes \mathcal{O}_X$, $A^\ell_\xi \in \Xi^c(A)$. Note that φ defines a morphism of

[64] Here F is a commutative algebra.

$\mathcal{D}_X$-algebras $j.(\varphi)$: $j.A^\ell \to j.(F \otimes \mathcal{O}_U)$, and $F \otimes \mathcal{O}_X$ is a $\mathcal{D}_X$-subalgebra of $j.(F \otimes \mathcal{O}_U)$. Set $A^\ell_\xi := j.(\varphi)^{-1}(F \otimes \mathcal{O}_X)$, $\varphi_X := j.(\varphi)\big|_{A^\ell_\xi}$. We are done. □

2.4.8. For the rest of the section our X is a curve. Let $x \in X$ be a (closed) point, $j_x : U_x \hookrightarrow X$ the complement to x. For $A^\ell \in \mathcal{C}omu_{\mathcal{D}}(U_x)$ consider the topology $\Xi^{as}_x = \Xi^{as}_x(j_{x*}A)$ on $j_{x*}A$ at x (see 2.1.13 for terminology) formed by all $\mathcal{D}_X$-subalgebras A^ℓ_ξ such that $j^*_x A_\xi = A$. We have

$$A^\ell_{\xi x} = i^!_x((j_{x*}A)/A_\xi) = h((j_{x*}A)/A_\xi)_x. \tag{2.4.8.1}$$

We see that for $A^\ell_{\xi'} \subset A^\ell_\xi$ the morphism of fibers $A^\ell_{\xi' x} \to A^\ell_{\xi x}$ is surjective. Denote by A^{as}_x the Ξ^{as}_x-projective limit of the commutative algebras $A^\ell_{\xi x}$; as a plain topological vector space it equals to the Ξ^{as}_x-completion of the vector space $h(j_{x*}A)_x$ (see 2.1.14).[65]

Remark. For an A^ℓ-module M let $\Xi^A_x(j_{x*}M)$ be the topology at x[66] on $j_{x*}M$ formed by $M_\xi \in \Xi_x(j_{x*}M)$ such that M_ξ is an A^ℓ_ξ-submodule of $j_{x*}M$ for some $A_\xi \in \Xi^{as}_x(j_{x*}A)$. Then the completion $\hat{h}^A_x(j_{x*}M)$ is a topological A^{as}_x-module.[67]

Let us show that the ind-affine ind-scheme $\operatorname{Spf} A^{as}_x := \varinjlim \operatorname{Spec} A^\ell_{\xi x}$, $\xi \in \Xi^{as}_x$, is the space of horizontal sections of $\operatorname{Spec} A^\ell$ over the formal punctured disc at x. Denote by K_x the field of fractions of $O_x := \varprojlim \mathcal{O}_x/\mathrm{m}^n_x$. For a commutative algebra F set $O_x \widehat{\otimes} F := \varprojlim (O_x/\mathrm{m}^n_x) \otimes F$, $K_x \widehat{\otimes} F := K_x \underset{O_x}{\otimes} (O_x \widehat{\otimes} F)$.[68] We consider $O_x \widehat{\otimes} F \subset K_x \widehat{\otimes} F$ as sheaves on X supported at x; these are *non-quasicoherent* commutative $\mathcal{D}_X$-algebras in the obvious way.

2.4.9 Lemma. There is a canonical identification

$$\operatorname{Hom}(A^{as}_x, F) = \operatorname{Hom}(j_{x*}A^\ell, K_x \widehat{\otimes} F). \tag{2.4.9.1}$$

Here the former Hom denotes the set of continuous morphisms of commutative algebras, the latter one that of commutative $\mathcal{D}_{U_x}$-algebras.

[65] The superscript "as" means "associative"; this notation will be clearified in 3.4.
[66] See 2.1.13.
[67] Since $h(j_{x*}M/M_\xi)$ is an $A^\ell_{\xi x}$-module.
[68] If t is a parameter at x then $O_x \widehat{\otimes} F = F[[t]]$, $K_x \widehat{\otimes} F = F((t))$.

Proof (cf. the proof of 2.4.7). The map $\mathrm{Hom}(j_{x*}A^\ell, K_x \widehat{\otimes} F) \to \mathrm{Hom}(A_x^{as}, F)$ is defined as follows. Take $\varphi\colon j_{x*}A^\ell \to K_x \widehat{\otimes} F$. Set $A^\ell_\varphi := \varphi^{-1}(O_x \widehat{\otimes} F) \subset j_{x*}A^\ell$. Since $K_x \widehat{\otimes} F/O_x \widehat{\otimes} F$ is a quasicoherent $\mathcal{O}_X$-module supported at x we see that $A^\ell_\varphi \in \Xi^{ch}_x$. We have the morhism of $\mathcal{D}_X$-algebras $\varphi\colon A^\ell_\varphi \to O_x \widehat{\otimes} F$. Let $\varphi_x\colon (A^\ell_\varphi)_x \to F$ be the morphism of fibers at x.

We leave it to the reader to check that the map $\varphi \mapsto (A^\ell_\varphi, \varphi_x) \in \mathrm{Hom}(A_x^{as}, F)$ is bijective. $\square$

Remark. The above lemma is essentially a particular case of a general statement about $\mathcal{D}$-modules: see Remark after Lemma 3.5.4.

2.4.10. Let $S \subset X$ be a finite set, $j_S : U_S \hookrightarrow X$ its complement. For a $\mathcal{D}$-algebra A on U_S set $A_S^{as} = \widehat{\bigotimes}_{s\in S} A_s^{as} := \varprojlim \bigotimes_{s\in S} A^\ell_{\xi_s s}$. Assume now that X is compact. Subalgebras from $\Xi^c(A)$ (see 2.4.6) are just intersections of subalgebras from $\Xi_s^{as}(j_{S*}A)$, $s \in S$, so, by 2.4.7, we have the canonical maps $A_s^{as} \to \langle A\rangle(U_S)$. They define a continuous morphism

$$v\colon A_S^{as} \to \langle A\rangle(U_S). \tag{2.4.10.1}$$

By 2.4.5 v identifies $\langle A\rangle(U_S)$ with the quotient of A_S^{as} modulo the closed ideal generated by the image of the morphism

$$r_S = \varprojlim \mathrm{Res}_S^{A_\xi}\colon\ H^0_{DR}(U_S, A) \to \bigoplus_S A_s^{as} \to A_S^{as}. \tag{2.4.10.2}$$

Remark. r_S obviously factors through the quotient $\Gamma(U_S, h(A))$ of $H^0_{DR}(U_S, A)$.

§5 Lie* algebras and algebroids

The notion (if not the name) of Lie* algebra has been, undoubtedly, well-known in mathematical physics for a long time. Since the first version of the present article appeared as a preprint, Lie* algebras migrated to mathematical literature, e.g. they present in [K] (under the alias "conformal algebras") and in [P] (as "vertex Lie algebras").[69]

For a list of basic examples see 2.5.7–2.5.10; an important example of Tate's extension is treated separately in the next section.

[69] The article [P] also refers to a preprint of Dong, Li, and Mason (which we were not able to locate) for "a more general notion of local vertex Lie algebra".

2.5.1. Having at hand the $*$ pseudo-tensor structure, we may consider for any k-operad $\mathcal{B}$ the category $\mathcal{B}^*(X)$ of $\mathcal{B}$ algebras in $\mathcal{M}(X)^*$, and for any $L \in \mathcal{B}^*(X)$ (and a $\mathcal{B}$-module operad $\mathcal{C}$) the corresponding category $\mathcal{M}(X, L) = \mathcal{M}(X)(L, \mathcal{C})$ of L-modules. We have the corresponding sheaves of categories $\mathcal{B}^*(X_{\text{ét}})$ and $\mathcal{M}(X_{\text{ét}}, L)$ on $X_{\text{ét}}$.

For a locally closed embedding $i: X \hookrightarrow Y$ we have the adjoint functors $\mathcal{B}^*(X) \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{B}^*(Y)$ (see 2.2.5). If X is a closed subscheme then i_* identifies $\mathcal{B}^*(X)$ with the full subcategory of $\mathcal{B}^*(Y)$ that consists of algebras supported on X.

2.5.2. The functor h sends $\mathcal{B}^*$ algebras on X to $\mathcal{B}$ algebras in the tensor category $Sh^{\otimes}(X)$ of sheaves of k-vector spaces on X. For $L \in \mathcal{B}^*(X)$ an L-module structure on a $\mathcal{D}$-module M defines on M the $\mathcal{C}(h(L))$-module structure (see 1.2.17). According to 2.2.16 the functor

$$\mathcal{M}(X, L) \to \mathcal{C}(h(L)) - \text{modules in } \mathcal{M}(X) \tag{2.5.2.1}$$

is a fully faithful embedding.

2.5.3. We will be mainly interested in Lie* algebras. Therefore a Lie* algebra is a $\mathcal{D}$-module L equipped with a Lie* bracket, i.e., a $*$-pairing $[\ \ ] \in P_2^*(\{L, L\}, L)$ which is skew-symmetric and satisfies the Jacobi identity. Then the sheaf $h(L)$ is a Lie algebra with bracket $[\bar{\ell}_1, \bar{\ell}_2] = \overline{[\ \ ](\ell_1 \boxtimes \ell_2)}$. Recall that $\mathcal{C}(h(L))$ is the universal enveloping algebra of $h(L)$ and a $\mathcal{C}(h(L))$-module is the same as an $h(L)$-module.

2.5.4. An L-module structure on $M \in \mathcal{M}(X)$ yields an $h(L)$-action on M given by

$$\bar{\ell}(m) = Tr_2(\ \cdot (\ell \boxtimes m)). \tag{2.5.4.1}$$

Here $\cdot \in P_2^*(\{L, M\}, M)$ is the L-action and $Tr_2: \Delta^{\cdot}\Delta_* M = pr_{2\cdot}\Delta_* M \to M$ is the trace map for the projection $pr_2: X \times X \to X$.

According to 2.2.16 this way we get a one-to-one correspondence between the L-module structures on $M \in \mathcal{M}(X)$ and such $h(L)$-actions on M (we call them *good* actions) that for every $m \in M$ the map $L \to M$, $\ell \mapsto \bar{\ell}m$, is a differential operator.

Remark. For an L-module M and a section $\bar{m}$ of $h(M)$ the stabilizer $L^{\bar{m}}$ of $\bar{m}$ (defined as the kernel of the morphism $L \to M$, $l \mapsto l\bar{m}$) is a Lie* subalgebra of L.

Examples. (i) Let V be a coherent $\mathcal{D}$-module. Then the $h(\mathcal{E}nd^* V)$-action on V coincides with the map $h(\mathcal{H}om(V, V \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X)) \to \mathrm{End}V$, which sends $\bar{\varphi}$, $\varphi \in \mathcal{H}om(V, V \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X)$, to the composition $V \xrightarrow{\varphi} V \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to V$, the second arrow is $v \otimes \partial \mapsto v\partial$

(ii) The adjoint action of L yields an $h(L)$-action on L denoted by $\bar{\ell} \mapsto ad_{\bar{\ell}} \in \mathrm{End}L$.

2.5.5 Proposition. For any $\mathcal{D}$-module L the Lie* brackets on L are in $1-1$ correspondence with morphism of sheaves $h(L) \to \mathrm{End}L$, $\bar{\ell} \mapsto ad_{\bar{\ell}}$, which satisfy the following properties (below we set $ad_{\bar{\ell}}(\bar{\ell'}) := \overline{ad_{\bar{\ell}}(\ell')}$):

(i) $\left[ad_{\bar{\ell}_1}, ad_{\bar{\ell}_2}\right] = ad_{ad_{\bar{\ell}_1}(\bar{\ell}_2)}$ for any $\bar{\ell}_1, \bar{\ell}_2 \in h(L)$,

(ii) $ad_{\bar{\ell}}(\ell) = 0$ for any $\ell \in L$,

(iii) For any $\ell' \in L$ the map $L \to L$, $\ell \mapsto ad_{\bar{\ell}}(\ell')$, is a differential operator (with respect to the $\mathcal{O}$-module structure on L).

Proof. We use 2.2.16. The condition (iii) is exactly the condition $(*)$ of 2.2.16, hence it means that ad comes from a $*$-pairing $[\ \ ] \in P_2^*(\{L, L\}, L)$. The property (i) is equivalent to the Jacobi identity by 2.2.16. The property (ii) means that $Tr_2[\ \ ]\colon L^{\otimes 2} \to L$ vanishes on the symmetric part $L^{\otimes 2+} \subset L^{\otimes 2}$. Set $N := \Delta^{\cdot}\Delta_* L$. Since $[\ \ ]\colon L^{\otimes 2} \to N$ is $\mathcal{D}_X^{\otimes 2}$-linear $M := [\ \ ](L^{\otimes 2+})$ is a $\mathcal{D}_X^{\otimes 2+}$-submodule of N such that $Tr_2(M) = 0$. So the skew-symmetry of $[\ \ ]$ follows from the next lemma:

2.5.6 Lemma. If $M \subset N$ is a $\mathcal{D}_X^{\otimes 2+}$-submodule such that $Tr_2(M) = 0$ then $M = 0$.

Proof. Take $m \in M$. For any $f \in \mathcal{O}_X$, $\theta \in \Theta_X$ we have $Tr_2(m \cdot (f \otimes \theta + \theta \otimes f)) = 0$. Since $Tr_2(n \cdot (\theta \otimes 1)) = 0$ and $Tr_2(n \cdot (1 \otimes \theta)) = Tr_2(n) \cdot \theta$ for any $n \in N$ we obtain $(Tr_2(m \cdot (f \otimes 1)))\theta = 0$. Since $\Theta_X \mathcal{D}_X = \mathcal{D}_X$ we have $Tr_2(m \cdot (f \otimes 1)) = 0$ for any $f \in \mathcal{O}_X$. Choose a coordinate system $t_1, \ldots, t_n$ on X and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^k$ set $t^\alpha := t_1^{\alpha_1} \ldots t_k^{\alpha_k} \in \mathcal{O}_X$, $\partial^\alpha := \partial_{t_1}^{\alpha_1} \ldots \partial_{t_k}^{\alpha_k} \in \mathcal{D}_X$. Since $N = L \cdot (\mathcal{D}_X \otimes 1)$ we can write m as $\sum_\alpha \ell_\alpha \cdot (\partial^\alpha \otimes 1)$, $\ell_\alpha \in L$. Then $Tr_2(m \cdot (f \otimes 1)) = \sum_\alpha \ell_\alpha \partial^\alpha(f)$. The equality $\sum_\alpha \ell_\alpha \partial^\alpha(f) = 0$ for all $f \in \mathcal{O}_X$ implies $\ell_\alpha = 0$ (set $f = t^\beta$). $\square$

Here are some basic examples of Lie* algebras:

2.5.7. (a) For any coherent $\mathcal{D}_X$-module M we have an associative*, hence Lie*, algebra

$\mathcal{E}nd^*(M)$ (see 2.2.12, 2.2.13). If M is a $\mathcal{D}_X$-vector bundle then $\mathcal{E}nd^*(M)$ considered as a Lie* algebra is denoted by $\mathfrak{gl}(M)$.

(b) By 2.2.4 (i) for an $\mathcal{O}$-module P a Lie bracket on P which is a bidifferential operator is the same as a Lie* bracket on the induced $\mathcal{D}$-module $P_{\mathcal{D}}$.

Examples. (i) The Lie bracket of vector fields defines a Lie* algebra structure on $\Theta_{\mathcal{D}} := \Theta_{X\mathcal{D}}$. For every $\mathcal{O}$-module F equipped with a Θ_X-action such that the action map is a differential operator with respect to Θ_X[70] the corresponding $\mathcal{D}$-module $F_{\mathcal{D}}$ is a $\Theta_{\mathcal{D}}$-module. E.g., $(\Omega_X^{\otimes j})_{\mathcal{D}}$ are $\Theta_{\mathcal{D}}$-modules.

(ii) If $\mathfrak{g}$ is a Lie algebra then $\mathfrak{g}_{\mathcal{O}} = \mathfrak{g} \otimes \mathcal{O}_X$ is a Lie $\mathcal{O}_X$-algebra, hence $\mathfrak{g}_{\mathcal{D}} := \mathfrak{g} \otimes \mathcal{D}_X = (\mathfrak{g}_{\mathcal{O}})_{\mathcal{D}}$ is a Lie* algebra.

(iii) If P is a coherent $\mathcal{O}$-module then then the sheaf of differential operators $\mathrm{Diff}(F,F)_X$ is an associative algebra in $\mathcal{D}iff(X)^*$ which acts on F. So $\mathrm{Diff}(F,F)_{\mathcal{D}} := \mathrm{Diff}(F,F)_{X\mathcal{D}}$ is an associative* (hence Lie*) algebra that acts on $F_{\mathcal{D}}$. This action yields a canonical isomorhism of associative* algebras $\mathrm{Diff}(F,F)_{\mathcal{D}} \xrightarrow{\sim} \mathcal{E}nd^*(F_{\mathcal{D}})$ (see 2.2.13, (2.2.11.1)).

Remark. One can show that every Lie* algebra can be represented X-locally as a quotient of a Lie* algebra which is free as a $\mathcal{D}$-module (hence, in particular, is an induced $\mathcal{D}$-module). For a proof see ???.

(c) Let R^ℓ be a finitely $\mathcal{D}_X$-presented $\mathcal{D}_X$-algebra.[71] Then $\Omega_{R/\mathcal{O}_X}$ is a finitely-presented $R^\ell[\mathcal{D}_X]$-module, so Θ_R is well-defined (see Remark (ii) in 2.3.12). This is a Lie* algebra (see 1.4.14). On the other hand, we have the Lie algebra algebra $E := \mathrm{Der}_{\mathcal{D}_X} R^\ell$ of horizontal $\mathcal{O}_X$-derivations of R^ℓ (experts in non-linear differential equations usually call E *the Lie-Bäcklund* algebra). We consider E as a sheaf on X. One has the canonical morphism of Lie algebras $h(\Theta_R) \to E$. If Ω_{R^ℓ} is a projective $R^\ell[\mathcal{D}_X]$-module (which happens if R^ℓ is $\mathcal{D}_X$-smooth) then this morphism is an isomorphism (see Remark (iii) in 2.3.12).

Θ_R is the first example of a Lie* R-algebroid (see 1.4.10 and 2.5.16 below).

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[70]We assume that Θ_X acts on F as on an $\mathcal{O}$-module, so the action is automatically a differential operator with respect to F.

[71]Such R^ℓ essentially the same as a finite system of non-linear pde's of finitely many variables, see Remark in 2.3.1.

Remark. If Ω_R is a free $\mathcal{R}^\ell[\mathcal{D}_X]$-module then, choosing its generators $\nu_1, .., \nu_n$, we get an identification $E \xrightarrow{\sim} R^{\ell n}$, $\tau \mapsto (\langle \nu_i, \tau \rangle)$. It yields, in particular, an $\mathcal{O}_X$-module structure on E. With respect to this structure the bracket on E is a bidifferential operator, so we have the Lie* algebra $E_{\mathcal{D}}$. The action of E on R^ℓ is good (see 2.5.4), so we have the $E_{\mathcal{D}}$-action on R^ℓ. It yields an isomorphism of Lie* algebras $E_{\mathcal{D}} \xrightarrow{\sim} \Theta_R$.

(d) According to 2.2.14 for a coherent $\mathcal{D}$-module N a Lie cobracket $N \to N \otimes^! N$ yields a Lie* bracket on the dual $\mathcal{D}$-module N'. Assume that N is a $\mathcal{D}_X$-vector bundle; then this gives a 1-1 correspondence between the Lie$^!$ cobrackets on N (= Lie cobrackets on the left $\mathcal{D}$-module N^ℓ) and Lie* brackets on N'.

If N' is an induced $\mathcal{D}$-module, $N' = F_{\mathcal{D}}$ for a vector bundle F on X, then $N^\ell = {}_{\mathcal{D}}F^*$ (see 2.2.14), and our Lie cobracket is a skew-symmetric morphism ${}_{\mathcal{D}}F^* \to {}_{\mathcal{D}}F^* \otimes {}_{\mathcal{D}}F^*$, i.e., a section of $F \otimes \Lambda^2({}_{\mathcal{D}}F^*)$. Sometimes it is convenient to write Lie* brackets on $F_{\mathcal{D}}$ in this format.

Examples. (i) If G is a group $\mathcal{D}_X$-scheme such that its Lie coalgebra $\mathrm{CoLie}(G)$ is a coherent $\mathcal{D}$-module, then $\mathrm{Lie}(G) = \mathrm{CoLie}(G)'$ is a Lie* algebra. An action of G on a $\mathcal{D}_X$-scheme Y yields the action of $\mathrm{Lie}(G)$ on Y.

(ii) Consider the standard Lie* algebra structure on $\Theta_{\mathcal{D}}$ from (b)(i) above. The corresponding section of[72] $\Theta_X \otimes \Lambda^2({}_{\mathcal{D}}\Omega)$ is $\sum\limits_{i,j} \partial_i \otimes ((1 \otimes dx^j) \wedge (\partial_j \otimes dx^i))$. Similarly, in the situation of (b)(ii) (we assume that $\dim \mathfrak{g} < \infty$) the corresponding section is the usual cobracket element in $\mathfrak{g}^* \otimes \Lambda^2 \mathfrak{g}^* \subset \mathfrak{g}^* \otimes \Lambda^2({}_{\mathcal{D}}\mathfrak{g}^*)$.

For a $\mathcal{D}$-module N set $h'(N) := \underline{\mathrm{Hom}}_{\mathcal{D}_X}(N, \omega_X) = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(N^\ell, \mathcal{O}_X)$; this is a sheaf of k-vector spaces. The functor h' extends to a pseudo-tensor functor $h' : \mathcal{M}(X)^{!\circ} \to Sh(X)^{\otimes}$ in the obvious way (cf. (2.2.7.3)). The augmentation functor structure on h yields for $\mathcal{D}_X$-coherent N a canonical morphism $\alpha_N : h(N') \to h'(N)$ which extends to a morphism of pseudo-tensor functors in the obvious way. If N is a $\mathcal{D}_X$-vector bundle then α_N is an isomorphism.

[72] Here $\Omega = \Omega^1_X = \Theta^*_X$.

Therefore if N is a Lie$^!$ coalgebra then $h'(N)$ is a sheaf of Lie algebras; if N is $\mathcal{D}_X$-coherent then $h(N') \to h'(N)$ is a morphism of Lie algebras.

Example. Assume that our X is a symplectic variety with symplectic form ν. It yields a Poisson bracket on $\mathcal{O}_X$, so the corresponding induced $\mathcal{D}$-module $\mathcal{O}_{\mathcal{D}} = \mathcal{D}_X$ becomes a Lie* algebra; the Hamiltonian action morphism $\mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{\sim} \Theta_X$ (the latter identification is defined by ν) yields an embedding of Lie* algebras $a : \mathcal{O}_{\mathcal{D}} \to \Theta_{\mathcal{D}}$ whose image is the stabilizer of ν.[73] However the dual morphism of Lie coalgebras $a' : {}_{\mathcal{D}}\Omega \to {}_{\mathcal{D}}\mathcal{O} = \mathcal{D}_X$ is *not* surjective. Its image is a Lie coalgebra ${}_{\mathcal{D}}\Omega^{\nu}$ such that the image of embedding $h'({}_{\mathcal{D}}\Omega^{\nu}) \hookrightarrow h'({}_{\mathcal{D}}\Omega) = h(\Theta_{\mathcal{D}}) = \Theta_X$ is the Lie algebra of symplectic vector fields.

We see that the Lie algebra of symplectic vector fields comes from a Lie$^!$ coalgebra, not a Lie* algebra. So the world of Lie$^!$ coalgebras differs from that of Lie* algebras if we stop to consider $\mathcal{D}_X$-vector bundles (of course, we bring the unity back passing to derived categories). The examples below provide Lie* algebras which do not come from Lie$^!$ coalgebras.

In the next three examples (see 2.5.8–2.5.10*) we assume that* $\dim X = 1$. We will define canonical ω_X-extensions of some natural Lie* algebras of the type considered in 2.5.7(b).[74] The constructions follow the same pattern. Namely, to define our ω_X-extension $L^\flat$ of L we first construct an L-module extension $L^\natural$ of the adjoint representation by an L-module which equals $\omega_{\mathcal{D}} := \omega_X \otimes \mathcal{D}_X$ as a plain $\mathcal{D}$-module and has the property that the canonical morphism $\omega_{\mathcal{D}} \twoheadrightarrow \omega_X$ is L-invariant. Now $L^\flat$ is the push-out of $L^\natural$ by this arrow. The L-action on $L^\flat$ yields a $*$ bracket on $L^\flat$ which lifts the Lie* bracket on L. It satisfies automatically the Jacobi identity; the skew-symmetry follows from separate reasons.

Remark. For a Lie* algebra V any lisse $\mathcal{D}_X$-submodule of V lies in the center of V (see 2.2.4(ii)). Therefore any ω_X-extension of a Lie* algebra L is automatically central.

2.5.8 The Kac-Moody extension. Assume we are in situation of 2.5.7(b)(ii). Let $\kappa = (\,,\,)$ be an ad-invariant symmetric bilinear form on $\mathfrak{g}$. We will define an ω_X-extension $\mathfrak{g}^{\kappa}_{\mathcal{D}}$ of $\mathfrak{g}_{\mathcal{D}}$ called *the Kac-Moody* Lie* algebra.

[73] With respect to the action of $\Theta_{\mathcal{D}}$ on $\Omega^2_{\mathcal{D}}$, see Remark in 2.5.4.

[74] Notice that due to the presence of ω_X the corresponding $\mathcal{D}$-modules are *not* induced.

Consider the pairing $\tilde{\phi} : \mathfrak{g}_{\mathcal{O}} \times \mathfrak{g}_{\mathcal{O}} \to \omega_X$, $\tilde{\phi}(a,b) := (da, b)$ and the corresponding $*$ pairing $\tilde{\phi}_{\mathcal{D}} \in P_2^*(\{\mathfrak{g}_{\mathcal{D}}, \mathfrak{g}_{\mathcal{D}}\}, \omega_{\mathcal{D}})$; let $\phi \in P_2^*(\{\mathfrak{g}_{\mathcal{D}}, \mathfrak{g}_{\mathcal{D}}\}, \omega_X)$ be the composition of $\tilde{\phi}_{\mathcal{D}}$ with the canonical morphism $\omega_{\mathcal{D}} = \omega_X \otimes \mathcal{D}_X \to \omega_X$. Since κ is ad-invariant one has $\tilde{\phi}([a,b],c) = \tilde{\phi}(a,[b,c]) - \tilde{\phi}(b,[a,c])$. Since κ is symmetric one has $\tilde{\phi}(a,b) + \tilde{\phi}(b,a) = d(a,b)$, hence ϕ is skew-symmetric. Therefore ϕ is a 2-cocycle of the Lie* algebra $\mathfrak{g}_{\mathcal{D}}$. Our $\mathfrak{g}_{\mathcal{D}}^{\kappa}$ is the extension corresponding to this cocycle.

If $\mathfrak{g}$ is the Lie algebra of an algebraic group G and κ is Ad_G-invariant then the adjoint action of the sheaf of groups $G(\mathcal{O}_X)$ on $\mathfrak{g}_{\mathcal{D}}$ lifts canonically to $\mathfrak{g}_{\mathcal{D}}^{\kappa}$.[75] Namely, this action comes from an action of $G(\mathcal{O}_X)$ on $\mathfrak{g}_{\mathcal{O}} \oplus \omega_X$ defined by $g(a+\nu) := \mathrm{Ad}_g(a) + ((g^{-1}dg, a) + \nu)$. Therefore for every G-torsor $\mathfrak{F}$ on X we have the corresponding $\mathfrak{F}$-twisted Kac-Moody Lie* algebra which is an extension of the $\mathfrak{F}$-twisted Lie* algebra $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$ by ω_X. It can be described in "geometric" terms as follows.

Notice that the $\mathfrak{F}$-twist $\mathfrak{g}(\mathfrak{F})$ of $\mathfrak{g}_{\mathcal{O}}$ is the Lie algebra of infinitesimal symmetries of $\mathfrak{F}$, i.e., the Lie algebra of G-invariant "vertical" vector fields on $\mathfrak{F}$. Its dual $\mathfrak{g}(\mathfrak{F})^*$ is the sheaf of relative G-invariant 1-forms on $\mathfrak{F}/X$. Consider the sheaf $\mathcal{P}$ of G-invariant 1-forms on $\mathfrak{F}$. We have an extension of $\mathcal{O}_X$-modules $0 \to \omega_X \to \mathcal{P} \to \mathfrak{g}(\mathfrak{F})^* \to 0$. Let

$$0 \to \omega_X \to \mathcal{P}^{\kappa} \to \mathfrak{g}(\mathfrak{F}) \to 0 \tag{2.5.8.1}$$

be the pull-back of this extension by the morphism $\mathfrak{g}(\mathfrak{F}) \to \mathfrak{g}(\mathfrak{F})^*$ corresponding to κ.

Here is another way to define $\mathcal{P}^{\kappa}$. Consider the $\mathfrak{g}(\mathfrak{F}) \otimes \omega_X$-torsor $\mathcal{C}onn(\mathfrak{F})$ of connections on $\mathfrak{F}$; let $0 \to \mathfrak{g}(\mathfrak{F}) \otimes \omega_X \to \mathcal{C}onn(\mathfrak{F})\check{} \to \mathcal{O}_X \to 0$ be the corresponding $\mathcal{O}$-module extension. Tensoring by $\mathfrak{g}(\mathfrak{F})$ we get an extension of $\mathfrak{g}(\mathfrak{F})$ by $\mathfrak{g}(\mathfrak{F})^{\otimes 2} \otimes \omega_X$. Its push-out by $-\kappa \otimes id_{\omega}$: $\mathfrak{g}(\mathfrak{F})^{\otimes 2} \otimes \omega_X \to \omega_X$ equals $\mathcal{P}^{\kappa}$. The identification comes from the map $\mathcal{C}onn(\mathfrak{F})\check{} \otimes \mathfrak{g}(\mathfrak{F}) \to \mathcal{P}$ which sends $\nabla \otimes g$, $\nabla \in \mathcal{C}onn(\mathfrak{F})$, $g \in \mathfrak{g}(\mathfrak{F})$, to the 1-form on $\mathfrak{F}$ which vanishes on the ∇-horizontal vectors and whose restriction to fibers equals $\kappa(g) \in \mathfrak{g}(\mathfrak{F})^*$.

The Lie algebra $\mathfrak{g}(\mathfrak{F})$ acts on all the above $\mathcal{O}_X$-modules by $\mathcal{O}$-linear transformations. The action map is a differential operator of order ≤ 1 with respect to $\mathfrak{g}(\mathfrak{F})$. Therefore the

[75] More generally, for a Harish-Chandra pair $(\mathfrak{g}, K)$ and Ad-invariant κ the group $K(\mathcal{O}_X)$ acts on $\mathfrak{g}_{\mathcal{D}}^{\kappa}$.

Lie* algebra $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$ acts on $\mathcal{P}^{\kappa}_{\mathcal{D}}$. We define the extension of $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$-modules

$$0 \to \omega_X \to \mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}} \xrightarrow{\pi} \mathfrak{g}(\mathfrak{F})_{\mathcal{D}} \to 0 \tag{2.5.8.2}$$

as the push-out of the extension $\mathcal{P}^{\kappa}_{\mathcal{D}}$ by the canonical morphism of $\mathcal{D}$-modules $\omega_X \otimes \mathcal{D} \to \omega_X$. The $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$-action yields a binary $*$ operation $[\ \ ]$ on $\mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}}$, $[a,b] := \pi(a)b$.

Every trivialization of $\mathfrak{F}$ yields an identification $\mathfrak{g}(\mathfrak{F}) \xrightarrow{\sim} \mathfrak{g}_{\mathcal{O}}$ and a splitting $\mathcal{P} \xrightarrow{\sim} \mathfrak{g}(\mathfrak{F})^* \oplus \omega_X$, hence an isomorphism $\mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}} \xrightarrow{\sim} \mathfrak{g}_{\mathcal{D}} \oplus \omega_X = \mathfrak{g}^{\kappa}_{\mathcal{D}}$. We leave it to the reader to check that it identifies $[\ \ ]$ with the Lie* bracket on $\mathfrak{g}^{\kappa}_{\mathcal{D}}$ (in particular, we see that $[\ \ ]$ is a Lie* bracket) and the action of $\mathrm{Aut}\mathfrak{F}$ on $\mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}}$ with the above $G(\mathcal{O}_X)$-action on $\mathfrak{g}^{\kappa}_{\mathcal{D}}$.[76] Thus $\mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}}$ is the above $\mathfrak{F}$-twisted Kac-Moody Lie* algebra.

Remark. $\mathfrak{g}(\mathfrak{F})$ is the $\mathcal{O}_X$-linear part of the Lie $\mathcal{O}_X$-algebroid $\mathcal{A}(\mathfrak{F})$ of all infinitesimal symmetries of $(\mathfrak{F}, X)$. Our $\mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}}$ comes from a canonical ω_X-extension of $\mathcal{A}(\mathfrak{F})_{\mathcal{D}}$. For every connection ∇ on $\mathfrak{F}$ this extension canonically splits over $\nabla(\Theta_{\mathcal{D}})$. See [BS] where a more general setting is discussed.

2.5.9 The trace extension. Let F be a vector bundle on X. Denote by $\mathcal{A}(F)$ the Lie algebroid of infinitesimal symmetries of (F, X) which is an extension of Θ_X by the Lie algebra $\mathfrak{gl}(F)$ of endomorphisms of F. So $\mathcal{A}(F)$ is a Lie subalgebra of the algebra $\mathrm{Diff}(F,F)_X$ of differential operators which consists of operators of first order whose principal symbol lies in $id_F \cdot \Theta_X \subset (\mathrm{End}F) \otimes \Theta_X$.

We will define a canonical ω_X-extension $\mathcal{A}(F)^{\flat}_{\mathcal{D}}$ of the Lie* algebra $\mathcal{A}(F)_{\mathcal{D}}$. The construction goes as follows (see [BS] for details).

Set $F^\circ := F^* \otimes \omega_X$. The adjunction of differential operators yields a canonical antiisomorphism $\mathrm{Diff}(F,F)_X \xrightarrow{\sim} \mathrm{Diff}(F^\circ, F^\circ)_X$, so F° is a right $\mathrm{Diff}(F,F)_X$-module. Recall that one has a canonical identification of $\mathrm{Diff}(F,F)_X$-bimodules

$$\mathrm{Diff}(F,F)_X \xrightarrow{\sim} (j_*j^*F \boxtimes F^\circ)/F \boxtimes F^\circ \tag{2.5.9.1}$$

which maps $1 \in \mathrm{Diff}(F,F)$ to $1 \in \mathrm{End}F = F \boxtimes F^\circ(\Delta)/F \boxtimes F^\circ$; the inverse map assigns to a "kernel" $k(x,y) \in j_*j^*F \boxtimes F^\circ$ the differential operator $f \mapsto k(f), k(f)(x) :=$

[76] $\mathrm{Aut}\mathfrak{F}$ is identified with $G(\mathcal{O}_X)$ by the trivialization of $\mathfrak{F}$.

$\mathrm{Res}_{y=x}^{(y)}\langle k(x,y), f(y)\rangle$. Therefore $(j_*j^*F\boxtimes F^\circ)/F\boxtimes F^\circ(-\Delta)$ is an extension of $\mathrm{Diff}(F,F)_X$ by $F\otimes F^\circ$. Pushing it out by the trace map $tr: F\otimes F^\circ \to \omega_X$ we get an ω_X-extension; restricting it to $\mathcal{A}(F)\subset \mathrm{Diff}(F,F)_X$ we get an extension

$$0\to\omega_X\to\mathcal{A}(F)^\natural \xrightarrow{\pi^\natural} \mathcal{A}(F)\to 0. \tag{2.5.9.2}$$

The above constructions were natural, so $\mathcal{A}(F)$ acts on all the above objects. Thus (2.5.9.2) is an extension of $\mathcal{A}(F)$-modules (where $\mathcal{A}(F)$ acts on ω_X via the standard Θ_X-action). By construction, $\mathcal{A}(F)^\natural$ is an $\mathcal{O}_{X\times X}$-module supported on the diagonal, so, by 2.1.8(i), it is an object of the category $\mathcal{D}iff(X)$ (see 2.1.7). The $\mathcal{A}(F)$-action on $\mathcal{A}(F)^\natural$ is given by a bidifferential operator (see 2.2.4), so we have the $\mathcal{A}(F)_\mathcal{D}$-module extension $\mathcal{A}(F)^\natural_\mathcal{D}$ of the adjoint representation of $\mathcal{A}(F)_\mathcal{D}$ by $\omega_\mathcal{D}$. Its push-out by the canonical morphism $\omega_\mathcal{D}\twoheadrightarrow\omega_X$ is our extension

$$0\to\omega_X\to\mathcal{A}(F)^\flat \xrightarrow{\pi} \mathcal{A}(F)_\mathcal{D}\to 0. \tag{2.5.9.3}$$

The $\mathcal{A}(F)_\mathcal{D}$-module structure on $\mathcal{A}(F)^\flat_\mathcal{D}$ yields a $*$ operation $[\ \ ]$ on it, $[a,b]:=\pi(a)b$. It is easy to see that $[\ \ ]$ is skew-symmetric,[77] so it is a Lie* bracket.

Remarks. (i) The trace extension satisfies the following additivity property. Namely, let $\mathcal{F}$ be a flag $F_1\subset ..\subset F_n=F$ such that gr_iF are vector bundles. Let $\mathcal{A}(F)_\mathcal{F}\subset\mathcal{A}(F)$ be the subalgebroid of symmetries preserving $\mathcal{F}$. Let $\mathcal{A}(F)^\flat_{\mathcal{F}\mathcal{D}}$ be the restriction of the extension $\mathcal{A}(F)^\flat_\mathcal{D}$, and let $\mathcal{A}(F)^{\flat'}_{\mathcal{F}\mathcal{D}}$ be the Baer sum of the pull-backs of the extensions $\mathcal{A}(\mathrm{gr}_iF)^\flat_\mathcal{D}$ by the obvious projections $\mathcal{A}(F)_\mathcal{F}\to\mathcal{A}(\mathrm{gr}_iF)$. now there is an obvious canonical isomorphism of ω_X-extensions

$$\mathcal{A}(F)^\flat_{\mathcal{F}\mathcal{D}}\xrightarrow{\sim}\mathcal{A}(F)^\flat_{\mathcal{F}\mathcal{D}}. \tag{2.5.9.4}$$

(ii) The canonical identification of Lie algebroids $\mathcal{A}(F)\xrightarrow{\sim}\mathcal{A}(F^\circ)$ lifts to a canonical identification of the ω_X-extensions

$$\mathcal{A}(F)^\flat_\mathcal{D}\xrightarrow{\sim}\mathcal{A}(F^\circ)^\flat_\mathcal{D} \tag{2.5.9.5}$$

[77] Indeed, for every $\alpha,\beta\in\mathcal{A}(F)^\natural$ one has $\pi^\natural(\alpha)\beta+\pi^\natural(\beta)\alpha\in\omega_X\subset\mathcal{A}(F)^\natural$ and this 1-form is exact, see [BS] ??.

induced by the transposition of coordinates symmetry of $X \times X$.

(iii) We have two natural ω_X-extensions of the Lie* algebra $\mathfrak{gl}(F)_{\mathcal{D}}$: the first one is the Kac-Moody extension $\mathfrak{gl}(F)^{\kappa}_{\mathcal{D}}$ where κ is form $a, b \mapsto -\mathrm{tr}(ab)$ on $\mathfrak{gl}$, the second one is the restriction $\mathfrak{gl}(F)^{\flat}_{\mathcal{D}}$ of $\mathcal{A}(F)^{\flat}_{\mathcal{D}}$ to $\mathfrak{gl}(F)_{\mathcal{D}} \subset \mathcal{A}(F)_{\mathcal{D}}$. Now there is a canonical isomorphism of ω_X-extensions

$$\mathfrak{gl}(F\omega_X^{\otimes 1/2})^{\flat}_{\mathcal{D}} \xrightarrow{\sim} \mathfrak{gl}(F)^{\kappa}_{\mathcal{D}} \tag{2.5.9.6}$$

which lifts the obvious identification $\mathfrak{gl}(F\omega_X^{\otimes 1/2}) \xrightarrow{\sim} \mathfrak{gl}(F)$. Namely, every trivialization of F yields the $\mathcal{D}$-module splittings of both extensions,[78] and (2.5.9.6) is compatible with these trivializations.

(iv) Assume that F is a Θ_X-equivariant bundle such that the Θ_X-action morphism $\Theta_X \to \mathcal{A}(F)$ is a differential operator.[79]Then we can restrict $\mathcal{A}(F)^{\flat}_{\mathcal{D}}$ to $\Theta_{\mathcal{D}} \subset \mathcal{A}(F)_{\mathcal{D}}$. Thus every such F yields an ω_X-extension of $\Theta_{\mathcal{D}}$ which is a Lie* subalgebra of $\mathcal{A}(F)^{\flat}_{\mathcal{D}}$.

2.5.10 The Virasoro extension. We will define a canonical ω_X-extension $\Theta^{v}_{\mathcal{D}}$ of the Lie* algebra $\Theta_{\mathcal{D}}$ called *the Virasoro* Lie* algebra. The construction is similar to that of the Kac-Moody extension with the torsor of connections on $\mathfrak{F}$ replaced by the torsor of projective connections on X. The definition of projective connection is recalled in (b) below after a technical digression of (a); the Virasoro extension is constructed in (c); it is identified in (d) with an extension from Remark (iii) of 2.5.9.

(a) Let $X^{(n)}$ the nth infinitexima l neighbourhood of the diagonal $X \hookrightarrow X \times X$ for some $n \geq 0$, so $\mathcal{O}_{X^{(n)}} := \mathcal{O}_{X\times X}/\mathcal{J}^{n+1}$ where $\mathcal{J}$ is the ideal of the diagonal. The scheme $X^{(n)}$ carries a canonical involution σ (transposition of coordinates); the action of the Lie algebra Θ_X on $X = X^{(0)}$ extends in the obvious (diagonal) way to an action on $X^{(n)}$ commuting with σ. An $\mathcal{O}$-module E on $X^{(n)}$ yields, by 2.1.8(i), the $\mathcal{D}$-module $E_{\mathcal{D}}$ on X. If E is σ-equivariant then σ acts on $E_{\mathcal{D}}$. We have the plain sheaf of σ-invariants E^{σ} and the $\mathcal{D}$-module one $E^{\sigma}_{\mathcal{D}} := (E_{\mathcal{D}})^{\sigma}$.

[78]The splitting of the Kac-Moody extension was discussed in 2.5.8. The splitting of the trace extension in case $\mathrm{rk}F = 1$ identifies $\mathfrak{gl}(\omega_X^{\otimes 1/2})_{\mathcal{D}}$ with the antiinvariants of the involution (2.5.9.5) of $\mathcal{A}(\omega_X^{\otimes 1/2})^{\flat}_{\mathcal{D}}$. Use (2.5.9.4) to reduce the general situation to the rank 1 case.

[79]i.e., the Θ_X-action satisfies the condition from 2.5.7(b)(i).

The projection $\mathcal{O}^*_X \twoheadleftarrow \mathcal{O}^*_{X^{(n)}}$ has a canonical section $\mathfrak{s} : \mathcal{O}^*_X \to \mathcal{O}^*_{X^{(n)}}$ uniquely defined by properties that it has étale local nature and $\mathfrak{s}(f^2) = f \boxtimes f|_{X^{(n)}}$. It is clear that $\mathfrak{s}$ is σ-invariant and commutes with the action of Θ_X. Explicit formula: $\mathfrak{s}(f)(x,y) = f(x) + \frac{1}{2}f'(x)(y-x) + \frac{1}{4}(f''(x) - f'(x)^2/f(x))(y-x)^2 + \dots$ So a line bundle $\mathcal{L}$ on X yields a σ-equivariant line bundle $\mathcal{L}^{\mathfrak{s}}$ on $X^{(n)}$. To construct $\mathcal{L}^{\mathfrak{s}}$ one should choose (locally) a square root $\mathcal{L}^{\otimes 1/2}$; then $\mathcal{L}^{\mathfrak{s}} = \mathcal{L}^{\otimes 1/2} \boxtimes \mathcal{L}^{\otimes 1/2}|_{X^{(n)}}$. Notice that the canonical filtration $\mathcal{J}^i\mathcal{L}^{\mathfrak{s}}$ has successive quotients equal to $\mathcal{L}$, $\mathcal{L} \otimes \omega_X$,.., $\mathcal{L} \otimes \omega_X^{\otimes n}$, and σ acts on ith one as multiplication by $(-1)^i$. If Θ_X acts on $\mathcal{L}$ then $\mathcal{L}^{\mathfrak{s}}$ is a (σ, Θ_X)-equivariant line bundle.

Denote by ν the restriction of $\mathcal{O}_{X\times X}(\Delta)$ on $X^{(n)}$. This is a (σ, Θ_X)-equivariant line bundle on $X^{(n)}$: the action of σ is the transposition action multiplied by -1, and Θ_X acts in the obvious way. One has $\nu^{\otimes m} = \mathcal{O}_{X\times X}(m\Delta)/\mathcal{O}_{X\times X}((m-n-1)\Delta)$.

(b) From now on $n = 2$. Assume that we have a (σ, Θ_X)-equivariant line bundle λ on $X^{(2)}$ whose restriction to X equals $\mathcal{O}_X$ (with the trivial action of σ and the standard action of Θ_X). The canonical filtration on λ has successive quotients $\mathcal{O}_X$, ω_X, $\omega_X^{\otimes 2}$, and σ acts on them as multiplication by, respectively, $1, -1, 1$. Let $\mathfrak{P}(\lambda) \subset \lambda^\sigma$ be the preimage of $1 \in \mathcal{O}_X$ by the projection $\lambda^\sigma \to \mathcal{O}_X$. This is a Θ_X-equivariant $\omega_X^{\otimes 2}$-torsor. It is clear that $\mathfrak{P}(\lambda^{\otimes m}) = m\mathfrak{P}(\lambda)$.

Our key example is $\lambda = \omega_X^{\mathfrak{s}} \otimes \nu$. The elements of $\mathfrak{P} := \mathfrak{P}(\omega_X^{\mathfrak{s}} \otimes \nu)$ are called *projective connections* on X.

Remarks. (i) A local coordinate t on X yields a projective connection ρ_t defined as restriction of $(x-y)^{-1}dx^{1/2}dy^{1/2}$ to $X^{(2)}$.[80] An immediate calculation shows that $(f(t)\partial_t)(\rho_t) = \frac{1}{12}f'''(x)dx^2$.

(ii) We have $(\omega_X^{\mathfrak{s}} \otimes \nu)^{\otimes m} = (\omega_X^{\otimes m/2} \boxtimes \omega_X^{\otimes m/2})(m\Delta)/((m-3)\Delta)$. So for $m = 3, 4$ one has embeddings $m\mathfrak{P} \hookrightarrow (j_*j^*\omega_X^{\otimes m/2} \boxtimes \omega_X^{\otimes m/2})/\omega_X^{\otimes m/2} \boxtimes \omega_X^{\otimes m/2} = \mathrm{Diff}(\omega_X^{\otimes 1-m/2}, \omega_X^{\otimes m/2})$ which identify projective connections[81] with symmetric differential operators $\omega_X^{\otimes -1/2} \to \omega_X^{\otimes 3/2}$ of order 2 and principal symbol 1 and, respectively, skew-symmetric differential operators $\omega_X^{\otimes -1} \to \omega_X^{\otimes 2}$ of order 3 and principal symbol 1. For an interpretation of projective

[80] Here x, y are coordinates on $X \times X$ corresponding to t.
[81] Here we identify $\mathfrak{P}$ with $3\mathfrak{P}, 4\mathfrak{P}$ by means of multiplications by 3 and 4.

connections as $\mathfrak{sl}_2$-opers see [BD].

(c) Tensoring Θ_X by $\mathfrak{P}$ we get an $\mathcal{O}$-module extension $\mathcal{P}^v$ of Θ_X by $\Theta_X \otimes \omega_X^{\otimes 2} = \omega_X$. It carries a canonical action of Θ_X which coincides with the standard action on the subquotients ω_X, Θ_X. According to Remark (i), every local coordinate t yields a splitting of our extension; in terms of this splitting the Θ_X-action is given by the cocycle $f(t)\partial_t \mapsto \frac{1}{12} f'''(t) dt^2$. Now the Virasoro extension

$$0 \to \omega_X \to \Theta_{\mathcal{D}}^v \xrightarrow{\pi} \Theta_{\mathcal{D}} \to 0 \tag{2.5.10.1}$$

is the push-out of the extension $\mathcal{P}^v_{\mathcal{D}}$ by the canonical morphism of $\mathcal{D}$-modules $\omega_{\mathcal{D}} \to \omega_X$.

The action of Θ_X on $\mathcal{P}^v$ is given by a bidifferential operator, so it yields a $*$ action of the Lie* algebra $\Theta_{\mathcal{D}}$ on $\mathcal{P}^v_{\mathcal{D}}$ and $\Theta^v_{\mathcal{D}}$ which lifts the adjoint action. The binary $*$ operation $[\ \ ]$, $[a,b] := \pi(a)b$, on $\Theta^v_{\mathcal{D}}$ is a Lie* bracket. Indeed, it obviously satisfies the Jacobi identity; one proves the skew-symmetry using the above explicit formula.

(d) In Remark (iv) of 2.5.9 consider $F = \mathcal{O}_X$. We get an ω_X-extension of $\Theta_{\mathcal{D}}$ which identifies canonically with the twice Virasoro extension[82] $\Theta^{2v}_{\mathcal{D}}$. Namely, the embedding $\Theta^{2v}_{\mathcal{D}} \hookrightarrow \mathcal{A}(\mathcal{O}_X)^\flat_{\mathcal{D}}$ comes from the morphism $\mathcal{P}^v \to \mathcal{A}(\mathcal{O}_X)^\natural$, $\tau \otimes \rho \mapsto (\tau \otimes 1)\rho^2$.[83]

Remark. More generally, the ω_X-extension of $\Theta_{\mathcal{D}}$ which comes from the Θ_X-equivariant line bundle $\omega_X^{\otimes j}$ identifies canonically with $\Theta_{\mathcal{D}}^{2(6j^2-6j+1)v}$. See e.g. [BS]?? for explicit "coordinate" formulas for this identification.

2.5.11. Let L be a Lie* algebra, $x \in X$ a (closed) point, $j_x : U_x := X \smallsetminus \{x\} \hookrightarrow X$. The vector space $h(L)_x$ carries the Ξ_x-topology (see 2.1.13).

Lemma. (i) The Lie bracket on $h(L)_x$ is Ξ_x-continuous with respect to each variable.
(ii) If $j_x^* L$ is a countably generated $\mathcal{D}_{U_x}$-module then the bracket is Ξ_x-continuous.
Proof. See 2.2.17(i),(ii). $\square$

Remark. If we decide to play with better objects $\tilde{h}_x$ from Remark (iii) of 2.1.14 then the countability condition becomes irrelevant.

[82] i.e., the push-out of the Virasoro extension by the multiplication by 2 map $\omega_X \to \omega_X$.
[83] Here $\rho^2 \in \omega_X^{\boxtimes 2}(2\Delta)/\omega_X^{\boxtimes 2}(-\Delta)$.

Examples. If we are in situation of 2.5.7(b) and the restriction of P to the complement of x is $\mathcal{O}$-coherent then $\hat{h}_x(P_{\mathcal{D}}) = P \otimes O_x$ (see 2.1.14(ii)) and the Lie bracket comes from the bidifferential operator defining the Lie bracket on P.

In the situation of 2.5.8 the Lie algebra $\hat{h}(j_{x*}j_x^*\mathfrak{g}_{\mathcal{D}}^{\kappa})$ is the central extension of $\mathfrak{g}(K_x) = \hat{h}(j_{x*}j_x^*\mathfrak{g}_{\mathcal{D}})$ by $k = \hat{h}(j_{x*}j_x^*\omega_X)$ defined by 2-cocycle $a, b \mapsto \mathrm{Res}_x(da, b)$, i.e., the usual affine Kac-Moody algebra.

2.5.12. Denote by $\Xi_x^{Lie}(L)$ the topology on L at x (see 2.1.13 for terminology) whose base consists of those $L_\xi \in \Xi_x(L)$ which are Lie* subalgebras of L. As follows from the Remark in 2.2.7, the latter condition amounts to the fact that $h(L_\xi)_x$ is a Lie subalgebra of $h(L)_x$. Therefore one can describe the Ξ_x^{Lie}-topology on $h(L)_x$ as the topology whose base is formed by all Ξ_x-open Lie subalgebras of $h(L)_x$.

The Lie bracket on $h(L)_x$ is automatically continuous with respect to Ξ_x^{Lie}-topology. Indeed, the base of our topology is formed by Lie subalgebras, and for every open Lie subalgebra V the adjoint action of V on $h(L)_x/V$ is continuous by 2.5.11(i).

Therefore the completion $\hat{h}_x^{Lie}(L)$ is a topological Lie algebra, and $\hat{h}_x(L)$ is a topological Lie algebra if L satisfies the condition of 2.5.11(ii). We have a canonical continuous morphism $\hat{h}_x(L) \to \hat{h}_x^{Lie}(L)$ of Lie algebras.

The two topologies do not differ if L satisfies a coherency condition:

2.5.13 Lemma. (i) Let M be an L-module such that its restriction to U_x is a coherent $\mathcal{D}$-module. Then for any $M_\xi, M_{\xi'} \in \Xi_x(M)$ such that $M_{\xi'}$ is a coherent $\mathcal{D}_X$-module there exists $L_\zeta \in \Xi_x^{Lie}(L)$ such that $L_\zeta(M_{\xi'}) \subset M_\xi$.

(ii) If the restriction of L to U_x is a coherent $\mathcal{D}$-module then the topologies $\Xi_x(L)$ and $\Xi_x^{Lie}(L)$ coincide.

Remark. Using the terminology of 2.6.9 the above statements can be reformulated as follows. Recall that $F := \hat{h}_x(M)$ is a Tate vector space (see 2.1.13 and 2.6.9 for terminology), and (i) just says that the action morphism $h(L)_x \to \mathrm{End}\hat{h}_x(M)$ is continuous with respect to the Ξ_x^{Lie}-topology and the topology on $\mathrm{End}F$ introduced in 2.6.9. In the situation of (ii) $\hat{h}_x(L)$ is a topological Lie algebra which is a Tate vector space. The argument below actually proves that any such object has a base of topology formed by open Lie subalgebras.

Proof. (i) Replacing M_ξ by $M_\xi \cap M_{\xi'}$ we may assume that $M_\xi \subset M_{\xi'}$. According to 2.2.17(iii) the maximal sub $\mathcal{D}$-module $L_\zeta \subset L$ such that $L_\zeta(M_{\xi'}) \subset M_\xi$ belongs to $\Xi_x(L)$. It is also a Lie* algebra, so we are done.

(ii) We want to show that every $L_\xi \in \Xi_x(L)$ contains some $L_\zeta \in \Xi_x^{Lie}(L)$. We can assume that L_ξ is a coherent $\mathcal{D}$-module. According to the proof of (i) the normalizer N of L_ξ is open. Set $L_\zeta := N \cap L_\xi$. □

In the non-coherent situation the topologies Ξ_x and Ξ_x^{Lie} can be very different:

2.5.14 Example. Consider the Lie* algebra $L = \mathfrak{gl}(\mathcal{D}_X)$. Then $h(L)$ is $\mathcal{D}_X$ considered as a sheaf of Lie algebras. For every $x \in X$ the completion $\hat{h}_x(L)$ is a Lie algebra because L satisfies the condition of 2.5.11(ii). Our L is an induced $\mathcal{D}_X$-module, so, by Example (i) of 2.1.13, the completion $\hat{h}_x(L)$ is the Lie algebra $D = \mathcal{D}_{O_x}$ of differential operators with coefficients in O_x (the formal completion of the local ring at x). Denote by $D^{\leq n}$ the subspace of differential operators of order $\leq n$; this is a free O_x-module of rank $n+1$. A vector subspace of D is open in Ξ_x-topology if its intersection with every $D^{\leq n}$ is open. The Ξ_x^{Lie}-topology is defined by Ξ_x-open Lie subalgebras. It is easy to see that for every open subspace $P \subset D^{\leq 2}$ the Lie subalgebra generated by P is open. Such subalgebras form a base of the topology Ξ_x^{Lie}; in particular, the topology Ξ_x^{Lie}, as opposed to Ξ_x, has a countable base.

Let us describe the Ξ_x^{Lie}-topology on D explicitly assuming that $\dim X = 1$. Let t be a local parameter at x, so $O_x = k[[t]]$. Set $\partial = \partial_t$. For $a \geq 0$, $b \geq 1$ let $D^{a,b} \subset D$ be the space of differential operators $\Sigma \phi_i(t)\partial^i$, $\phi_i(t) \in t^{a+bi}O_x$.

Lemma. $D^{a,b}$ form a basis of the Ξ_x^{Lie}-topology on D.

Proof. Indeed, $D^{a,b}$ are open in the Ξ_x-topology and they are associative, hence Lie, subalgebras of D. It remains to check that any open Lie subalgebra $Q \subset D$ contains some $D^{a,b}$. Set $D_i^{a,b} := D^{a,b} \cap D^{\leq i}$. A simple calculation shows that the operator

$A_b := ad_{t^{b+1}\partial^2} \in \mathrm{End}D$ preserves $D^{a,b}$. It sends $D^{\leq i}$ to $D^{\leq i+1}$ and the corresponding operator $\mathrm{gr}_i D^{a,b} \to \mathrm{gr}_{i+1} D^{a,b}$ is an isomorphism if $a \geq 1$, $i \geq 0$. Therefore $D^{a,b} = \underset{i\geq o}{\oplus} A_b^i(t^a O_x)$. Now choose $a, b \geq 1$ such that $t^a O_x \subset Q$ and $t^{b+1}\partial^2 \in Q$. Then $D^{a,b} \subset Q$. □

We see that the completion $\hat{h}_x^{Lie}(L)$ consists of "differential operators of infinite order" $\Sigma\phi_i(t)\partial^i$ such that the number of the first non-vanishing coefficient of $\phi_i(t)$ grows faster than any linear function of i.

Remark. We will see in 2.6.13 that the above L carries an important topology which is weaker then Ξ_x^{Lie}.

2.5.15. Let L be any Lie* algebra. Notice that for an $h(L)_x$-module V the action map $h(L)_x \times V \to V$ is continuous with respect to the Ξ_x^{Lie}-topology (and the discrete topology on V) if and and only if it is continuous with respect to the Ξ_x-topology (since the stabilizer of every $v \in V$ is a Lie subalgebra).[84] Such $h(L)_x$-modules are called *discrete*; the corresponding category is denoted by $\hat{h}_x(L)$mod.

Let $\mathcal{M}(X, L)_x$ be the category of L-modules supported at x. For $M \in \mathcal{M}(X, L)_x$ the Lie algebra $h(L)_x$ acts canonically on M, hence $i_x^! M = h(M)$ is an $h(L)_x$-module. By 2.5.13(i) it is discrete.

Lemma. The functor $h : \mathcal{M}(X, L)_x \to \hat{h}_x(L)$mod is an equivalence of categories.

Proof. Follows from 2.5.4. □

2.5.16 Lie* algebroids. Let R^ℓ be a commutative $\mathcal{D}_X$-algebra. According to 1.4.10 we have the notion of Lie* R-algebroid. As follows from the second lemma of 2.3.12 these objects have étale local nature. Therefore we know what is a Lie* algebroid $\mathcal{L}$ on an algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ and what are $\mathcal{L}$-modules. Then $h(\mathcal{L})$ is a sheaf of Lie algebras (on $\mathcal{Y}_{\acute{e}t}$) acting on $\mathcal{O}_\mathcal{Y}$ by horizontal $\mathcal{O}_X$-derivations.

The constructions of 1.4.13 are étale local as well (for the same reason). So for a Lie* algebroid $\mathcal{L}$ on $\mathcal{Y}$ which is a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$ (see 2.3.11) we have a commutative DG superalgebra $\mathcal{C}(\mathcal{L})_\mathcal{Y}$ called *the de Rham-Chevalley complex of* $\mathcal{L}$, and for an $\mathcal{L}$-module M

[84] In fact, one can carachterize Ξ_x^{Lie} as the weakest topology on $h(L)_x$ which satisfies this property.

we have a DG $\mathcal{C}(\mathcal{L})_{\mathcal{Y}}$-module $\mathcal{C}(\mathcal{L},M)_{\mathcal{Y}}$. If we forget about the differential then $\mathcal{C}(\mathcal{L})_{\mathcal{Y}} = \mathrm{Sym}(\mathcal{L}'[-1])$, $\mathcal{C}(\mathcal{L},M)_{\mathcal{Y}} = \mathrm{Sym}(\mathcal{L}'[-1]) \otimes M$ where $\mathcal{L}'$ is the $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$ dual to $\mathcal{L}$.

For example, if $\Omega_{\mathcal{Y}}$ is finitely-presented then the tangent Lie algebroid $\Theta_{\mathcal{Y}}$ is well-defined (see 1.4.14 and Remarks in 2.3.12). If $\Omega_{\mathcal{Y}}$ is a $\mathcal{D}_{\mathcal{Y}}$-vector bundle, e.g. if $\mathcal{Y}$ is smooth (see 2.3.13–2.3.16), then $h(\Theta_{\mathcal{Y}})$ is the Lie algebra of all horizontal $\mathcal{O}_X$-derivations of $\mathcal{O}_{\mathcal{Y}}$. One has $\mathcal{C}(\Theta_{\mathcal{Y}})_{\mathcal{Y}} = DR_{\mathcal{Y}/X}$ (the relative de Rham complex).

More generally, if V is a $\mathcal{D}_X$-vector bundle on such $\mathcal{Y}$ then the Lie* algebroid $\mathcal{E}(V)$ on $\mathcal{Y}$ (see 1.4.14) is well-defined. The morphism $\tau\colon \mathcal{E}(V) \to \Theta_{\mathcal{Y}}$ is surjective, so $\mathcal{E}(V)$ is an extension of $\Theta_{\mathcal{Y}}$ by $\mathfrak{g}l(V)$.

The material of 2.5.11–2.5.13 generalizes to the setting of Lie* algebroids as follows. We assume that $\dim X = 1$, R^ℓ is a commutative $\mathcal{D}$-algebra on U_x, $\mathcal{L}$ a Lie* R-algebroid. Being an R^ℓ-module, $j_{x*}\mathcal{L}$ carries the topology Ξ_x^R at x (see Remark in 2.4.8).

2.5.17 Lemma. (i) The action of $h_x(j_{x*}\mathcal{L})$ on $h_x(j_{x*}R)$ is Ξ_x^R -Ξ_x^{as}-continuous with respect to each variable. The Lie bracket on $h_x(j_{x*}\mathcal{L})_x$ is Ξ_x^R-continuous with respect to each variable.

(ii) If R^ℓ is a countably generated $\mathcal{D}_{U_x}$-algebra and $\mathcal{L}$ is a countably generated $R^\ell[\mathcal{D}_{U_x}]$-module then the action is Ξ_x^R -Ξ_x^{as}-continuous and the bracket is Ξ_x^R-continuous.

Proof. (i) Take $R_\xi \in \Xi_x^{as}(j_{x*}R)$. For $\bar{l} \in h_x(j_{x*}\mathcal{L})$ one has $\{r \in R_\xi : \bar{l}^{-1}(r) \in R_\xi\} \in \Xi_x^{as}$. For $\bar{r} \in h_x(R)$ one has $\{l \in \mathcal{L} : \bar{l}(\bar{r}) \in h_x(R_\xi)\} \in \Xi_x^R$.

(ii) Take R_ξ as above and $\mathcal{L}_\zeta \in \Xi_x^R$ which is an R_ξ-module. It suffices to find $\mathcal{L}_\eta \in \Xi_x^R$, $R_\nu \in \Xi_x^{as}$ such that the action map on $\mathcal{L}_\eta$, R_ν takes values in R_ξ, and the bracket on $\mathcal{L}_\eta$ takes values in $\mathcal{L}_\zeta$. Together with (i) this proves the continuity.

Let $P \subset R_\xi$, $Q \subset \mathcal{L}_\zeta$ be countably generated $\mathcal{D}$-submodules such that $P|_{U_x}$ generates R^ℓ as a $\mathcal{D}_{U_x}$-algebra and $Q|_{U_x}$ generates $\mathcal{L}$ as an $R^\ell[\mathcal{D}_{U_x}]$-module. According to 2.2.17(ii) there exists $P' \in \Xi_x(P)$, $Q' \in \Xi_x(Q)$ such that the action map on Q', P' takes values in R_ξ, and the bracket on Q' takes values in $\mathcal{L}_\zeta$. Now let R_ν be the $\mathcal{D}_X$-subalgebra of $j_{x*}R$ generated by P', $\mathcal{L}_\eta$ be the R_ν-submodule of $j_{x*}\mathcal{L}$ generated by Q'. $\square$

We see that countability conditions of 2.5.17(ii) imply that $\hat{h}^R_x(\mathcal{L})$ is a topological Lie R^{as}_x-algebroid.

Remark. If we decide to play with better objects $\tilde{h}_x$ from Remark (iii) of 2.1.14 then the countability condition becomes irrelevant.

The module $j_{x*}\mathcal{L}$ also carries a weaker topology $\Xi^{R\,Lie}_x := \Xi^R_x \cap \Xi^{Lie}_x$.

2.5.18 Lemma. Assume that R^ℓ is finitely generated $\mathcal{D}_{U_x}$-algebra and $\mathcal{L}$ is a finitely generated $R^\ell[\mathcal{D}_{U_x}]$-module. Then the Ξ^R_x- and $\Xi^{R\,Lie}_x$-topologies on $h_x(\mathcal{L})$-coincide.

Proof. Similar to that of 2.5.13; left to the reader. □

§6 The Tate extension

In this section it is assumed that dim $X = 1$. We are going to define a canonical central extension of a matrix Lie* algebra. A linear algebra version of this ubiquitous extension first appeared (implicitly) in Tate's remarkable note [T]; it was rediscovered and studied under various disguises and names[85] in the beginning of 80s (see [DJKM], [KP], and [SW]).

We begin in 2.6.1 with a general format for construction of central extensions of Lie algebras. The Tate extension in $\mathcal{D}$-module setting is defined in 2.6.2, 2.6.3. Tate's original linear algebra construction is presented in 2.6.9–2.6.11; the two constructions are related in 2.6.12–2.6.15.

We will see in 3.8.6, 3.8.18 that the Tate extension arises naturally as a part of (chiral) Clifford algebra.

2.6.1 Some concrete nonsense. We live in an abelian pseudo-tensor k-category. For simplicity of notation we write formulas in terms of "symbolic elements" of our objects.

(i) Let L be a Lie algebra. Denote by $\mathcal{E}(L)$ the category of pairs $(L^\natural, \pi)$ where $L^\natural$ is an L-module, $\pi : L^\natural \to L$ a surjective morphism of L-modules (we consider L as an L-module with respect to the adjoint action). We often write $L^\natural$ for $(L^\natural, \pi)$ and set $L^\natural_0 := \mathrm{Ker}\pi$.

[85] For example, in Moscow they used to call it "the Japanese extension"; cf. A. Brehm's discourse in " The Life of Animals" on the names of the common cockroach (Blatta germanica) in different tongues: Russians call the fellow Prussian, in Austrian highlands it is known as Russian, etc.

An object $L^\natural \in \mathcal{E}(L)$ yields a symmetric L-invariant pairing $(\quad) \in P_2(\{L^\natural, L^\natural\}, L_0^\natural)$, $(a,b) := \pi(a)b + \pi(b)a$. Notice that the L-action on $L_0^\natural$ is trivial if and only if $(\quad) \in P_2(\{L, L\}, L_0^\natural) \subset P_2(\{L^\natural, L^\natural\}, L_0^\natural)$. We say that $L^\natural$ is *central* if $(\quad) = 0$.

Denote by $\mathcal{CE}(L)$ the category of central Lie algebra extensions of L. Notice that for every $L^\flat \in \mathcal{CE}(L)$ the adjoint action of $L^\flat$ factors through L. So $L^\flat$ is an L-module; the structure projection $L^\flat \to L$ makes it an object of $\mathcal{E}(L)$.

Lemma. The functor $\mathcal{CE}(L) \to \mathcal{E}(L)$ we have defined is fully faithful. It identifies $\mathcal{CE}(L)$ with the category of central objects in $\mathcal{E}(L)$. $\square$

Assume we have $L^\natural \in \mathcal{E}(L)$ and an L-invariant morphism $\mathrm{tr} : L_0^\natural \to F$ (the L-action on F is trivial). We say that tr is *central* if $\mathrm{tr}(\quad) \in P_2(\{L, L\}, F)$ equals 0. Denote by $L^\natural_{\mathrm{tr}}$ the push-forward of $L^\natural$ by tr; this is an object of $\mathcal{E}(L)$ which is an extension of L by F. It is clear that $L^\natural_{\mathrm{tr}} \in \mathcal{CE}(L)$, if and only if tr is central.

Remark. If every L-invariant symmetric F-valued pairing on L is trivial then every tr as above is automatically central.

We suggest the reader to skip the rest of this subsection returning to it when necessary.

(ii) Here is an example of extensions that arise in the above manner:

Lemma. Let L be a Lie algebra, $L_\pm \subset L$ ideals such that $L_+ + L_- = L$, and $\mathrm{tr} : L_0 := L_+ \cap L_- \to F$ a morphism such that $\mathrm{tr}([l_+, l_-]) = 0$ for every $l_\pm \in L_\pm$. Then there exists a central extension $L^\flat$ of L by F together with sections $s_\pm : L_\pm \to L^\flat$ such that the images $s_\pm(L_\pm)$ are ideals in $L^\flat$[86] and for any $l \in L_0$ one has $(s_+ - s_-)l = \mathrm{tr}(l) \in F \subset L^\flat$. Such $(L^\flat, s_\pm)$ is unique (up to a unique isomorphism).

Proof. The uniqueness of $(L^\flat, s_\pm)$ is clear. Let us construct $L^\flat$. Set $L^\natural := L_+ \oplus L_-$, and consider the extension of L-modules $0 \to L_0 \xrightarrow{\alpha} L^\natural \xrightarrow{\beta} L \to 0$ where $\alpha(l) = (l, -l)$, $\beta(l_+, l_-) = l_+ + l_-$ (the action of L is the adjoint one). Then $L^\natural \in \mathcal{E}(L)$ and tr is a central morphism (the L-invariance of tr and vanishing of $\mathrm{tr}(\quad)$ follow from the property $\mathrm{tr}([L_+, L_-]) = 0$). Now set $L^\flat := L^\natural_{\mathrm{tr}}$. $\square$

Remark. In the situation of the corollary one has $L_+/L_0 \xrightarrow{\sim} L/L_-$. The Lie algebra

[86] Or, equivalently, that $s_\pm$ are morphisms of L-modules (with respect to the adjoint actions).

L_+/L_0 has a central extension $(L_+/L_0)^\flat$ defined as the push-out of the extension $0 \to L_0 \to L_+ \to L_+/L_0 \to 0$ by $\mathrm{tr} : L_0 \to F$. Now s_+ yields an isomorphism $(L_+/L_0)^\flat \xrightarrow{\sim} L^\flat/s_-(L_-)$.

Similarly, s_- yields an isomorphism $(L_-/L_0)^\flat \xrightarrow{\sim} L^\flat/s_+(L_+)$ where the central extension $(L_-/L_0)^\flat$ of L_-/L_0 is the push-out of the extension $0 \to L_0 \to L_- \to L_-/L_0 \to 0$ by $-\mathrm{tr} : L_0 \to F$.

(iii) Let A be an associative algebra. Denote by A^{Lie} our A considered as a Lie algebra.

If $A^\flat$ is a central extension of A^{Lie} by F then its bracket comes from a skew-symetric pairing $[\ \ ]^\flat \in P_2(\{A, A\}, A^\flat$. We say that $A^\flat$ satisfies *the cyclic property* if for every $a, b, c \in A$ one has $[ab, c]^\flat + [bc, a]^\flat + [ca, b]^\flat = 0$.

Denote by $\mathcal{E}(A)$ the category of pairs $(A^\natural, \pi)$ where $A^\natural$ is an A-bimodule, $\pi : A^\natural \to A$ a surjective morphism of A-bimodules. Notice that any A-bimodule is an A^{Lie}-module with respect to the commutator action. Thus every object of $\mathcal{E}(A)$ can be considered as an object of $\mathcal{E}(A^{Lie})$, i.e., we have a faithful functor $\mathcal{E}(A) \to \mathcal{E}(A^{Lie})$.

Take $A^\natural \in \mathcal{E}(A)$ and let $\mathrm{tr} : A_0^\natural \to F$ be a central A^{Lie}-invariant morphism, so $A^\flat := A^\natural_{\mathrm{tr}}$ is a central Lie algebra extension of A^{Lie} by F.

Lemma. $A^\flat$ satisfies the cyclic property.

Proof. Let $\tilde{c}$ be a lifting of c. We have a tautological identity $[ab, \tilde{c}] + [b\tilde{c}, a] + [\tilde{c}a, b] = 0$ in $A^\natural$ (it holds in any A-bimodule).[87] Now notice that $\mathrm{tr}([bc, \tilde{a}] - [b\tilde{c}, a]) = \mathrm{tr}(a, bc) = 0$ (since $b\tilde{c}$ is a lifting of bc) and similarly $\mathrm{tr}([ca, \tilde{b}] - [\tilde{c}a, b]) = \mathrm{tr}(ca, b) = 0$. □

For example, if we have two-sided ideals $A_\pm \subset A$ and a morphism $\mathrm{tr} : A_0 := A_+ \cap A_- \to F$ which vanishes on $[A_+, A_-] \subset A_0$ then the central extension $A^\flat$ of A^{Lie} defined in (ii) above satisfies the cyclic property.

Remarks. (i) Assume that our pseudo-tensor category is a tensor unital category (say, the category of k-modules) and A is a commutative unital algebra. Consider the A-

[87] This identity can be interpreted as follows. For any A-bimodule M and $a \in A$ denote by l_a, r_a the left, resp. right, multiplication by a; set $ad_a := l_a - r_a$. The identity says that $ad_{ab} = ad_a l_b + ad_b r_a$. Define the A-bimodule structure on $\mathrm{End}(M)$ by $af := fr_a$, $fb := fl_b$. Then the identity says that $ad_{ab} = ad_a b + a ad_b$, i.e., the map $A \to \mathrm{End}(M)$, $a \mapsto ad_a$, is a derivation.

module of differentials Ω_A, so we have a universal derivation $d : A \to \Omega_A$. For every object F the map $\mathrm{Hom}(\Omega_A, F) \to P_2(\{A, A\}, F)$, which sends $\varphi : \Omega_A \to F$ to the operation $a, b \mapsto \varphi(adb)$ is injective, and its image consists of those operations ψ that $\psi(a, bc) = \psi(ab, c) + \psi(ac, b)$. Notice that φ vanishes on the image of d if and only if the corresponding ψ is skew-symmetric. Now for $(A^\natural, \mathrm{tr})$ as above one has $[\ \ ]^\flat \in P_2(\{A, A\}, F)$ (since A is commutative), and our lemma shows that $[\ \ ]^\flat$ yields a morphism $\Omega_A/d(A) \to F$. This is one of the principal ideas of Tate's work [T].

(ii) ?? TSYGAN'S PICTURE -DOPISAT'

(iv) Assume our pseudo-tensor category is augmented. For A, $A^\natural$ as above $h(A)$ is an associative algebra and $A^\natural$ is an $h(A)$-bimodule (see 1.2.8). The A-actions also yield morphisms $A \otimes h(A^\natural) \to A^\natural$, $h(A^\natural) \otimes A \to A^\natural$ which are morphisms of, respectively, $(A, h(A))$- and $(h(A), A)$-bimodules.

If $z^\natural \in h(A)$ is such that $z := \pi(z^\natural) \in h(A)$ is an $A^\natural$-central element (i.e., left and right multiplications by z on $A^\natural$ coincide) then the morphism $\partial_{z^\natural} : A \to A_0^\natural$, $a \mapsto az^\natural - z^\natural a$, is a derivation. If $\mathrm{tr} : A_0^\natural \to F$ is an A^{Lie}-invariant morphism then the composition $\mathrm{tr}\partial_{z^\natural} : A \to F$ depends only on z, so we denote it by $\mathrm{tr}\partial_z$. It is clear that $\mathrm{tr}\partial_z$ kills the commutant $[A, A]$.

Let us assume that A is a unital associative algera (see 1.2.8), $A^\natural$ is a unital bimodule, and $1 = 1_A \in h(A)$ lifts to $h(A^\natural)$. We get a canonical morphism $\mathrm{tr}\partial_1 : A \to F$.

Lemma. One has $\mathrm{tr}(a, b) = \mathrm{tr}\partial_1(ab)$. Therefore tr is central if and only if $\mathrm{tr}\partial_1 = 0$.

Proof. Choose $\tilde{1} \in h(A^\natural)$. We have two sections $A \to A^\natural$, $a \mapsto 1^\natural a$, $a1^\natural$. One has $(1^\natural a, b1^\natural) = ab1^\natural - 1^\natural ab$. Applying tr we get the lemma. □

In particular, if A^{Lie} is perfect, i.e., $[A, A] = A$, then every tr is central.

2.6.2 The Tate extension in $\mathcal{D}$-module setting. Let V, V' be $\mathcal{D}$-modules on X, $\langle\ \ \rangle \in P_2^*(\{V, V'\}, \omega_X) = P_2^*(\{V', V\}, \omega_X)$ a $*$ pairing. By 1.4.2 the $\mathcal{D}$-module $\mathcal{G} := V \otimes V'$ is an associative* algebra which acts on V, V'. We have the corresponding Lie* algebra $\mathcal{G}^{Lie}$; the pairing is $\mathcal{G}^{Lie}$-invariant.

The Tate extension is a canonical central extension of Lie* algebras

$$0 \to \omega_X \to \mathcal{G}^\flat \xrightarrow{\pi^\flat} \mathcal{G}^{Lie} \to 0. \tag{2.6.2.1}$$

To define $\mathcal{G}^\flat$ consider the exact sequence[88] of $\mathcal{D}$-modules on $X \times X$

$$V \boxtimes V' \xrightarrow{\varepsilon} j_* j^* (V \boxtimes V') \xrightarrow{\pi} \Delta_* (V \otimes V') \to 0. \tag{2.6.2.2}$$

Here $\Delta: X \hookrightarrow X \times X$ is the diagonal embedding, $j: U := X \times X \smallsetminus \Delta(X) \hookrightarrow X \times X$ is the complementary embedding, and π is the canonical arrow. Namely, according to (2.2.2.1), one has $V \otimes V' = H^1 \Delta^! (V \boxtimes V') = \operatorname{Coker} \varepsilon$; explicitly, π sends $(t_2 - t_1)^{-1} v \boxtimes v' \in j_* j^* V \boxtimes V'$ to $v \otimes v' (dt)^{-1} \in \Delta_\cdot (V \otimes V') \subset \Delta_* (V \otimes V')$. Note that $\langle \quad \rangle$ vanishes on $\operatorname{Ker} \varepsilon$.[89] So, pushing out the above exact sequence by $\langle \quad \rangle$, we get an extension of $\Delta_*(V \otimes V')$ by $\Delta_* \omega_X$. This extension is supported on the diagonal; applying $\Delta^!$ we get the Tate extension $\mathcal{G}^\flat$. We denote the canonical morphism $j_* j^* (V \boxtimes V') \to \Delta_* \mathcal{G}^\flat$ by $\mu = \mu_{\mathcal{G}}$.

Let us define now the Lie* bracket $[\quad]_{\mathcal{G}^\flat}$ on $\mathcal{G}^\flat$.

Let L be a Lie* algebra that acts on V and V' preserving the pairing. Then the sheaf of Lie algebras $h(L)$ acts on $\mathcal{G}$. This action lifts canonically to an action on $\mathcal{G}^\flat$. Indeed, global sections of $h(L)$ act on V, V' preserving the $*$ pairing, hence they act on $j_* j^* (V \boxtimes V')$ and $\mathcal{G}^\flat$ by functoriality, and everything is compatible with localization of X. In particular, the canonical action of $\mathcal{G}$ on V, V' yields a canonical action of $h(\mathcal{G}^{Lie})$ on $\mathcal{G}^\flat$.

2.6.3 Proposition. There is a unique Lie* algebra structure on $\mathcal{G}^\flat$ such that the adjoint action of $h(\mathcal{G}^\flat)$ on $\mathcal{G}^\flat$ is the above action composed with $h(\mathcal{G}^\flat) \to h(\mathcal{G}^{Lie})$. The projection $\mathcal{G}^\flat \to \mathcal{G}^{Lie}$ is a morphism of Lie* algebras, so $\mathcal{G}^\flat$ is a central extension of $\mathcal{G}^{Lie}$. This extension satisfies the cyclic property (see 2.6.1(iii)).

Proof. The uniqueness is clear, as well as the latter property (notice that the commutator on $\mathcal{G}^\flat$ comes from the action of $\mathcal{G}^{Lie}$ on $\mathcal{G}^\flat$ that lifts the adjoint action).

Let us show that our picture fits into the format of 2.6.1(iii); this implies the existence statement. The cyclic property follows then from the lemma in 2.6.1(iii).

[88] It is also exact from the left if V, V' have no $\mathcal{O}_X$-torsion.

[89] Indeed, $\operatorname{Ker} \varepsilon = \Delta_* H^{-1} (V \overset{L}{\otimes} V') = \Delta_* \operatorname{Tor}_1^{\mathcal{O}_X} (V, V' \omega_X^{-1})$, hence its sections have finite supports.

Consider the ordered set of all submodules $N_\alpha \subset j_*j^*V \boxtimes V'$ such that $j_*j^*V \boxtimes V'/N_\alpha$ is supported on the diagonal. Set $\mathcal{G}^\natural_\alpha := \Delta^!(j_*j^*V \boxtimes V'/N_\alpha)$ and $\mathcal{G}^\natural := \varprojlim \mathcal{G}^\natural_\alpha$.

The pro$\mathcal{D}_X$-module $\mathcal{G}^\natural$ carries a canonical structure of topological $\mathcal{G}$-bimodule. This means that for every *coherent* $\mathcal{D}_X$-submodule $\mathcal{G}_\beta \subset \mathcal{G}$ we have mutually commuting $*$ pairings (left and right action) in $\cdot^l \in P^*_2(\{\mathcal{G}_\beta, \mathcal{G}^\natural\}, \mathcal{G}^\natural)$ and $\cdot^r \in P^*_2(\{\mathcal{G}^\natural, \mathcal{G}_\beta\}, \mathcal{G}^\natural)$ which are compatible with respect to embeddings of $\mathcal{G}_\beta$'s and satisfy the left and right $\mathcal{G}$-action properties.[90]

To define $\cdot^l$ consider the morphism $\cdot_V \boxtimes id_{V'} : \mathcal{G}\boxtimes V \boxtimes V' \to \Delta^{1=2}_*(V\boxtimes V')$. Here $\cdot_V$ is the $*$ action of $\mathcal{G}$ on V and $\Delta^{1=2}$ is the diagonal embedding $(x,y)\mapsto(x,x,y)$. Localizing we get the morphism $\tilde{\cdot}^l : \mathcal{G}\boxtimes j_*j^*(V\boxtimes V') \to \Delta^{1=2}_* j_*j^*(V\boxtimes V')$. It satisfies the following continuity property: for every N_α, $\mathcal{G}_\beta$ as above there exists $N_{\alpha'}$ such that $\tilde{\cdot}^l(\mathcal{G}_\beta \boxtimes N_\alpha) \subset \Delta^{1=2}_*(N_\alpha)$.[91] Now $\cdot^l$ is the completion of $\tilde{\cdot}^l$ with respect to the N_α-topology.

One defines $\cdot^r$ in the similar way using the action of $\mathcal{G}$ on V' instead of V. It is clear that $\cdot^{l,r}$ define on $\mathcal{G}^\natural$ a topological $\mathcal{G}$-bimodule structure.

The projection π from (2.6.2.2) yields the projection $\mathcal{G}^\natural \twoheadrightarrow \mathcal{G}$ which we also denote by π. This is a morphism of (topological) $\mathcal{G}$-bimodules. The $*$ pairing $\langle\quad\rangle$ yields a $\mathcal{G}^{Lie}$-invariant morphism $\mathcal{G}^\natural_0 := \operatorname{Ker}\pi \to \omega_X$ which we denote by tr.

One has[92] $\mathcal{G}^\flat = \mathcal{G}^\natural_{\mathrm{tr}}$, and the $h(\mathcal{G}^{Lie})$-module structure on $\mathcal{G}^\flat$ defined in 2.6.2 comes from the $\mathcal{G}^\flat$-module structure on $\mathcal{G}^\natural_{\mathrm{tr}}$. Therefore, by 2.6.1(i), our proposition follows from the next Lemma:

Lemma. tr is a central morphism.

Proof of Lemma. (a) It suffices to check our Lemma in case $V = \mathcal{D}^n_X$, $V' = \omega\mathcal{D}^n_X$ is its dual. Indeed, our problem is local, so we can choose a surjective morphism $\mathcal{D}^n_X \twoheadrightarrow V$. Then $\langle\quad\rangle$ yields a morphism $V' \to \omega\mathcal{D}^n_X$. Our constructions are functorial with respect

[90] This makes sense since for every two coherent $\mathcal{D}_X$-submodules $\mathcal{G}_\beta$, $\mathcal{G}_{\beta'}$ of $\mathcal{G}$ their product (i.e., the image in $\mathcal{G}$ of $\mathcal{G}_\beta \boxtimes \mathcal{G}_{\beta'}$ by the product morphism) is coherent.

[91] Let g_i be a finite set of generators of $\mathcal{G}_\beta$, so we have the correponding classes $\bar{g}_i \in h(\mathcal{G}_\beta)$ and endomorphisms $\bar{g}_i\tilde{\cdot}^l$ of $j_*j^*V\boxtimes V'$. Our $N_{\alpha'}$ is the intersection of preimages of N_α by these endomorphisms.

[92] We use notation of 2.6.1(i); the material of 2.6.1 renders to the setting of topological modules in the obvious way.

to morphisms of $(V, V', \langle\ \ \rangle)$'s. Looking at $(V, V') \twoheadleftarrow (\mathcal{D}_X^n, V') \to (\mathcal{D}_X^n, \omega\mathcal{D}_X^n)$ we get the desired reduction.

(b) For such V, V' our $\mathcal{G}$ is the unital associative algebra and $\mathcal{G}^\natural$ is a unital $\mathcal{G}$-module. One has $\mathcal{G}^{Lie} = \mathfrak{gl}(\mathcal{D}_X^n)$ (see 2.5.7(a)), so, by 2.6.1(iv), it remains to check the following fact:

Sublemma. The Lie* algebra $\mathfrak{gl}(\mathcal{D}_X^n)$ is perfect.

Proof of Sublemma. It suffices to consider the case $n = 1$. One has $\mathfrak{gl}(\mathcal{D}_X) = D_{\mathcal{D}}^{Lie}$ where D^{Lie} is the sheaf $D = \mathcal{D}_X$ of differential operators considered as a Lie algebra in $\mathcal{D}iff(X)^*$, see 2.5.7(b)(iii). We want to show that the adjoint action of $D^{Lie} = h(D_{\mathcal{D}}^{Lie})$ on $D_{\mathcal{D}}$ has trivial coinvariants. In fact, the action of the subalgebra $\Theta_X \subset D^{Lie}$ of vector fields has trivial coinvariants. To see this notice that this action preserves the filtration by the degree of the differential operator, and $\mathrm{gr} D_{\mathcal{D}} = \underset{n\geq 0}{\oplus} \Theta_{X\mathcal{D}}^{\otimes n}$. Let us show that the coinvariants of the Θ_X action on $\Theta_{\mathcal{D}}^{\otimes n}$ are trivial for $n \neq -1$. Since we deal with a coherent $\mathcal{D}$-module (see Lemma(i) in 2.1.6) this amounts to the equality $[\Theta_X, \Theta_X^{\otimes n}] = \Theta_X^{\otimes n}$. If $v \in \Theta_X$, $\eta \in \omega_X$ then $[v, v^{n+1}\eta] = v^{n+1}[v,\eta] = v^{n+1} d(v,\eta)$. Such sections for $v(x) \neq 0$ generate the stalk of $\Theta_X^{\otimes n}$ at x if $n \neq -1$. □

2.6.4. If V is a $\mathcal{D}_X$-vector bundle (see 2.2.13), V' its dual, and $\langle\ \ \rangle$ is the canonical pairing, then $\mathcal{G}^{Lie} = \mathfrak{gl}(V)$ (see 2.5.7(a)). Therefore we have a canonical central extension $\mathfrak{gl}(V)^\flat$ of $\mathfrak{gl}(V)$ by ω_X. Passing to homology, we get a central extension $\bar{\mathfrak{gl}}(V)^\flat$ of the sheaf of Lie algebras $\bar{\mathfrak{gl}}(V) = h(\mathfrak{gl}(V))$ (which is the sheaf of endomorphisms of V considered as a Lie algebra) by $h(\omega_X)$.

The Tate extension satisfies an important additivity property. Namely, let $\mathcal{F}$ be a finite flag $V_1 \subset .. \subset V_n = V$ such that every $\mathrm{gr}_i V$ is a $\mathcal{D}_X$-vector bundle. Let $\mathfrak{gl}(V)_{\mathcal{F}} \subset \mathfrak{gl}(V)$ be the corresponding parabolic Lie* subalgebra, so we have the projections $\mathfrak{gl}(V)_{\mathcal{F}} \to \mathfrak{gl}(\mathrm{gr}_i V)$. Let $\mathfrak{gl}(V)_{\mathcal{F}}^\flat$ be the restriction of the Tate extension $\mathfrak{gl}(V)^\flat$ to $\mathfrak{gl}(V)_{\mathcal{F}}$, and let $\mathfrak{gl}(V)_{\mathcal{F}}^{\flat'}$ be the sum of the pull-backs of the Tate extensions $\mathfrak{gl}(\mathrm{gr}_i V)^\flat$. Now there is a canonical isomorphism of extensions

$$\mathfrak{gl}(V)_{\mathcal{F}}^\flat \xrightarrow{\sim} \mathfrak{gl}(V)_{\mathcal{F}}^{\flat'}. \tag{2.6.4.1}$$

To see this consider the submodule $P := \Sigma V_j \boxtimes (V/V_{j-1})' \subset V \boxtimes V'$; we have the obvious projections $\pi_i : P \to \mathrm{gr}_i V \boxtimes (\mathrm{gr}_i V)'$. Now $\mathfrak{gl}(V)_{\mathcal{F}} = \Sigma V_i \otimes (V/V_{i-1})' \subset V \otimes V'$, i.e., $\mathfrak{gl}(V)_{\mathcal{F}} = \Delta^! P \subset \Delta^!(V \boxtimes V') = V \otimes V'$. So the exact sequence $0 \to \Delta_* \omega_X \to \mathfrak{gl}(V)^\flat_{\mathcal{F}} \to \mathfrak{gl}(V)_{\mathcal{F}} \to 0$ is the push-out of $0 \to P \to j_* j^* P \to \Delta^! P \to 0$ by $f : P \to \Delta_* \omega_X$, where f is the restriction of the canonical pairing $V \boxtimes V' \to \Delta_* \omega_X$. Since f is the sum of the canonical pairings $\mathrm{gr}_i V \boxtimes (\mathrm{gr}_i V)' \to \Delta_* \omega_X$ composed with π_i we get (2.6.4.1). We leave it to the reader to check compatibility with Lie* brackets.

2.6.5. Assume that V is induced, so $V = F_{\mathcal{D}} := F \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$, F is a locally free $\mathcal{O}_X$-module of finite rank. We have $\mathfrak{gl}(V) = \mathrm{Diff}(F,F)_{\mathcal{D}}$ and $\bar{\mathfrak{gl}}(V) = \mathrm{Diff}(F,F)_X$ (see 2.5.7(b)(iii)). Let us the $h(\omega_X)$-extension $\bar{\mathfrak{gl}}(V)^\flat$. The dual $\mathcal{D}$-module V' equals $F^\circ_{\mathcal{D}}$ where $F^\circ = F^* \omega_X$, so $j_* j^*(V \boxtimes V') = (j_* j^* F \boxtimes F^\circ)_{\mathcal{D}}$ and $\Delta_*(V \otimes V') = (j_* j^*(F \boxtimes F^\circ)/F \boxtimes F^\circ)_{\mathcal{D}}$.

Applying h, we see that $\Delta_\cdot \bar{\mathfrak{gl}}(V)^\flat$ is equal to the push-forward of the extension $0 \to F \boxtimes F^\circ \to j_* j^* F \boxtimes F^\circ \to j_* j^* F \boxtimes F^\circ / F \boxtimes F^\circ \to 0$ by the map $F \boxtimes F^\circ \to \Delta_\cdot \omega_X$, $f(x) \boxtimes g(y) \mapsto \langle f(x), g(x) \rangle$. The identification $\Delta^\cdot j_* j^*(F \boxtimes F^o)/F \boxtimes F^o \xrightarrow{\sim} \mathrm{Diff}(F,F)_X$ is the isomorphism (2.5.9.1). The left and right action of $\bar{\mathfrak{gl}}(V) = \mathrm{Diff}(F,F)_X$ on $\Delta^\cdot h(j_* j^*(V \boxtimes V')) = \Delta^\cdot j_* j^*(F \boxtimes F^\circ)$ come from the usual left and right action of $\mathrm{Diff}(F,F)_X$ on F and F^o, respectively; the adjoint action yields our Lie bracket on the quotient $\bar{\mathfrak{gl}}(V)^\flat$. Therefore $\bar{\mathfrak{gl}}(V)^\flat$ coincides with the canonical extension of the Lie algebra $\mathrm{Diff}(F,F)_X$ as defined in, say, [BS] ??.

Notice that, by construction, the embedding $\mathcal{A}(F) \hookrightarrow \mathrm{Diff}(F,F)_X$ lifts to the one of ω_X-extensions $\mathcal{A}(F)^\flat_{\mathcal{D}} \hookrightarrow \mathfrak{gl}(V)^\flat$. In particular, by 2.5.10 (d), $\mathfrak{gl}(\mathcal{D}_X)$ contains the Virasoro extension $\Theta^{2v}_{\mathcal{D}}$.

2.6.6. The construction of Tate's extension renders easily to the case of A^ℓ-modules, $A^\ell \in \mathcal{C}omu^!(X)$. Assume we have $V, V' \in \mathcal{M}(X, A^\ell)$ and $\langle \quad \rangle \in P^*_{A2}(\{V, V'\}, A)$. We have the associative* matrix algebra[93] $V \underset{A}{\otimes} V' \in \mathcal{M}(X, A)$; denote by $\mathcal{G}_A$ the corresponding Lie* algebra. We will define a canonical central extension of Lie* algebras in $\mathcal{M}(X, A)$

$$0 \to A \to \mathcal{G}^\flat_A \to \mathcal{G}_A \to 0. \tag{2.6.6.1}$$

[93] As usual, we write the tensor product of right $A[\mathcal{D}_X]$-modules as $\underset{A}{\otimes}$ instead of $\underset{A}{\otimes}^!$.

Let us construct $\mathcal{G}_A^\flat$ as a plain $A^\ell[\mathcal{D}_X]$-module.

To construct $\mathcal{G}_A^\flat$ we first define an extension of $\mathcal{D}$-modules $0 \to A \to \mathcal{G}_A'^\flat \to V \otimes^! V' \to 0$. Namely, $\Delta_* \mathcal{G}_A'^\flat$ is push-forward of $V \boxtimes V' \to j_* j^* V \boxtimes V' \to \Delta_*(V \otimes^! V') \to 0$ by $\langle \quad \rangle : V \boxtimes V' \to \Delta_* A$. This $\mathcal{G}_A'^\flat$ is an $A \otimes^! A$-module; set $\mathcal{G}_A^\flat = \mathcal{G}_A'^\flat / I \mathcal{G}_A'^\flat$ where $I := \mathrm{Ker}(A \otimes^! A \to A)$. One defines the bracket on $\mathcal{G}_A^\flat$ as in 2.6.2, 2.6.3; again, we refer to 3.8 for a natural Clifford realization of $\mathcal{G}_A^\flat$.

The above construction is compatible with étale localization, so it works in the setting of algebraic $\mathcal{D}_X$-spaces. Namely, let $\mathcal{Y}$ be a algebraic $\mathcal{D}_X$-space, V, V' be $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules equipped with an $\mathcal{O}_\mathcal{Y}$-valued $*$ pairing $\langle \quad \rangle$; assume that $\mathcal{Y}$ is flat over X and $\mathcal{T}or_{\mathcal{O}_\mathcal{Y}}(V, V') = 0$. We have a Lie* algebra $\mathcal{G}_\mathcal{Y} = V \otimes^! V' \in \mathcal{M}_(\mathcal{Y})$ and its Tate (central) extension $\mathcal{G}_\mathcal{Y}^\flat$ by $\mathcal{O}_\mathcal{Y}$. Notice that the construction of $\mathcal{G}_\mathcal{Y}$, $\mathcal{G}_\mathcal{Y}^\flat$ is compatible with base change (with respect to morphisms of $\mathcal{Y}$'s).

Assume now that $\mathcal{Y}$ is a smooth algebraic $\mathcal{D}_X$-space, V is a $\mathcal{D}_X$-vector bundle on $\mathcal{Y}$, V' is its $*$ dual, so $\mathcal{G}_\mathcal{Y} = \mathfrak{gl}(V)$. The algebroid $\mathcal{E}(V)$ (see 2.3.16) acts on V, V'. As in 2.6.2 its action on $\mathfrak{gl}(V)$ lifts canonically to an action on $\mathfrak{gl}(V)^\flat$. On $\mathfrak{gl}(V) \subset \mathcal{E}(V)$ this action coincides with the "adjoint" one.

Remark. One may see that $\mathfrak{gl}(V)$ coincides with its commutant, so every central extension of $\mathfrak{gl}(V)$ has no non-trivial automorphisms.

The meaning of the following definition will become clear in 3.8.18.

2.6.7 Definition. A *Tate structure* on V is the following datum (i), (ii):

(i) An extension $\mathcal{E}(V)^\flat$ of the Lie* algebroid $\mathcal{E}(V)$ on $\mathcal{Y}$ by $\mathcal{O}_\mathcal{Y}$.

The corresponding Lie* $\mathcal{O}_\mathcal{Y}$-algebra $\mathcal{E}(V)^{\flat\circ} = \mathrm{Ker}(\tau)$ is an extension of $\mathfrak{gl}(V)$ by $\mathcal{O}_\mathcal{Y}$.

(ii) An identification of this extension with $\mathfrak{gl}(V)^\flat$ compatible with $\mathcal{E}(V)^\flat$ actions.

Here $\mathcal{E}(V)^\flat$ acts on $\mathcal{E}(V)^{\flat\circ}$ via the adjoint action, and on $\mathfrak{gl}(V)^\flat$ via the canonical action of $\mathcal{E}(V)$.

Tate structures form a groupoid $\mathcal{T}ate(V) = \mathcal{T}ate(\mathcal{Y}, V)$. They have étale local nature, so we have a sheaf of groupoids $\mathcal{T}ate(V)_{\mathcal{Y}_{\text{ét}}}$ on $\mathcal{Y}_{\text{ét}}$.

For a Lie* algebroid L on $\mathcal{Y}$ we denote by $\mathcal{P}^{cl}(L) = \mathcal{P}^{cl}(\mathcal{Y}, L)$[94] the groupoid of Lie* algebroid extensions of L by $\mathcal{O}_{\mathcal{Y}}$. This isa Picard groupoid, or rather a k-vector space in categories, in the obvious way (the sum is the Baer sum operation). The objects of $\mathcal{P}^{cl}(L)$ have étale local nature, so we have a sheaf $\mathcal{P}^{cl}(L)_{\mathcal{Y}_{\text{ét}}}$ of Picard groupoids on $\mathcal{Y}_{\text{ét}}$.

There is a canonical Action of $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$ on $\mathcal{T}ate(V)$. Namely, an object Θ^ν of $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$ amounts, by pull-back, to an extension $\mathcal{E}(V)^\nu$ of $\mathcal{E}(V)$ by $\mathcal{O}_{\mathcal{Y}}$ trivialized over $\mathfrak{gl}(V)$. The corresponding automorphism of $\mathcal{T}ate(V)$ sends $\mathcal{E}(V)^\flat$ to its Baer sum with $\mathcal{E}(V)^\nu$.

2.6.8 Lemma. $\mathcal{T}ate(V)$ is a $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$-Torsor. □

In the rest of this section we present a version of the original Tate's linear algebra construction and relate it then with the above Lie* algebra one.

2.6.9 Tate's linear algebra. Let F be a k-vector space equipped with a separated complete linear[95] topology. A closed vector subspace $L \subset F$ is *compact*, resp. *cocompact*, if for any open vector subspace $U \subset F$ one has $\dim L/(L \cap U) < \infty$, resp. $\dim F/(L+U) < \infty$. A *c-lattice* is an open compact subspace; dually, a *d-lattice* is a discrete cocompact subspace.

It is easy to see that F has a c-lattice if and only if it has a d-lattice. Such F is called *Tate vector space.*[96] For a Tate vector space F its dual[97] F^* is again a Tate space; the canonical map $F \to F^{**}$ is an isomorphism.

Any discrete vector space[98] is a Tate space. Duality interchanges discrete and compact Tate spaces. Any Tate vector space can be represented as a direct sum of a discrete and a compact space.

A morphism of Tate vector spaces $A : F \to G$ is *compact* if the closure of $\mathrm{Im}A$ is compact; it is *discrete* if $\mathrm{Ker}A$ is open. The composition of a compact operator with

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[94] Here the superscript "cl" means "classical", see 3.8.8 for an explanation.

[95] i.e., the topology has a base formed by vector subspaces of F.

[96] Tate vector spaces, as well as their key attribute – the Tate extension of $\mathfrak{gl}$ – appear in [T] implicitly. In fact, they were introduced by Lefschetz ([Lef], p. 78–79) under the name of locally linearly compact spaces and used by Chevalley (these authors did not conceive, however, the Tate extension). So a proper name would be LCT vector space (which may be also understood as "locally compact topological").

[97] := the space of all continuous linear functionals $F \to k$; open subspaces in F^* are orthogonal complements to compact subspaces of F.

[98] i.e., a vector space equipped with a discrete topology.

an arbitrary one is compact; same for discrete operators. A is compact if and only if $A^* : G^* \to F^*$ is discrete. We denote by $\mathrm{Hom}_+(F,G), \mathrm{Hom}_-(F,G) \subset \mathrm{Hom}(F,G)$ the subspaces of compact, resp. discrete, operators, and by $\mathrm{Hom}_0(F,G)$ the intersection of these subspaces. Any operator may be represented as sum of a compact and a discrete operator.

The above vector spaces of operators carry natural topologies. Notation: for subspaces $U \subset F$ and $V \subset G$ set $\mathrm{Hom}(F,G)_{U,V} := \{A \in \mathrm{Hom}(F,G) : A(U) \subset V\}$. Now bases of topologies on $\mathrm{Hom}(F,G)$, $\mathrm{Hom}_+(F,G)$, $\mathrm{Hom}_-(F,G)$, and $\mathrm{Hom}_0(F,G)$ are formed, respectively, by subspaces $\mathrm{Hom}(F,G)_{U,V}$, $\mathrm{Hom}(F,V)$, $\mathrm{Hom}(F/U,G)$, and $\mathrm{Hom}(F/U,V)$ where $U \subset F$ and $V \subset G$ are c-lattices. These topologies are complete and separated. The exact sequence

$$0 \to \mathrm{Hom}_0(F,G) \xrightarrow{\alpha} \mathrm{Hom}_+(F,G) \oplus \mathrm{Hom}_-(F,G) \xrightarrow{\beta} \mathrm{Hom}(F,G) \to 0, \tag{2.6.9.1}$$

where $\alpha(A) := (-A, A)$, $\beta(A,B) := A + B$, is strongly compatible with the topologies.[99]

Notice that $\mathrm{Hom}_\pm$ and Hom_0 are two-sided ideals in the obvious sense, and all the composition maps like $\mathrm{Hom}_-(G,H) \times \mathrm{Hom}_+(F,G) \to \mathrm{Hom}_0(F,H)$, etc., are continuous. The maps $\mathrm{Hom}(F,G) \xrightarrow{\sim} \mathrm{Hom}(G^*,F^*)$, $\mathrm{Hom}(F,G)_+ \xrightarrow{\sim} \mathrm{Hom}(G^*,F^*)_-$, etc., which send A to A^*, are homeomorphisms.

Exercise. Consider an (abstract) vector space $G \otimes F$. It carries 4 natural topologies with bases of opens formed, respectively, by subspaces $V \otimes F + G \otimes U$, $V \otimes F$, $G \otimes U$, and $V \otimes U$, where $V \subset G$, $U \subset F$ are c-lattices. Now replace F by F^* and consider the obvious canonical map $G \otimes F^* \to \mathrm{Hom}_0(F,G)$. Show that it yields homeomorphisms between the completions of $G \otimes F^*$ with respect to these topologies and, respectively, $\mathrm{Hom}(F,G)$, $\mathrm{Hom}_+(F,G)$, $\mathrm{Hom}_-(F,G)$, and $\mathrm{Hom}_0(F,G)$.

So for a Tate space F we have an associative algebra $\mathrm{End}F$ together with two-sided ideals $\mathrm{End}_0F, \mathrm{End}_\pm F \subset \mathrm{End}F$ such that $\mathrm{End}_+F + \mathrm{End}_-F = \mathrm{End}F$. An operator A belongs to $\mathrm{End}_0F := \mathrm{End}_+F \cap \mathrm{End}_-F$ if and only if there are c-lattices $V \subset U$ such

[99] i.e., α is a homeomorphism onto a closed subspace, β is continuous and open.

that $A(V) = 0$, $A(F) \subset U$. For such A its *trace* $\mathrm{tr}A$ is the trace of the induced operator on U/V; it does not depend on the auxiliary choice of lattices. For any $A_\pm \in \mathrm{End}_\pm F$ one has $\mathrm{tr}([A_+, A_-]) = 0$. Since $\mathrm{End}F = \mathrm{End}_+F + \mathrm{End}_-F$ one has $\mathrm{tr}[A,B] = 0$ for any $A \in \mathrm{End}_0 F$, $B \in \mathrm{End}F$. The functional tr is continuous with respect to our topology on $\mathrm{End}_0 F$. In terms of the above exercise, tr is simply the continuous extension of the canonical pairing $F \otimes F^* \to k$.

2.6.10 The Tate extension in the linear algebra setting. For a Tate vector space F let $\mathfrak{gl}(F)$ be $\mathrm{End}F$ considered as a Lie algebra. By 2.6.9 we have ideals $\mathfrak{gl}_0(F) \subset \mathfrak{gl}_\pm(F) \subset \mathfrak{gl}(F)$ and $\mathrm{tr} : \mathfrak{gl}_0(F) \to k$ that fit into the setting of 2.6.1(ii). The corresponding central extension $\mathfrak{gl}(F)^\flat$ is called the *Tate extension*. It satisfies the cyclic property (see 2.6.1(iii)). We have canonical sections $s_\pm = s_\pm^F : \mathfrak{gl}_\pm(F) \to \mathfrak{gl}(F)^\flat$.

According to 2.6.9 the Lie algebras $\mathfrak{gl}(F)$, $\mathfrak{gl}_\pm(F)$, $\mathfrak{gl}_0(F)$ carry canonical topologies, and the functional tr is continuous. Our $\mathfrak{gl}(F)^\flat$ is also a topological Lie algebra: the topology is defined by the condition that the map $s_+ + s_- : \mathfrak{gl}_+(F) \times \mathfrak{gl}_-(F) \to \mathfrak{gl}(F)^\flat$ is continuous and open. It is complete and separated; the projection $\mathfrak{gl}(F)^\flat \to \mathfrak{gl}(F)$ is continuous and open.

Remark. The identification $\mathfrak{gl}(F) \xrightarrow{\sim} \mathfrak{gl}(F^*)$, $A \mapsto -A^*$, lifts canonically to an identification of topological k-extensions $\mathfrak{gl}(F)^\flat \xrightarrow{\sim} \mathfrak{gl}(F^*)^\flat$ which interchanges s_+ and s_-.

The Tate extension satisfies an important additivity property. Let $\mathcal{F}$ be a finite flag $F_1 \subset F_2 \subset \dots \subset F_n = F$ of closed subspaces of F.[100] Then F_i are Tate vector spaces, as well as $\mathrm{gr}_i F$. Let $\mathfrak{gl}(F)_{\mathcal{F}} \subset \mathfrak{gl}(F)$ be the stabilizer of our flag, so one has the projections $\pi_i : \mathfrak{gl}(F)_{\mathcal{F}} \to \mathfrak{gl}(\mathrm{gr}_i F)$. There are two natural k-extensions $\mathfrak{gl}(F)^\flat_{\mathcal{F}}$ and $\mathfrak{gl}(F)^{\flat\prime}_{\mathcal{F}}$ of $\mathfrak{gl}(F)_{\mathcal{F}}$: the first one is the restriction of $\mathfrak{gl}(F)^\flat$ to $\mathfrak{gl}(F)_{\mathcal{F}}$, the second one is the Baer sum of the π_i-pull-backs of the extensions $\mathfrak{gl}(\mathrm{gr}_i F)^\flat$.

2.6.11 Lemma. The topological central extensions $\mathfrak{gl}(F)^\flat_{\mathcal{F}}$, $\mathfrak{gl}(F)^{\flat\prime}_{\mathcal{F}}$ are canonically isomorphic.

[100] Any such flag splits, i.e., it comes from a grading.

Proof. Consider the ideals $\mathfrak{gl}_\pm(F)_{\mathcal{F}} := \mathfrak{gl}(F)_{\mathcal{F}} \cap \mathfrak{gl}_\pm(F)$; one has $\mathfrak{gl}_+(F)_{\mathcal{F}} + \mathfrak{gl}_-(F)_{\mathcal{F}} = \mathfrak{gl}(F)_{\mathcal{F}}$ and $\pi_i(\mathfrak{gl}_\pm(F)_{\mathcal{F}}) \subset \mathfrak{gl}_\pm(\mathrm{gr}_i F)$. Both $\mathfrak{gl}(F)^\flat_{\mathcal{F}}$ and $\mathfrak{gl}(F)^{\flat'}_{\mathcal{F}}$ are equipped with sections over $\mathfrak{gl}_\pm(F)_{\mathcal{F}}$: these are, respectively, $s_\pm := s^F_\pm$ restricted to $\mathfrak{gl}_\pm(F)_{\mathcal{F}}$ and $s'_\pm := \Sigma\pi_i^*(s_\pm^{\mathrm{gr}_i F})$. Over $\mathfrak{gl}_0(F)_{\mathcal{F}} := \mathfrak{gl}_+(F)_{\mathcal{F}} \cap \mathfrak{gl}_-(F)_{\mathcal{F}}$ one has $s_+ - s_- = s'_+ - s'_- = \mathrm{tr}$. Now our lemma follows from the uniqueness statement of 2.6.1(ii). □

Remark. Assume that in the situation of the above lemma every $\mathrm{gr}_i F$ is either compact or discrete. Then the corresponding sections $s_\pm^{\mathrm{gr}_i F}$ (together with the lemma) provide a splitting of $\mathfrak{gl}(F)^\flat_{\mathcal{F}}$. For example, if $\mathcal{F}$ consists of a single c-lattice $P \subset F$ then s^P_+ and $s_-^{F/P}$ yield a canonical splitting s_P of $\mathfrak{gl}(F)^\flat_P$. If $P' \subset P$ is another c-lattice then on $\mathfrak{gl}(F)_P \cap \mathfrak{gl}(F)_{P'}$ one has $s_P - s_{P'} = \mathrm{tr}_{P/P'}$.

2.6.12. We return to the geometric setting of 2.6.2. Let $x \in X$ be a (closed) point, $j_x : U_x \hookrightarrow X$ the complement to x, $O_x \subset K_x$ is the completion of the local ring at x and its field of fractions. Let V be a coherent $\mathcal{D}$-module on U_x. Set $V_{(x)} := \hat{h}_x(j_{x*}V)$ (see 2.1.13). This is a Tate vector space (see Remark in 2.1.14). Applying 2.1.14(ii) to $M = j_{x*}V$ we get a canonical isomorphism of topological vector spaces

$$V \underset{\mathcal{D}_{U_x}}{\otimes} K_x \xrightarrow{\sim} V_{(x)}. \tag{2.6.12.1}$$

Here $V \underset{\mathcal{D}_{U_x}}{\otimes} K_x := \Gamma(U_x, V) \underset{\Gamma(U_x,\mathcal{D})}{\otimes} K_x = V_\eta \underset{\mathcal{D}_\eta}{\otimes} K_x$ (where η is the generic point of X) and its topology is formed by the images of $V_\xi \underset{\mathcal{D}_X}{\otimes} O_x$, $V_\xi \in \Xi_x(j_{x*}V)$ (see 2.1.14(i)).

We assume from now on that V is a $\mathcal{D}$-vector bundle on U_x;[101] let V' be its dual. The $*$ pairing $j_{x*}V \boxtimes j_{x*}V' \to \Delta_* j_{x*}\omega_{U_x}$ composed with the canonical projection $j_{x*}\omega_{U_x} \twoheadrightarrow i_{x*}k$ yields a canonical k-valued pairing between the Tate vector spaces $V_{(x)}$ and $V'_{(x)}$. We claim that this pairing is non-degenerate, so $V_{(x)}$ and $V'_{(x)}$ are mutually dual Tate vector spaces. To prove this it is enough to consider the following example:

Example. If $V = \mathcal{D}_{U_x}$ then $V' = \omega_{U_x} \otimes \mathcal{D}_{U_x}$. One has $V_{(x)} = K_x$, $V'_{(x)} = \omega_{K_x}$, and the above pairing is $f, \nu \mapsto \mathrm{Res}_x(f\nu)$.

[101] In fact, V becomes a free $\mathcal{D}$-module after replacing X by some neighbourhood of x. This follows from the fact that every ideal of $\mathcal{D}_\eta$ is principal (here η is the generic point of X).

2.6.13. Consider now the Lie* algebra $j_{x*}\mathfrak{gl}(V)$. It acts on $j_{x*}V$ and $j_{x*}V'$.

Assume we have $V_\xi \in \Xi_x(j_{x*}V)$, $V'_{\xi'} \in \Xi_x(j_{x*}V')$ such that the canonical $*$ pairing sends $V_\xi \boxtimes V'_{\xi'}$ to $\Delta_*\omega_X \subset \Delta_* j_{x*}\omega_{U_x}$. Then the matrix algebra $V_\xi \otimes V'_{\xi'}$ is a Lie* subalgebra of $j_{x*}\mathfrak{gl}(V)$ which equals to $\mathfrak{gl}(V)$ on U_x. We call such a subalgebra *special.* Special Lie* subalgebras form a base of a topology Ξ_x^{sp} on $j_{x*}\mathfrak{gl}(V)$ at x (called *the special topology*) which is weaker than the topology Ξ_x^{Lie} (see 2.5.12).[102] It is clear that the action of $j_{x*}\mathfrak{gl}(V)$ on $j_{x*}V$ is continuous with respect to the Ξ_x^{sp}- and Ξ_x-topologies.[103]

Denote by $\mathfrak{gl}(V)_{(x)}$ the completion of the Lie algebra $h_x(j_{x*}\mathfrak{gl}(V))$ with respect to Ξ_x^{sp}-topology (see 2.5.12 and Remark in 2.5.13). This is a topological Lie algebra[104] which acts continuously on the Tate vector space $V_{(x)}$.

2.6.14 Lemma. This action yields an isomorphism of topological Lie algebras

$$r:\ \mathfrak{gl}(V)_{(x)} \xrightarrow{\sim} \mathfrak{gl}(V_{(x)}). \tag{2.6.14.1}$$

Proof. It suffices to consider the case $V = \mathcal{D}_{U_x}$. Then $h_x(j_{x*}\mathfrak{gl}(V))$ is the vector space $\mathcal{D}_\eta$ of all meromorphic differential operators. The special topology has a base formed by the subspaces $\mathcal{D}_{m,n} := \{\partial : \partial(t^{-m}\mathcal{O}_x) \subset t^n\mathcal{O}_x\}$; here t is a parameter at x. This is exactly the topology induced from the topology on $\mathfrak{gl}(V_{(x)})$ (see 2.6.10, 2.6.9). It remains to show that $\mathcal{D}_\eta/\mathcal{D}_{m,n} \xrightarrow{\sim} \mathrm{Hom}(t^{-m}O_x, K_x/t^nO_x)$ (where Hom means continuous linear maps); this is a refreshing exercise for the reader. $\square$

2.6.15. Now consider the central extension $j_{x*}\mathfrak{gl}(V)^\flat$ of $j_{x*}\mathfrak{gl}(V)$ by $j_{x*}\omega_{U_x}$. Notice that for any special subalgebra $\mathcal{G} = V_\xi \otimes V'_{\xi'} \subset j_{x*}\mathfrak{gl}(V)$ (see 2.6.13) its Tate extension[105] $\mathcal{G}^\flat$ is a Lie* subalgebra of $j_{x*}\mathfrak{gl}(V)^\flat$. We call such Lie* subalgebra of $j_{x*}\mathfrak{gl}(V)^\flat$ *special.* Special subalgebras form a base of a topology $\Xi_x^{sp\flat}$ on $j_{x*}\mathfrak{gl}(V)^\flat$ at x (called *the special topology*) which is weaker then the Ξ_x^{Lie}-topology (see 2.5.12).

[102] In fact, as follows from 2.5.14 and 2.6.14 the special topology is stricty weaker than Ξ_x^{Lie}.

[103] In fact, Ξ_x^{sp} is the weakest topology on $j_{x*}\mathfrak{gl}(V)$ such that the $j_{x*}\mathfrak{gl}(V)$-action on $j_{x*}V$ equipped with Ξ_x-topology is continuous.

[104] In fact, this is a topological associative algebra.

[105] With respect to the $*$ pairing $\in P_2^*(\{V_\xi, V'_{\xi'}\}, \omega_X)$.

Let $\mathfrak{gl}(V)^\flat_{(x)}$ be the completion of $h_x(j_{x*}\mathfrak{gl}(V)^\flat)$ with respect to $\Xi_x^{sp\flat}$-topology. The adjoint action of $j_{x*}\mathfrak{gl}(V)$ on $j_{x*}\mathfrak{gl}(V)^\flat$ is continuous with respect to special topologies, so $\mathfrak{gl}(V)^\flat_{(x)}$ is a topological Lie algebra. For every special subalgebra $\mathcal{G}^\flat \subset j_{x*}\mathfrak{gl}(V)^\flat$ we have $\mathcal{G}^\flat \cap j_{x*}\omega_{U_x} = \omega_X$, so the identification $\mathrm{Res}_x : j_{x*}\omega_{U_x}/\omega_X \xrightarrow{\sim} k$ tells us that $\mathfrak{gl}(V)^\flat_{(x)}$ is a topological central extension of $\mathfrak{gl}(V)_{(x)}$ by k.

2.6.16 Proposition. The isomorphism r of 2.6.14 lifts canonically to an isomorphism of topological central extensions

(2.6.16.1) $$r^\flat : \mathfrak{gl}(V)^\flat_{(x)} \xrightarrow{\sim} \mathfrak{gl}(V_{(x)})^\flat.$$

Notice that such lifting $r^\flat$ is unique since the extension $\mathfrak{gl}(V_{(x)})^\flat$ has no automorphisms.

We give two constructions of $r^\flat$. The one right below is a version of considerations from [BS]. A more conceptual and simple approach uses chiral Clifford algebras; it can be found in 3.8.23.

Proof. (i) Let us compare the constructions of Tate extensions:

(a) Let η be the generic point of X. Apply the functor h to (2.6.2.2) and consider sections of the resulting sheaves over $\eta \times \eta$. We get an extension of $\bar{\mathfrak{gl}}(V)_\eta$-modules[106]

(2.6.16.2) $$0 \to h(V)_\eta \otimes h(V')_\eta \to \bar{\mathfrak{gl}}(V)^\natural_\eta \to \bar{\mathfrak{gl}}(V)_\eta \to 0.$$

Consider the push-forward of this extension by the map $h(V)_\eta \otimes h(V')_\eta \xrightarrow{(\;)} h(\omega)_\eta \xrightarrow{\mathrm{Res}_x} k$ (see 2.6.2). This is a central extension of $\bar{\mathfrak{gl}}(V)_\eta$ by k; the bracket comes from the $\bar{\mathfrak{gl}}(V)_\eta$-module structure on $\bar{\mathfrak{gl}}(V)^\natural_\eta$. Our $\mathfrak{gl}(V)^\flat_{(x)}$ is the completion of this Lie algebra with respect to the topology defined by the images of vector spaces $h_{(x,x)}(j_*j^*V_\xi \boxtimes V'_{\xi'})$ where V_ξ,$V'_{\xi'}$ are as in 2.6.13.

(b) Look at (2.6.9.1) for $F = G = V_{(x)}$. This is an extension

(2.6.16.3) $$0 \to \mathfrak{gl}_0(V_{(x)}) \xrightarrow{\alpha} \mathfrak{gl}(V_{(x)})^\natural \xrightarrow{\beta} \mathfrak{gl}(V_{(x)}) \to 0$$

of topological $\mathfrak{gl}(V_{(x)})$-modules; here $\mathfrak{gl}(V_{(x)})^\natural := \mathfrak{gl}_+(V_{(x)}) \oplus \mathfrak{gl}_-(V_{(x)})$ and $\alpha = (+,-)$, $\beta = (+,+)$. Now $\mathfrak{gl}(V_{(x)})^\flat$ is the push-forward of this extension by $\mathrm{tr} : \mathfrak{gl}_0(V_{(x)}) \to k$; the bracket comes from the $\mathfrak{gl}(V_{(x)})$-module structure on $\mathfrak{gl}(V_{(x)})^\natural$.

[106] We use the notation from 2.6.4.

(ii) The action of $\bar{\mathfrak{gl}}(V)_\eta$ on the completion $V_{(x)}$ of $h(V)_\eta$ yields the morphism $r : \bar{\mathfrak{gl}}(V)_\eta \to \mathfrak{gl}(V_{(x)})$ of Lie algebras. We will lift it to a continuous morphism of extensions

$$r^\natural : \bar{\mathfrak{gl}}(V)^\natural_\eta \to \mathfrak{gl}(V_{(x)})^\natural \tag{2.6.16.4}$$

such that $r^\natural$ is an r-morphism of $\mathfrak{gl}(V)_\eta$-modules and its restriction to $h(V)_\eta \otimes h(V')_\eta$ equals the obvious morphism $h(V)_\eta \otimes h(V')_\eta \to V_{(x)} \otimes V'_{(x)} \to \mathfrak{gl}_0(V_{(x)})$. It is clear from (a), (b) above[107] that such $r^\natural$ yields the desired $r^\flat$.

(iii) Take $u^\natural \in \bar{\mathfrak{gl}}(V)^\natural_\eta = hj_*j^*(V \boxtimes V')_{\eta\times\eta}$; let $u \in \bar{\mathfrak{gl}}(V)_\eta$ be the image of $u^\natural$. To define $r^\natural(u^\natural) \in \mathfrak{gl}(V_{(x)})^\natural = \mathfrak{gl}_+(V_{(x)}) \oplus \mathfrak{gl}_-(V_{(x)})$ we must represent $r(u) \in \mathfrak{gl}(V_{(x)})$ as $r_+(u^\natural) + r_-(u^\natural)$, $r_\pm(u^\natural) \in \mathfrak{gl}_\pm(V_{(x)})$. Notice that $r(u) : h(V)_\eta \to h(V)_\eta$ equals $R_\Delta \circ f_{u^\natural}$ where $f_{u^\natural} : h(V)_\eta \to hj_*j^*(V \boxtimes \omega_X)_{\eta\times\eta}$ maps $s \in h(V)_\eta$ to $(id_V \boxtimes s)(u^\natural)$ (here we consider s as a morphism of $\mathcal{D}_X$-modules $V' \to \omega_X$ at η) and $R_\Delta : hj_*j^*(V \boxtimes \omega_X)_{\eta\times\eta} \to h(V)_\eta$ is the residue at $\eta \times x$ along the second variable. We have $R_\Delta = -R_x + \int_\eta$ where $\int_\eta$ denotes integration over the second variable and R_x is the residue at $\eta \times x$ along the second variable. It remains to prove the following lemma which is left to the reader:

Lemma. (a) The map $\int_\eta \circ f_{u^\natural} : h(V)_\eta \to h(V)_\eta$ extends to a compact operator $r^\natural_+(u^\natural) : V_{(x)} \to V_{(x)}$.

(b) The map $-R_x \circ f_{u^\natural} : h(V)_\eta \to h(V)_\eta$ extends to a discrete operator $r^\natural_-(u^\natural) : V_{(x)} \to V_{(x)}$.

(c) The map $r^\natural : \bar{\mathfrak{gl}}(V)^\natural_\eta \to \mathfrak{gl}(V_{(x)})^\natural$ defined by $r^\natural := (r^\natural_+, r^\natural_-)$ satisfies the properties promised in (ii). $\square$

§7 Coisson algebras

2.7.1. According to 1.4.15 we have the category $\mathrm{Cois}(X) = \mathrm{Cois}(\mathcal{M}^{*!}(X))$ of coisson algebras on X. Explicitly, a coisson algebra is a commutative $\mathcal{D}_X$-algebra A^ℓ together with a Lie* bracket $\{\ \ \}$ on A (the coisson bracket) such that for any local sections $a, b, c \in A^\ell$ and $\nu \in \omega_X$ one has $\{ab\nu, c\nu\} = p_1^*a \cdot \{b\nu, c\nu\} + p_1^*b \cdot \{a\nu, c\nu\} \in \Delta_*A$ (we use the left $p_1^*A^\ell$-module structure on Δ_*A).

[107] Use the interpretation of tr at the end of 2.6.9.

The coisson brackets on a $\mathcal{D}_X$-algebra A^ℓ are étale local objects. Therefore we have the notion of coisson structure on any algebraic $\mathcal{D}_X$-space.

Let $x \in X$ be a (closed) point, $j_x : U_x := X \smallsetminus \{x\} \hookrightarrow X$ the complementary open. Let A^ℓ be a coisson algebra on U_x. Consider the commutative topological algebra A_x^{as} defined in 2.4.8. Recall that A_x^{as} is the completion of the vector space $h(j_{x*}A)_x$ with respect to the Ξ_x^{as}-topology. The coisson bracket defines a Lie algebra structure on $h(j_{x*}A)_x$. The following lemma, as well as its proof, is parallel to 2.5.11:

2.7.2 Lemma. (i) The Lie bracket is Ξ_x^{as}-continuous with respect to each variable.

(ii) If A^ℓ is a countably generated $\mathcal{D}_{U_x}$-algebra then the Lie bracket is Ξ_x^{as}-continuous. Its continuous extension to A_x^{as} is a Poisson bracket.

Proof. (i) We want to show that for every $\bar{l} \in h(j_{x*}A)$ the operator $ad_{\bar{l}}$ is Ξ_x^{as}-continuous. By 2.2.17(i) it is Ξ_x-continuous, hence for every $A_\xi \in \Xi_x^{as}(A)$ its preimage $ad_{\bar{l}}^{-1}(A_\xi)$ is Ξ_x-open. Since $ad_{\bar{l}}$ is a derivation, $ad_{\bar{l}}^{-1}(A_\xi) \cap A_\xi \in \Xi_x^{as}$, q.e.d.

(ii) We will find for every $A_\xi \in \Xi_x^{as}(A)$ some $A_\zeta \in \Xi_x^{as}(A)$ such that $\{A_\zeta, A_\zeta\} \subset A_\xi$. Together with (i) it proves the Ξ_x^{as}-continuity of the bracket. Let $L \subset A_\xi$ be a countably generated $\mathcal{D}$-module such that its restriction to U_x generates A^ℓ as a $\mathcal{D}_{U_x}$-algebra. According to 2.2.17(ii) the restriction of the coisson bracket $h_x(L) \otimes h_x(L) \to h_x(j_{x*}A)$ is Ξ_x-continuous. So for some $L' \in \Xi_x(L)$ one has $\{h_x(L'), h_x(L')\} \subset h_x(A_\xi)$. Now take for A_ζ the subalgebra generated by L'. $\square$

Assume that A satisfies the countability condition of 2.7.2(ii). We define the bracket on A_x^{as} extending the bracket on $h_x(j_{x*}A)$ by continuity. It is automatically a Poisson bracket since $h(j_{x*}A)$ acts on $j_{x*}A$ by derivations.

2.7.3. Let A^ℓ be an arbitrary (not necessary countably generated) coisson algebra. Denote by $\Xi_x^{\mathcal{P}ois}$ the topology at x formed by all $A_\xi \in \Xi_x^{as}(A)$ which are coisson subalgebras of $j_{x*}A$. The coisson bracket on $h_x(j_{x*}A)$ is continuous with respect to this topology,[108] so the corresponding completion is a topological Poisson algebra.

[108]Indeed, for every $A_\xi \in \Xi_x^{\mathcal{P}ois}(A)$ the action of the Lie algebra $h_x(A_\xi)$ on $A^\ell_{\xi x}$ is continuous with respect to the topology on $h_x(A_\xi)$ formed by $h_x(A_\zeta)$ where $A_\zeta \in \Xi_x^{\mathcal{P}ois}(A)$, $A_\zeta \subset A_\xi$.

If we are in situation of 2.7.2 then $A_x^{\mathcal{P}ois}$ is the completion of A_x^{as} with respect to topology formed by those open ideals which are Lie subalgebras of A_x^{as}. It seems (though we do not know an example) that A_x^{Pois} can be a small quotient of A_x^{as}.

2.7.4. ¿From now on we assume that X is a curve.

Let $S \subset X$ be a finite subset, $j_S : U_S \hookrightarrow X$ the complementary open embedding. Let A be a finitely generated $\mathcal{D}_{U_S}$-algebra equipped with a coisson structure, so, by 2.7.2, $A_S^{as} = \widehat{\bigotimes}_{s\in S} A_s^{as}$ (see 2.4.10) is a topological Poisson algebra. The canonical morphism $\Gamma(U_S, h(A)) \to A_S^{as}$ from 2.4.10 commutes with brackets, i.e., it is a Hamiltonian action of the Lie algebra $\Gamma(U_S, h(A))$ on A_S^{as}. If X is compact, then, according to 2.4.5, 2.4.10, the zero fiber of the momentum map coincides with $\operatorname{Spec}\langle A\rangle(U) \subset \operatorname{Spf} A_S^{as} = \prod \operatorname{Spf} A_s^{as}$.

2.7.5. Let $\mathcal{Y}$ be a smooth $\mathcal{D}_X$-scheme. A coisson bracket $\{\ \ \}$ on $\mathcal{Y}$ defines a Lie* algebroid structure on $\Omega_{\mathcal{Y}}$ (see 1.4.15), so we have the action map $\tau = \tau_{\{\ \}}\colon \Omega_{\mathcal{Y}} \to \Theta_{\mathcal{Y}}$. One says that $\{\ \ \}$ is *symplectic* if $\tau_{\{\ \}}$ is an isomorphism. This situation occurs fairly rarely. A better notion is *elliptic* coisson structure which we are going to define. We assume that $\dim X = 1$.

2.7.6 Definition. A Lie* algebroid $\mathcal{L}$ on $\mathcal{Y}$ is *elliptic* if the action map $\tau_{\mathcal{L}}\colon \mathcal{L} \to \Theta_{\mathcal{Y}}$ is injective and $\Phi_{\mathcal{L}} := \operatorname{Coker}(\tau_{\mathcal{L}})$ is a locally free $\mathcal{O}_{\mathcal{Y}}$-module of finite rank. A coisson structure on $\mathcal{Y}$ is *elliptic* if the corresponding Lie* algebroid $\Omega_{\mathcal{Y}}$ is elliptic. □

§8 Other settings

We define a canonical compound tensor structure on the categories of modules over a Lie algebroid and of Harish–Chandra modules.

2.8.1. The compound tensor structure on the category of $\mathcal{D}$-modules is a particular case of a canonical compound tensor structure on the category of modules over Lie algebroid.[109] Let us recall the basic definitions (for details see, e.g., [M] or first two sections of [BB]).

Let X be any k-scheme, $\mathcal{L}$ a Lie algebroid on X. This means that $\mathcal{L}$ is a sheaf of k-vector spaces on X equipped with Lie algebra and $\mathcal{O}_X$-module structures, and we also

[109] Subject to a condition from 2.8.2.

have an action τ of the Lie algebra $\mathcal{L}$ on $\mathcal{O}_X$.[110] We demand that for $f, g \in \mathcal{O}_X$, $l, l' \in \mathcal{L}$ one has $\tau(fl)g = f(\tau(l)g)$ and $[l, fl'] = \tau(l)(f)l' + f[l, l']$. We also assume that $\mathcal{L}$ is a quasicoherent $\mathcal{O}_X$-module.

Below we denote by $\mathcal{L}^-$ our $\mathcal{L}$ considered as a plain sheaf of Lie algebras equipped with an action τ on $\mathcal{O}_X$.[111]

An $\mathcal{L}^-$-*module* is a quasicoherent $\mathcal{O}_X$-module M equipped with an action of $\mathcal{L}^-$ which is compatible with the $\mathcal{L}^-$-action on $\mathcal{O}_X$, i.e., for every $f \in \mathcal{O}_X$, $m \in M$, and $l \in \mathcal{L}$ one has $l(fm) = l(f)m + f(lm)$. We say that M is a *left* (resp. *right*) $\mathcal{L}$-*module* if, in addition, one has $(fl)m = f(lm)$ (resp. $(fl)n = l(fn)$).

The category $\mathcal{M}(X, \mathcal{L}^-)$ of $\mathcal{L}^-$-modules is an abelian k-category. The $\mathcal{O}_X$-tensor product makes it a tensor category. The full subcategories $\mathcal{M}^\ell(X, L)$, $\mathcal{M}^r(X, \mathcal{L})$ of left and right, $\mathcal{L}$-modules are closed under subquotients. The tensor product of two left $\mathcal{L}$-modules is a left $\mathcal{L}$-module, and that of a left and a right $\mathcal{L}$-module is a right $\mathcal{L}$-module. Therefore $\mathcal{M}^\ell(X, \mathcal{L})$ is a tensor category, and $\mathcal{M}^r(X, \mathcal{L})$ is an $\mathcal{M}^\ell(X, \mathcal{L})$-Module.

Let $U(\mathcal{L})$ be the enveloping algebra of $\mathcal{L}$. This is a sheaf of associative algebras on X equiped with a morphism of algebras $i_{\mathcal{O}} : \mathcal{O}_X \to U(\mathcal{L})$ and a morphism of Lie algebras $i_{\mathcal{L}} : \mathcal{L} \to U(\mathcal{L})$ such that $i_{\mathcal{O}}$ is a morphism of Lie algebra $\mathcal{L}$-modules ($\mathcal{L}$ acts on $U(\mathcal{L})$ by $i_{\mathcal{L}}$ composed with the adjoint action) and $i_{\mathcal{L}}$ is a morphism of $\mathcal{O}_X$-modules ($\mathcal{O}_X$ acts on $U(\mathcal{L})$ by $i_{\mathcal{O}}$ and left multiplication), and $U(\mathcal{L})$ is universal with respect to this datum. Then left, resp. right, $\mathcal{L}$-modules are the same as left (right) $U(\mathcal{L})$-modules which are quasicoherent as $\mathcal{O}_X$-modules.

2.8.2. ¿From now on we assume that $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of finite rank n. The adjoint action of $\mathcal{L}$ on $\mathcal{L}$ makes it an $\mathcal{L}^-$-module. One checks in a moment that the line bundle $\nu_{\mathcal{L}} := \det(\mathcal{L})^*$ is a right $\mathcal{L}$-module. There is a canonical equivalence of categories

$$\mathcal{M}^\ell(X, \mathcal{L}) \xrightarrow{\sim} \mathcal{M}^r(X, \mathcal{L}), \qquad M \mapsto M^r := M \otimes \nu_{\mathcal{L}}. \tag{2.8.2.1}$$

We denote these categories thus identified by $\mathcal{M}(X, \mathcal{L})$.

[110] i.e., τ is a morphism of Lie algebras $\mathcal{L} \to \Theta_X := \mathrm{Der}(\mathcal{O}_X)$.

[111] i.e., we forget about the $\mathcal{O}$-module structure on $\mathcal{L}$.

Remark. Assume that our algebroid is rigidified, i.e., we have a Lie algebra L and a morphism of Lie algebras $L \to \mathcal{L}$ which yields an isomorphism $\mathcal{O}_X \underset{k}{\otimes} L \xrightarrow{\sim} \mathcal{L}$. Then L acts on $\mathcal{O}_X$, so we have the category $\mathcal{M}(X, L)$ of quasicoherent $\mathcal{O}_X$-modules equipped with an L-action compatible with the L-action on $\mathcal{O}_X$. Then the obvious "identity" functors $\mathcal{M}^\ell(X, \mathcal{L}) \to \mathcal{M}(X, L)$, $\mathcal{M}^r(X, \mathcal{L}) \to \mathcal{M}(X, L)$ are equivalences of categories. The corresponding composition equivalence $\mathcal{M}^\ell(X, \mathcal{L}) \xrightarrow{\sim} \mathcal{M}(X, \mathcal{L})$ differs from (2.8.2.1) by twist by $\det L$.

2.8.3. Let us define on $\mathcal{M}(X, \mathcal{L})$ the structure of compound tensor category. We already have the tensor structure. It remains to define the $*$ operations and the compatibility morphisms.

2.8.1. Recall that *a Harish–Chandra pair* $(\mathfrak{g}, K)$ consists of a Lie algebra $\mathfrak{g}$, an affine group scheme K together with an embedding $i : \mathfrak{k} := \mathrm{Lie} K \hookrightarrow \mathfrak{g}$ and a K-action on $\mathfrak{g}$

CHAPTER THREE: CHIRAL BASICS

Алгебраисты обычно определяют группы как множества с операциями, удовлетворяющими длинному ряду трудно-запоминаемых аксиом. Понять такое определение, на мой взгляд, невозможно.

Член-корреспондент АН СССР В. И. Арнольд, "Математика с человеческим лицом", Природа, №3, 1988, 117–118 †

§1 Chiral operations

¿From now on we assume that X is a smooth curve.

3.1.1. For a finite non-empty set I put $U^{(I)} := \{(x_i) \in X^I : x_{i_1} \neq x_{i_2} \text{ for every } i_1 \neq i_2\}$. So $U^{(I)}$ is the complement to the diagonal divizor if $|I| \geq 2$. Denote by $j^{(I)} : U^{(I)} \hookrightarrow X^I$ the open embedding. We write $U^{(n)}$ for $U^{(\{1,..,n\})}$.

Let L_i, $i \in I$, be an I-family of (right) $\mathcal{D}$-modules on X. For a $\mathcal{D}_X$-module M set

$$P_I^{ch}(\{L_i\}, M) := \mathrm{Hom}_{\mathcal{M}(X^I)}(j_*^{(I)} j^{(I)*}(\boxtimes_I L_i), \Delta_*^{(I)} M). \tag{3.1.1.1}$$

The elements of this vector space are called *chiral I-operations.* They have local nature with respect to the étale topology of X. As usual, we write P_n^{ch} for $P_{\{1,..,n\}}^{ch}$; the elements of P_2^{ch} are called *chiral pairings.*

Remark. Let F_i ,G be quasicoherent $\mathcal{O}_X$-modules. According to 2.1.7, 2.1.8(iii) we have $P_I^{ch}(\{F_{i\mathcal{D}}\}, G_{\mathcal{D}}) = \mathrm{Diff}(j_*^{(I)} j^{(I)*}(\boxtimes F_i), \Delta.G)$; elements of this vector space are called *chiral polydifferential operators.*

† Algebraists define a group as a set equipped with operations, subject to a long series of axioms which are difficult to remember. Understanding such a definition is, in my opinion, impossible. V. I. Arnold, Corresponding Member of the Academy of Sciences of the USSR, "Mathematics with a Human Face", Priroda, No. 3, 1988, 117–118.

3.1.2. Our P_I^{ch} are left exact k-polylinear functors on $\mathcal{M}(X)$. Let us define the composition of chiral operations. Let $\pi\colon J \twoheadrightarrow I$ be a surjective map of finite non-empty sets, $\{K_j\}$ a J-family of $\mathcal{D}_X$-modules. The composition map

$$P_I^{ch}(\{L_i\}, M) \otimes (\otimes_I P_{J_i}^{ch}(\{K_j\}, L_i)) \longrightarrow P_J^{ch}(\{K_j\}, M) \tag{3.1.2.1}$$

sends $\varphi \otimes (\otimes \psi_i)$ to $\varphi(\psi_i)$ defined as the composition

$$j_*^{(J)} j^{(J)*}(\boxtimes_J K_j) = j_*^{(\pi)} j^{(\pi)*}(\boxtimes_I (j_*^{(J_i)} j^{(J_i)*} \boxtimes_{J_i} K_j)) \xrightarrow{\boxtimes \psi_i} j_*^{(\pi)} j^{(\pi)*}(\boxtimes_I \Delta_*^{(J_i)} L_i)$$
$$= \Delta_*^{(\pi)} j_*^{(I)} j^{(I)*}(\boxtimes_I L_i) \xrightarrow{\Delta_*^{(\pi)}(\varphi)} \Delta_*^{(\pi)} \Delta_*^{(I)} M = \Delta_*^{(J)} M.$$

Here $j^{(\pi)}\colon U^{(\pi)} := \{(x_j) \in X^J : x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2)\} \hookrightarrow X^J$; for the rest of notation see 2.2.3.

The composition is associative, so P_I^{ch} define on $\mathcal{M}(X)$ an abelian pseudo-tensor structure on $\mathcal{M}(X)$. We call it the *chiral structure* and denote by $\mathcal{M}(X)^{ch}$. The pseudo-tensor categories $\mathcal{M}(U)^{ch}$, $U \in X_{\text{ét}}$, form a sheaf of pseudo-tensor categories $\mathcal{M}(X_{\text{ét}})^{ch}$ on the étale topology of X.

The de Rham functor $h = h_X$ (see 2.1.6) is an augmentation functor on $\mathcal{M}(X_{\text{ét}})^{ch}$ (see 1.2.5).[112] The structure map

$$h_{I,i_0}\colon P_I^{ch}(\{L_i\}, M) \otimes h(L_{i_0}) \to P_{I \smallsetminus \{i_0\}}^{ch}(\{L_i\}, M) \tag{3.1.2.2}$$

sends $\varphi \otimes \bar{\ell}_{i_0}$, where $\varphi \in P_I^{ch}(\{L_i\}, M)$, $\ell_{i_0} \in L_{i_0}$, to the chiral $I \smallsetminus \{i_0\}$-operation $a \mapsto Tr_{i_0}\varphi(a \boxtimes \ell_{i_0})$ (cf.2.2.7). Here, as in 2.2.7, $Tr_{i_0}\colon pr_{I,i_0 \cdot}\Delta_*^{(I)} M \to \Delta_*^{(I \smallsetminus \{i_0\})} M$ is the trace map for the projection $pr_{I,i_0}\colon X^I \to X^{I \smallsetminus \{i_0\}}$. Compatibilities 1.2.5(i),(ii) are immediate.

3.1.3. Consider the operad $P^{ch}(\omega_X)$ (see 1.1.6) of chiral operations acting on ω_X. One has $P^{ch}(\omega_X)_I := \mathrm{Hom}(j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I}, \Delta_*^{(I)} \omega_X)$. Set $\lambda_I := \det(k^I)$; this is a line on which Aut I acts by the sgn character. One has a canonical (in particular, Aut I-equivariant) isomorphism $\varepsilon_I\colon \omega_X^{\boxtimes I} \otimes \lambda_I \xrightarrow{\sim} \omega_{X^I}$, $(\boxtimes \nu_i) \otimes (e_{i_1} \wedge .. \wedge e_{i_n}) \mapsto pr_{i_1}^* \nu_{i_1} \wedge .. \wedge pr_{i_n}^* \nu_{i_n}$; here $\nu_i \in \omega_X$, $\{i_1, .., i_n\}$ is an ordering of I, and $\{e_i\}$ is the standard base of k^I. If $|I| = 2$ the residue Res: $j_*^{(I)} \omega_{U^{(I)}} \to \Delta_*^{(I)} \omega_X$ yields a canonical map $r_I\colon \lambda_I \to \mathrm{Hom}(j_*^{(I)} \omega_{U^{(I)}}, \Delta_*^{(I)} \omega_X) = P^{ch}(\omega_X)_I$.

[112] Contrary to the $*$ situation, in the chiral setting h is highly degenerate.

3.1.4 Proposition. There is a unique isomorphism of operads

$$\kappa : \mathcal{L}ie \xrightarrow{\sim} P^{ch}(\omega_X)$$

which coincides with r_I for $|I| = 2$.

Beginning of the proof. Our basic tool is the Cousin complex C_I for ω_{X^I} with respect to the diagonal stratification. One has $C_I^m := \underset{T \in Q(I,|I|-m)}{\oplus} \Delta_*^{(I/T)} j_*^{(T)} \omega_{U(T)}$ where $Q(I,t)$ is the set of quotient sets $I \twoheadrightarrow T$, $|T| = t$ (see 1.3.1), and $\Delta^{(I/T)}$ is the diagonal embedding $X^T \hookrightarrow X^I$. The non-zero components of the differential $d^m : C_I^m \to C_I^{m+1}$ are the morphisms $\Delta_*^{(I/T)}(\mathrm{Res}_{T'}) : \Delta_*^{(I/T)} j_*^{(T)} \omega_{U(T)} \to \Delta_*^{(I/T')} j_*^{(T')} \omega_{U(T')}$ for $X^{T'} \subset X^T$ where $\mathrm{Res}_{T'}$ is the residue morphism. The complex C_I is a resolution of ω_{X^I}.

It is easy to prove the existence of κ: the map $r_I : \lambda_I \to P^{ch}(\omega_X)_I$, $|I| = 2$, is $\mathrm{Aut}I$-equivariant, so it defines a skew-symmetric operation $[\quad] \in P^{ch}(\omega_X)_2$, and we have only to show that the operation $\alpha := [[1,2],3] + [[2,3]1] + [[3,1],2] \in P^{ch}(\omega_X)_3$ is zero. In fact, $\alpha = d^1 d^0$ where d^j are the differentials in the complex C_I for $I = \{1,2,3\}$. The uniqueness of κ is clear. We will show that κ is bijective in 3.1.5-3.1.8.

3.1.5. Let M be a $\mathcal{D}_{X^I}$-module. We say that a finite increasing filtration $W.$ on M is *special* if for every l the successive quotient $\mathrm{gr}_l^W M$ is a finite sum of copies of $\Delta_*^{(I/T)} \omega_{X^T}$ for $T \in Q(I, -l)$. Such a filtration is unique if it exists. We call M a *special* $\mathcal{D}_{X^I}$-module if it has a special filtration. Let $\mathcal{M}(X^I)^{sp} \subset \mathcal{M}(X^I)$ be the full subcategory of special $\mathcal{D}$-modules. It is closed under finite direct sums and subquotients, so it is an abelian category. Any morphism in $\mathcal{M}(X^I)^{sp}$ is strictly compatible with $W.$, i.e., gr^W is an exact functor.

3.1.6 Lemma. $j_*^{(I)} \omega_{U(I)}$ is a special $\mathcal{D}_{X^I}$-module.

Proof. We proceed by induction with respect to $|I|$ using the exact sequence $0 \to \omega_{X^I} \to C_I^0 \xrightarrow{d^0} C_I^1 \xrightarrow{d^1} C_I^2$ (see 3.1.4). Since $C_I^m \in \mathcal{M}(X^I)^{sp}$ for $m > 0$ by the induction assumption, the $\mathcal{D}$-module $K := \mathrm{Ker}\, d^1$ is special. Since $j^{(I)*}K = 0$ one has $W_{-|I|}K = 0$, so the following filtration $W.$ on $j_*^{(I)} \omega_{U(I)} = C_I^0$ is special: $W_l C_I^0 := (d^0)^{-1}(W_l K)$ for $l \geq -|I|$, $W_l C_I^0 = 0$ for $l < -|I|$. $\square$

3.1.7. So for any $T \in Q(I)$ the $\mathcal{D}_{X^I}$-module $\Delta_*^{(I/T)} j_*^{(T)} \omega_{U(T)}$ is special. In fact, this is an injective object of $\mathcal{M}(X^I)^{sp}$. This follows from exactness of the functor $P \mapsto P/W_{-|T|-1}P$ since $\Delta_*^{(I/T)} j_*^{(T)} \omega_{U(T)}$ is obviously an injective object of the subcategory $\mathcal{M}(X^I)^{sp}_{\geq -|T|} := \{P : W_{-|T|-1}P = 0\}$. In particular, $\Delta_*^{(I)} \omega_{U(I)}$ is an injective object of $\mathcal{M}(X^I)^{sp}$.

Assume that $|I| > 1$. Since $C_I^{\cdot} \in \mathcal{M}(X^I)^{sp}$ and $\mathrm{Hom}(\omega_{X^I}, \Delta_*^{(I)} \omega_X) = 0$, the complex $\mathrm{Hom}(C_I, \Delta_*^{(I)} \omega_X)$ is acyclic. Using the canonical isomorphism $\omega_{X^I} = \omega_X^{\boxtimes I} \otimes \lambda_I$ (see 3.1.3) we may rewrite this exact complex as follows:

$$.. \to \bigoplus_{T' \in Q(I,|I|-2)} P^{ch}(\omega_X)_{T'} \otimes \lambda_{T'} \xrightarrow{d_1} \bigoplus_{T \in Q(I,|I|-1)} P^{ch}(\omega_X)_T \otimes \lambda_T \xrightarrow{d_0} P^{ch}(\omega_X)_I \otimes \lambda_I \to 0.$$

Let us explain the operad meaning of the differential. Take $T' \in Q(T, |T|-1)$; let $\alpha', \alpha" \in T$ be the two distinct T'-equivalent elements of T, $\alpha \in T'$ the corresponding element of T'. Then $T \smallsetminus \{\alpha', \alpha''\} = T' \smallsetminus \{\alpha\}$, so we have a canonical identification

$$\lambda_{\{\alpha', \alpha''\}} \xrightarrow{\sim} \lambda_{T'} \otimes \lambda_T^{\otimes -1}, \quad e_{\alpha'} \wedge e_{\alpha''} \mapsto (\gamma \wedge e_\alpha) \otimes (\gamma \wedge e_{\alpha'} \wedge e_{\alpha''})^{-1}, \tag{3.1.7.1}$$

where γ is a generator of $\lambda_{T' \smallsetminus \{\alpha\}} = \lambda_{T \smallsetminus \{\alpha', \alpha''\}}$. If S is a set of two elements, then the above complex for $I = S$ reduces to the isomorphism $P^{ch}(\omega_X)_S = \lambda_S$. Therefore we may rewrite any non-zero component of d as the morphism

$$P^{ch}(\omega_X)_{T'} \otimes \lambda_{\{\alpha', \alpha''\}} = P^{ch}(\omega_X)_{T'} \otimes P^{ch}(\omega_X)_{\{\alpha', \alpha''\}} \to P^{ch}(\omega_X)_T.$$

This arrow coincides with the composition of operations map for $T \twoheadrightarrow T'$.

3.1.8. *End of the proof of* 3.1.4. The surjectivity of d_0 means that for $n \geq 2$ every operation in $P^{ch}(\omega_X)_n$ is a sum of $(n-1)$-operations composed with binary operations. Therefore $P^{ch}(\omega_X)$ is generated by $P^{ch}(\omega_X)_2$. Exactness of the complex from 3.1.7 means that all the relations in our operad come from the Jacobi relation for $[\ \ ] \in P^{ch}(\omega_X)_2$. $\square$

3.1.9 Remarks. (i) For $S, T \in Q(I)$ let $\mathcal{L}ie_{S/T}$ be the vector space $\bigotimes_T \mathcal{L}ie_{S_t}$ if $S \geq T$ and 0 otherwise. Let us define a canonical isomorphism

$$\alpha(I,S)_l \colon \mathrm{gr}^W_{-l}(\Delta_*^{(I/S)} j_*^{(S)} j^{(S)*} \omega_X^{\boxtimes S}) \xrightarrow{\sim} \bigoplus_{T \in Q(I,l)} \Delta_*^{(I/T)} \omega_X^{\boxtimes T} \otimes \mathcal{L}ie^*_{S/T}. \tag{3.1.9.1}$$

First notice that both parts are supported on $X^S \subset X^I$, so (applying $\Delta^{(I/S)!}$ and replacing S by I) we may assume that $S = I$. The case of top quotient $l = 1$ is just the isomorphism κ_I of 3.1.4. Assume that $l > 1$. It suffices to define $\alpha(I,I)_l = \alpha(I)_l$ over the complement to diagonals of dimension $< l$. Its T-component $\alpha(I)_T$ is determined by the restriction to a neighbourhood of X^T. There $j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I}$ coincides with $\underset{T}{\boxtimes}\, j_*^{(I_t)} j^{(I_t)*} \omega_X^{\boxtimes I_t}$, and $\alpha(I)_T := \underset{T}{\boxtimes}\, \alpha(I_t)_1$.

(ii) Let us describe $\mathcal{M}(X^I)^{sp}$ explicitly. Its irreducible objects are $\Delta_*^{(I/T)} \omega_X^{\boxtimes T}$, $T \in Q(I)$. The object $\mathcal{I}_T := \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*} \omega_X^{\boxtimes T}$ is injective by 3.1.7; it contains $\Delta_*^{(I/T)} \omega_X^{\boxtimes T}$.[113] Isomorphism 3.1.9.1 yields a canonical identification $\mathrm{Hom}(\mathcal{I}_S, \mathcal{I}_T) = \mathcal{L}ie_{S/T}$. The composition of morphisms corresponds to (a tensor product) of compositions of Lie operations.

So let $\mathcal{Q}(I)$ be the k-category whose objects are elements of $Q(I)$ and $\mathrm{Hom}_{\mathcal{Q}(I)}(S,T) = \mathcal{L}ie_{S/T}$; the composition of morphisms is the tensor product of compositions of Lie operations. We have defined a fully faithful embedding $\mathcal{Q}(I) \to \mathcal{M}(X^I)^{sp}$, $T \mapsto \mathcal{I}_T$. A $\mathcal{Q}(I)$-*module* is a k-linear functor $\mathcal{Q}(I) \to$ (finite-dimensional vector spaces); $\mathcal{Q}(I)$-modules form an abelian category $\mathcal{V}(I)$. There is an obvious functor $\mathcal{M}(X^I)^{sp} \to \mathcal{V}(I)^\circ$ which sends a $\mathcal{D}$-module M to the $\mathcal{Q}(I)$-module $T \mapsto \mathrm{Hom}(M, \mathcal{I}_T)$. This functor is an equivalence of categories.

3.1.10 Remarks. We assume that $k = \mathbb{C}$.

(i) For any special $\mathcal{D}_{X^I}$-module P one has $H^l DR_{an}(P) = H^l DR_{an}(\mathrm{gr}_l^W P)$; this sheaf is isomorphic to a direct sum of constant sheaves supported on X^S, $|S| = -l$. Here DR_{an} denotes the analytic de Rham complex (used in the Riemann-Hilbert correwpondence). If $l = -1$ then $H^l DR_{an}(P)$ is the constant sheaf on the diagonal $X \subset X^I$ with fiber V^* where $V := \mathrm{Hom}(P, \Delta_*^{(I)} \omega_X)$.

(ii) The filtration $W.$ on $j_*^{(I)} \omega_{U(I)}$ coincides with the weight filtration on the mixed Hodge module $j_*^{(I)} \mathbb{C}(|I|)_{U(I)}[|I|]$.

(iii) By 3.1.3, one has $P^{ch}(\omega_X)_I = \mathrm{Hom}(j_*^{(I)} \omega_{U(I)}, \Delta_*^{(I)} \omega_X) \otimes \lambda_I$. So remark (i) above implies that $P^{ch}(\omega_X)_I \otimes \lambda_I = H_{|I|-1}(Y_I, \mathbb{C})$ where Y_I is the configuration space of I-tuples

[113] $\mathcal{I}_T$ is the injective envelope of $\Delta_*^{(I/T)} \omega_X^{\boxtimes T}$ because $\dim \mathrm{Hom}(\Delta_*^{(I/T')} \omega_X^{\boxtimes T'}, \mathcal{I}_T)$ equals 0 if $T' \neq T$ and 1 if $T' = T$.

of different points on $\mathbb{C} = \mathbb{R}^2$. Up to homotopy the Y_I's are the spaces of the "small disc" operad. The composition law in the operad $P^{ch}(\omega_X)$ coincides with the one on the homology operad of this topological operad. So 3.1.4 amounts to Cohen's theorem [C].

3.1.11. For $M \in \mathcal{M}(X)$ we define *the unit operation* $\varepsilon_M \in P_2^{ch}(\{\omega_X, M\}, M)$ as the composition $j_*j^*\omega_X \boxtimes M \to (j_*j^*\omega_X \boxtimes M)/\omega_X \boxtimes M \xrightarrow{\sim} \Delta_* M$ where the last arrow comes from the canonical isomorphism $\omega_X \otimes^! M = M$.

3.1.12 Lemma. (i) For $M_i, N \in \mathcal{M}(X)$ there is a canonical isomorphism[114]

$$P_I^{ch}(\{M_i\}, N)^I \xrightarrow{\sim} P_{\tilde{I}}^{ch}(\{\omega_X, M_i\}, N), \quad (\varphi_i) \mapsto \sum_{i \in I} \varphi_i(\varepsilon_{M_i}, id_{M_{i'}})_{i' \neq i}.$$

(ii) Chiral operations satisfy Leibnitz rule with respect to the unit operations: for every $\varphi \in P_I^{ch}(\{M_i\}, N)$ one has $\varepsilon_N \varphi = \sum\limits_{i \in I} \varphi(\varepsilon_{M_i}, id_{M_{i'}})_{i' \neq i} \in P_{\tilde{I}}^{ch}(\{\omega_X, M_i\}, N)$.

Proof. (i) Let $\Delta^i : X^I \hookrightarrow X^{\tilde{I}}$ be the i^{th} diagonal section of the projection $X^{\tilde{I}} \to X^I$. The images of the Δ^i's are disjoint over $U^{(I)}$, so one has a short exact sequence

$$(3.1.12.1) \quad 0 \to \omega_X \boxtimes j_*^{(I)} j^{(I)*}(\boxtimes M_i) \to j_*^{(\tilde{I})} j^{(\tilde{I})*} \omega_X \boxtimes (\boxtimes M_i) \to \oplus \Delta_*^i j_*^{(I)} j^{(I)*} \boxtimes M_i \to 0.$$

Here we use the standard identifications $j_*j^*\omega_X \boxtimes M_i / \omega_X \boxtimes M_i = \Delta_*(\omega_X \otimes^! M_i = \Delta_* M_i$. Since $\mathrm{Hom}(\omega_X \boxtimes j_*^{(I)} j^{(I)*}(\boxtimes M_i), \Delta_*^{(\tilde{I})} N) = 0$, (3.2.4.1) yields a canonical isomorphism $P_I^{ch}(\{M_i\}, N)^I \xrightarrow{\sim} P_{\tilde{I}}^{ch}(\{\omega_X, M_i\}, N)$. One checks in a moment that it coincides with the map of 3.1.12(i).

(ii) Let t be a local coordinate on X, t_0, t_i the corresponding local coordinates on $X^{\tilde{I}}$. For $i \in I$ set $\nu^i := \frac{dt_o}{t_0 - t_i}$. For any $m \in j_*^{(I)} j^{(I)*} \boxtimes M_i$ one has $\varepsilon_N \varphi(\nu^i \cdot m) = \Delta_\cdot^i(\varphi(m)) = \varphi(\varepsilon_{M_i}, id_{M_{i'}})_{i' \neq i}(\nu_i \cdot m) \in \Delta_*^{(\tilde{I})} N$ and $\varphi(\varepsilon_{M_{i''}}, id_{M_{i'}})_{i' \neq i''} = 0$ for $i'' \neq i$. Now use (i). $\square$

3.1.13 Variants. (i) Let L be any Lie* algebra, $M_i, N \in \mathcal{M}(X, L)$. A chiral operation $\varphi \in P_I^{ch}(\{M_i\}, N)$ is *L-compatible* if it satisfies the Leibnitz rule with respect to the L-actions. Explicitly, we demand that the composition $\cdot_N \varphi \in \mathrm{Hom}(L \boxtimes j_*^{(I)} j^{(I)*} \boxtimes M_i, \Delta_*^{(\tilde{I})} N)$

[114] Here $\tilde{I} := I \sqcup \cdot$ (see 1.2.1).

equals the sum of I operations $\varphi(\cdot_{M_i}, id_{M_{i'}})_{i'\neq i}$, $i \in I$.[115] This amounts to the fact that φ satisfies the Leibnitz rule with respect to the $h(L)$-action on our $\mathcal{D}$-modules. We denote the set of such operations by P^{ch}_{LI}; they define on $\mathcal{M}(X, L)$ a pseudo-tensor structure $\mathcal{M}(X, L)^{ch}$.

(ii) Let R^ℓ be a commutative $\mathcal{D}_X$-algebra, M_i, N are $R^\ell[\mathcal{D}_X]$-modules. Consider the $\mathcal{D}_{X^I}$-algebra $R^{\ell\boxtimes I}$. Then $j^{(I)}_* j^{(I)*} \boxtimes M_i$ and $\Delta^{(I)}_* N$ are $R^{\ell\boxtimes I}$-modules in the obvious way.[116] We say that a chiral operation $\varphi : j^{(I)}_* j^{(I)*} \boxtimes M_i \to \Delta^{(I)}_* N$ is R^ℓ-*polylinear* if it is a morphism of $R^{\ell\boxtimes I}$-modules.

R^ℓ-polylinear chiral operations are closed under composition, so they define a pseudo-tensor structure on $\mathcal{M}(X, R^\ell)$, which we denote by $\mathcal{M}(X, R^\ell)^{ch}$.

§2 Relation to "classical" operations

In this section we show that "classical" operations from Ch 2 can be considered as "symbols", or "classical limits" of chiral operations.

3.2.1. The obvious morphism $\boxtimes L_i \to j^{(I)}_* j^{(I)*} \boxtimes L_i$ yields a map

$$\beta_I\colon P^{ch}_I(\{L_i\}, M) \to P^*_I(\{L_i\}, M). \tag{3.2.1.1}$$

It is clear that β's are compatible with composition of operations, so we have a pseudo-tensor functor

$$\beta\colon \mathcal{M}(X)^{ch} \to \mathcal{M}(X)^* \tag{3.2.1.2}$$

which extends the identity functor on $\mathcal{M}(X)$. It is compatible with the augmentations, so β is an augmented pseudo-tensor functor.

3.2.2. Consider the tensor category $\mathcal{M}(X)^!$ and the pseudo-tensor category $\mathcal{M}(X)^! \otimes \mathcal{L}ie$ (see 1.1.10). So $\mathcal{M}(X)^! \otimes \mathcal{L}ie$ coincides with $\mathcal{M}(X)$ as a usual category, and the corresponding operations are $P^{!\mathcal{L}ie}_I(\{L_i\}, M) := \mathrm{Hom}(\otimes^! L_i, M) \otimes \mathcal{L}ie_I$ where $\mathcal{L}ie$ is the Lie algebras operad.

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[115] Here $\cdot_{M_i}$, $\cdot_N$ are the $*$ actions of L.

[116] The $R^{\ell\boxtimes I}$-module structure on $\Delta^{(I)}_* N$ comes from the $R^{\ell\otimes I}$-module structure on N (defined by the product morphism of algebras $R^{\ell\otimes I} \to R^\ell$) and the fact that $\Delta^{(I)}_*$ sends $R^{\ell\otimes I} = \Delta^{(I)*} R^{\ell\boxtimes I}$-modules to $R^{\ell\boxtimes I}$-modules (to define the module structure use (2.1.3.1) for $i = \Delta^{(I)}$, $L = R^{\ell\boxtimes I}$, $M = N$).

There is a canonical faithful pseudo-tensor functor

$$\alpha\colon \mathcal{M}(X)^! \otimes \mathcal{L}ie \to \mathcal{M}(X)^{ch} \tag{3.2.2.1}$$

which extends the identity functor on $\mathcal{M}(X)$. Namely, the map

$$\alpha_I\colon \mathrm{Hom}(\otimes^! L_i, M) \otimes \mathcal{L}ie_I \hookrightarrow P_I^{ch}(\{L_i\}, M) \tag{3.2.2.1}$$

sends $\phi \otimes \nu \in \mathrm{Hom}(\otimes^! L_i, M) \otimes \mathcal{L}ie_I$ to a chiral operation equal to composition

$$j_*^{(I)} j^{(I)*}(\boxtimes L_i) = (\boxtimes L_i^\ell) \otimes j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I} \to (\boxtimes L_i^\ell) \otimes \Delta_*^{(I)} \omega_X = \Delta_*^{(I)}(\otimes^! L_i) \to \Delta_*^{(I)} M.$$

Here the first arrow is $\kappa(\nu)$ (see 3.1.4) tensored by the identity morphism of $\boxtimes L_i^\ell$, the second one is $\Delta_*^{(I)}(\phi)$, and the equalities are standard canonical isomorphisms. We leave it to the reader to check injectivity of α_I (notice that the canonical map $j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I} \to \Delta_*^{(I)} \omega_X \otimes \mathcal{L}ie_I^*$ coming from κ is surjective) and compatibility of α_I's with composition of operations.

3.2.3 Remark. The composition $\beta_I \alpha_I : P_I^{!\mathcal{L}ie} \to P_I^*$ vanishes for $|I| \geq 2$. The sequence $0 \to P_2^{!\mathcal{L}ie} \to P_2^{ch} \to P_2^*$ is exact. Notice that $P_2^! \xrightarrow{\sim} P_2^{!\mathcal{L}ie}$, $\phi \mapsto \phi \otimes [\ \ ]$.

The above pseudo-tensor functors α, β are parts of the following general picture:

3.2.4. Notice that $j_*^{(I)} j^{(I)*} \boxtimes L_i = (j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I}) \otimes (\boxtimes L_i^\ell)$. So the special filtration $W.$ on $j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I}$ (see 3.1.5, 3.1.6) yields a finite filtration on $j_*^{(I)} j^{(I)*} \boxtimes L_i$ which we denote also by $W.$. The canonical identifications $gr_{-l}^W j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I} = \underset{T \in Q(I,l)}{\oplus} \Delta_*^{(I/T)} \omega_X^{\boxtimes T} \otimes \mathcal{L}ie_{I/T}^*$ of 3.1.9(i) yields canonical surjections

$$\underset{T \in Q(I,l)}{\oplus} \Delta_*^{(I/T)} \big(\underset{t \in T}{\boxtimes} \big(\overset{!}{\underset{I_t}{\otimes}} L_i \big) \otimes \mathcal{L}ie_{I_t}^* \big) \twoheadrightarrow gr_{-l}^W j_*^{(I)} j^{(I)*} \boxtimes L_i \tag{3.2.4.1}$$

which are isomorphisms if L_i are $\mathcal{O}_X$-flat.

The above filtration yields a canonical filtration on k-modules of chiral operations:

$$P_I^{ch}(\{L_i\}, M)^n := \mathrm{Hom}(j_*^{(I)} j^{(I)*} \boxtimes L_i / W_{n-1-|I|} j_*^{(I)} j^{(I)*} \boxtimes L_i,\ \Delta_*^{(I)} M). \tag{3.2.4.2}$$

The morphisms (3.2.4.1) yield canonical embeddings

$$gr^n P_I^{ch}(\{L_i\}, M) \hookrightarrow \bigoplus_{T\in Q(I,|I|-n)} P_T^*(\{\otimes^!_{I_t} L_i\}, M) \otimes \mathcal{L}ie_{I/T} \tag{3.2.4.3}$$

which are isomorphisms if L_i are projective $\mathcal{D}_X$-modules.

Notice that the maps α, β from 3.2.1, 3.2.2 occure as "boundary" morphisms of (3.2.4.3). Namely, the morphism (3.2.4.3) for $n = |I|-1$ is always an isomorphism. Its inverse (composed with the embedding $gr^{|I|-1}P_I^{ch} = P_I^{ch\,|I|-1} \hookrightarrow P_I^{ch}$) coincides with α_I of (3.2.2.1). Simlarly, β_I of (3.2.1.1) is the morphism (3.2.4.3) for $n = 0$ composed with the projection $P_I^{ch} \twoheadrightarrow P_I^{ch}/P_I^{ch\,1} = gr^0 P_I^{ch}$.

3.2.5. The above filtration is compatible with composition of chiral operations. Namely, for $J \twoheadrightarrow I$ and a J-family of $\mathcal{D}_X$-modules K_j the composition map sends the tensor product $P_I^{ch}(\{L_i\}, M)^n \otimes (\otimes_I P_{J_i}^{ch}(\{K_j\}, L_i)^{m_i}$ to $P_J^{ch}(\{K_j\}, M)^{n+\Sigma m_i}$. Therefore we can define a new pseudo-tensor structure on $\mathcal{M}$ with operations $P^{cl} := gr^{\cdot} P^{ch}$; this is *the classical limit* of chiral structure. We can rewrite (3.2.4.3) as a canonical embedding $P_I^{cl}(\{L_i\}, M) \hookrightarrow P_I^{c}(\{L_i\}, M)$ (recall that c operations were defined in 1.4.26 using the compound pseudo-tensor structure $\mathcal{M}^{*!}$) which is isomorphism if L_i are projective $\mathcal{D}_X$-modules. It follows from definitions that this embedding is compatible with composition of operations. So we defined a canonical fully faithful embedding of pseudo-tensor categories

$$\mathcal{M}(X)^{cl} \hookrightarrow \mathcal{M}(X)^{c} \tag{3.2.5.1}$$

which extends the identity functor on $\mathcal{M}$.

§3 Chiral algebras and modules

3.3.1. For a k-operad $\mathcal{B}$ a $\mathcal{B}^{ch}$ *algebra* on X is a $\mathcal{B}$ algebra in the pseudo-tensor category $\mathcal{M}(X)^{ch}$. The category $\mathcal{B}(\mathcal{M}(X)^{ch})$ of $\mathcal{B}^{ch}$ algebras will also be denoted by $\mathcal{B}^{ch}(X)$. The canonical pseudo-tensor functors α, β from §3.2 yield the functors between the categories of $\mathcal{B}$ algebras

$$\mathcal{B}(\mathcal{M}(X)^! \otimes \mathcal{L}ie) \xrightarrow{\alpha^{\mathcal{B}}} \mathcal{B}^{ch}(X) \xrightarrow{\beta^{\mathcal{B}}} \mathcal{B}^*(X) \tag{3.3.1.1}$$

We will be interested in Lie^{ch} algebras. According to 1.1.10 one has $\mathcal{L}ie(\mathcal{M}(X)^! \otimes \mathcal{L}ie) = \mathcal{C}om^!(X) := \mathcal{C}om(\mathcal{M}(X)^!)$, so we may rewrite (3.3.1.1) as

$$\mathcal{C}om^!(X) \xrightarrow{\alpha^{\mathcal{L}ie}} \mathcal{L}ie^{ch}(X) \xrightarrow{\beta^{\mathcal{L}ie}} \mathcal{L}ie^*(X). \tag{3.3.1.2}$$

Note that $\beta^{\mathcal{L}ie}$ is a faithful functor, and $\alpha^{\mathcal{L}ie}$ is a fully faithful embedding (since α is a faithful pseudo-tensor functor).

3.3.2. For a Lie^{ch} algebra A we denote by $\mu = \mu_A \in P_2^{ch}(\{A, A\}, A)$ the commutator and by $[\ ] = [\ ]_A \in P_2^*(\{A, A\}, A)$ the $*$ bracket $\beta(\mu)$. We usually call μ the *chiral product.* Set $A^{(*)} := \beta^{\mathcal{L}ie}(A) \in \mathcal{L}ie(\mathcal{M}^*)$, i.e., $A^{(*)}$ is A considered as a Lie* algebra with bracket $[\ ]_A$.

Consider the sheaf of Lie algebras $h(A) = h(A^{(*)})$ (see 2.5.3). It acts on the $\mathcal{D}$-module A (the adjoint action) by derivations of μ_A (see the end of 1.2.18).

A Lie^{ch} algebra A is said to be *commutative* if $[\ ]_A = 0$. Denote by $\mathcal{L}ie^{ch}_{com}(X) \subset \mathcal{L}ie^{ch}(X)$ the full subcategory of commutative Lie^{ch} algebras. It follows from 3.2.3 that $\alpha^{\mathcal{L}ie}(\mathcal{C}om^!(X)) = \mathcal{L}ie^{ch}_{com}(X)$, i.e., we have the equivalence of categories

$$\alpha^{\mathcal{L}ie}\colon \mathcal{C}om^!(X) \xrightarrow{\sim} \mathcal{L}ie^{ch}_{com}(X). \tag{3.3.2.1}$$

3.3.3. Let A be a Lie^{ch} algebra. A *unit in* A is a morphism of $\mathcal{D}$-modules $1 = 1_A\colon \omega_X \to A$ (i.e., 1 is a horizontal section of A^ℓ) such that $\mu_A(1, id_A) \in P_2^{ch}(\{\omega_X, A\}, A)$ is the unit operation ε_A (see 3.1.11). A unit in A is unique if it exists.

A Lie^{ch} algebra with unit is called *unital Lie^{ch} algebra* or simply *chiral algebra.*[117] Denote by $\mathcal{CA}(X) \subset \mathcal{L}ie^{ch}(X)$ the subcategory of chiral algebras and morphisms preserving units. One has an obvious "adding of unit" functor[118] $\mathcal{L}ie^{ch}(X) \to \mathcal{CA}(X)$, $\quad B \mapsto B \oplus \omega_X$, left adjoint to the embedding $\mathcal{CA}(X) \hookrightarrow \mathcal{L}ie^{ch}(X)$.

Chiral algebras form a sheaf of categories $\mathcal{CA}(X_{\text{ét}})$ on the étale topology of X.

Let L be a Lie* algebra. An *L-action on A* is a Lie* action of L on the $\mathcal{D}$-module A such that μ_A is L-compatible (see 3.1.13(i)).[119] E.g., we have the adjoint $A^{(*)}$-action on A.

[117] We will also refer to Lie^{ch} algebras as chiral algebras without unit.
[118] The Jacobi identity for $B \oplus \omega_X$ follows from 3.1.12(ii).
[119] The unit is automatically fixed by any L-action (as well as any horizontal section of A^ℓ).

Let $\mathcal{CA}(X)_{com} \subset \mathcal{CA}(X)$ be the full subcategory of commutative chiral algebras. The above equivalence (3.3.2.1) identifies the corresponding subcategories of unital algebras, so we have a canonical equivalence (see 2.3.1 for notation)

$$\mathcal{C}omu_{\mathcal{D}}(X) = \mathcal{C}omu^{!}(X) \xrightarrow{\sim} \mathcal{CA}(X)_{com}. \tag{3.3.3.1}$$

Every chiral algebra A admits the maximal commutative quotient.[120]

3.3.4. For a chiral algebra A an *A-module* (or *chiral A-module*) always means *unital* A-module, i.e., this is an A-module M (here A is considered as a Lie^{ch} algebra) such that for the A-action $\mu_M \in P_2^{ch}(\{A, M\}, M)$ the operation $\mu_M(1_A, id_M) \in P_2^{ch}(\{\omega_X, M\}, M)$ is the unit operation ε_M (see 3.1.11). We denote the category of A-modules by $\mathcal{M}(X, A)$. According to 1.2.18, $\mathcal{M}(X, A)$ carries an abelian augmented pseudo-tensor structure; we denote it $\mathcal{M}(X, A)^{ch}$. For $M_i, N \in \mathcal{M}(X, A)$ the corresponding vector space of *chiral A-operations* is denoted by $P_{AI}^{ch}(\{M_i\}, N) \subset P_I^{ch}(\{M_i\}, N)$. For any A-module M the Lie algebra $h(A)$ acts canonically on M (considered as a plain $\mathcal{D}$-module); every operation from P_{AI}^{ch} satisfies Leibnitz rule with respect to this action. The augmentation functor on $\mathcal{M}(X, A)^{ch}$ is $M \mapsto h^A(M) := h(M)^{h(A)}$.

Example. For the unit chiral algebra ω_X the forgetful functor $\mathcal{M}(X, \omega_X)^{ch} \to \mathcal{M}(X)^{ch}$ is an equivalence of augmented pseudo-tensor categories (see 3.1.12(ii)).

Remarks. (i) If N is a left $\mathcal{D}_X$-module, M a chiral A-module then $N \otimes M$ is a chiral A-module. So $\mathcal{M}(X, A)^{ch}$ is a Module over the tensor category $\mathcal{M}^{\ell}(X)$.

(ii) A-modules localize in the obvious manner, so we have a sheaf of augmented pseudo-tensor categories $\mathcal{M}(X_{\text{ét}}, A)^{ch}$.

(iii) Let $a : P \hookrightarrow A$ be a sub$\mathcal{D}$-module which generates A as a chiral algebra. Then for M_i, N as above an operation $\varphi \in P_I^{ch}(\{M_i\}, N)$ is a chiral A-operation if and only if the operations $\mu_N(a, \varphi), \sum\limits_{i \in I} \varphi(id_{M_{j \neq i}}, \mu_{M_i}(a, id_{M_i})) \in P_{\tilde{I}}^{ch}(\{P, M_i\}, N)$ coincide.

Any A-module is automatically (via the functor β) an $A^{(*)}$-module. Precisely, we have

[120] which is the quotient of A modulo the ideal generated by the image of $[\ ]_A$.

the pseudo-tensor functor

$$\beta_A\colon \mathcal{M}(X,A)^{ch} \to \mathcal{M}(X,A^{(*)})^*. \tag{3.3.4.1}$$

Remark. For $M_i, N \in \mathcal{M}(X,A)$ one has $P^{ch}_{AI}(\{M_i\},N) \subset P^{ch}_{A^{(*)}I}(\{M_i\},N)$ (see 3.1.13(i) for notation), so (3.3.4.1) is the composition of pseudo-tensor functors

$$\mathcal{M}(X,A)^{ch} \to \mathcal{M}(X,A^{(*)})^{ch} \to \mathcal{M}(X,A^{(*)})^*. \tag{3.3.4.2}$$

3.3.5 Remark. An A-module structure on M amounts to a structure of chiral algebra on $A \oplus M$ such that $A \hookrightarrow A \oplus M$ and $A \oplus M \twoheadrightarrow A$ are morphisms of chiral algebras and the restriction of the chiral product to $M \times M$ vanishes.

3.3.6. Let M be an A-module and $\gamma \in \Gamma(X,M)$ a section.

We say that $a \in A$ *kills* γ if $\mu_M(fa \boxtimes \gamma) \in \Delta_* M$ vanishes for every $f \in j_*j^*\mathcal{O}_{X\times X}$. Such a form a subsheaf $\mathcal{A}nn(\gamma) \subset A$ called *the annihilator of* γ. This is an ideal in A.

The *centralizer* $\mathcal{C}ent(\gamma)$ of γ consists of sections $a \in A$ that commute with γ, i.e., $\mu_M(a \boxtimes \gamma) = 0$. This is a chiral subalgebra of A. We say that γ is an *A-central* section of M if $\mathcal{C}ent(\gamma) = A$.

The annihilator and centralizer of a section are compatible with étale localization of X.

Exercise. Show that the map $\varphi \mapsto \varphi(1)$ identifies $\mathrm{Hom}_{\mathcal{M}(X,A)}(A,M)$ with the vector space of all horizontal A-central sections of M^ℓ.

The *annihilator* $\mathcal{A}nn(M) = \mathcal{A}nn(A,M)$ *of* M is the subsheaf of A whose sections over an affine open U are those a that kill every section γ of M over U. Similarly, the *centralizer* $\mathcal{C}ent(M) = \mathcal{C}ent(A,M)$ *of* M is the subsheaf of A whose sections over U as above are those a that commute with every γ. Both these subsheaves are $\mathcal{O}_X$-submodules of A. If we know that $\mathcal{A}nn(M)$ (resp. $\mathcal{C}ent(M)$) is a quasicoherent $\mathcal{O}_X$-module then it is an ideal (resp. a chiral subalgebra) in A. This always happens if M is $\mathcal{O}_X$-flat.

We say that M is *commutative* (or the A-action on M is commutative) if $\mathcal{C}ent(M) = A$, or, equivalently, if every $m \in M$ is A-central.

The centralizer of A (considered as an A-module) is called the *center of* A; we denote it by $Z(A)$. If $Z(A)$ is $\mathcal{O}_X$-quasicoherent (see above) then this is a commutative subalgebra of A.

We denote by $\mathcal{M}(X,A)^{ch}_{com} \subset \mathcal{M}(X,A)^{ch}$ the full pseudo-tensor subcategory of commutative modules. The subcategory $\mathcal{M}(X,A)_{com} \subset \mathcal{M}(X,A)$ is closed under direct sums and subquotients. Any A-module M has maximal commutative sub- and quotient modules which we denote, respectively, by M^{com} and M_{com}. Notice that M_{com} is the quotient of M modulo the A-submodule generated by the image of the $*$ action of $A^{(*)}$ on M. So the functor $M \mapsto M_{com}$ is compatible with the étale localization.

Every chiral algebra A admits the maximal commutative quotient algbera A_{com}. As an A-module it equals the maximal commutative A-module quotient of A.

3.3.7 Lemma. The obvious functor $\mathcal{M}(X,A_{com})^{ch}_{com} \to \mathcal{M}(X,A)^{ch}_{com}$ is an equivalence of pseudo-tensor categories.

Proof. An equivalent statement: the action of A on any commutative A-module factors through A_{com}. The proof is left to the reader. □

3.3.8. Let R be a commutative chiral algebra, so R^ℓ is a commutative $\mathcal{D}_X$-algebra. The functor α identifies R^ℓ-actions on a $\mathcal{D}_X$-module M with *commutative* chiral R-actions. We get a canonical equivalence of categories

$$\mathcal{M}(X,R^\ell) \xrightarrow{\sim} \mathcal{M}(X,R)_{com}. \tag{3.3.8.1}$$

We leave it to the reader to check that a chiral operation φ between $R^\ell[\mathcal{D}_X]$-modules is R^ℓ-polylinear (see 3.1.13(ii)) if and only if it satisfies Leibnitz rule with respect to the chiral actions of R on these modules.[121] So we have a canonical equivalence of pseudo-tensor categories

$$\mathcal{M}(X,R^\ell)^{ch} \xrightarrow{\sim} \mathcal{M}(X,R)^{ch}_{com}. \tag{3.3.8.2}$$

3.3.9. Everything said in this section remains valid if we replace $\mathcal{M}(X)^{ch}$ by $\mathcal{M}(X,R^\ell)^{ch}$. We leave it to the reader to check that (3.3.8.2) identifies chiral algebras in $\mathcal{M}(X,R^\ell)$ with pairs (A,i) where A is a chiral algebra on X and $i: R \to A$ is a morphism of chiral algebras such that A is a commutative R-module; we call such pair (A,i) a *chiral R^ℓ-algebra* or a

[121] For a generalization of this statement to the case of arbitrary chiral algebras see 3.3.19.

chiral algebra on SpecR^ℓ. An A-module in $\mathcal{M}(X,R^\ell)^{ch}$ is the same as an A-module M such that the R-action on M (via the structure morphism i) is commutative; such objects are called *R-commutative A-modules* or *A-modules on* Spec$R$.

Recall that $R^\ell[\mathcal{D}_X]$-modules have local nature with respect to étale topology of SpecR^ℓ (see 2.3.6). R^ℓ-polylinear chiral operations also have local nature. So for every algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the pseudo-tensor category $\mathcal{M}(\mathcal{Y})^{ch}$ of $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules (see 2.3.6), the category $\mathcal{CA}(\mathcal{Y})$ of chiral algebras on $\mathcal{Y}$, and for $\mathcal{A} \in \mathcal{CA}(\mathcal{Y})$ the pseudo-tensor category $\mathcal{M}(\mathcal{Y},\mathcal{A})^{ch}$ of $\mathcal{A}$-modules on $\mathcal{Y}$, as well as the corresponding sheaves of categories on $\mathcal{Y}_{\acute{e}t}$.

Example. If $\mathcal{Y}$ is constant, i.e., $\mathcal{Y} = X \times Y$ with a "trivial" connection, then $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules are the same as Y-families of $\mathcal{D}$-modules on X. Similarly, chiral algebras on $\mathcal{Y}$ are the same as Y-families of chiral algebras on X, etc.

3.3.10. Let us explain why coisson algebras are "classical limits" of chiral algebras.

Let A_t be a one-parameter flat family of chiral algebras, i.e., A_t is a chiral $k[t]$-algebra which is flat as a $k[t]$-module (see 3.3.9). Assume that $A_{t=0} := A/tA$ is a commutative chiral algebra. This means that the $*$ bracket $[\ \ ]_t$ on A_t is divisible by t. Thus $\{\ \ \}_t := t^{-1}[\ \ ]$ is a Lie* bracket on A_t; the corresponding Lie* algebra acts on the chiral algebra A_t (see 3.3.3) by adjoint action. Reducing this picture modulo t we see[122] that $A^\ell_{t=0}$ is a commutative $\mathcal{D}_X$-algebra, and $\{\ \ \} := \{\ \ \}_{t=0}$ is a coisson bracket on it.

Remark. The above construction is also a direct consequence of 3.2.5 and 1.4.27.

3.3.11. Let A be a Lie^{ch} algebra. A *filtration* on A is an increasing sequence of $\mathcal{D}$-submodules $A_0 \subset A_1 \subset .. \subset A$ such that $\cup A_i = A$ and $\mu(j_*j^*A_i \boxtimes A_j) \subset \Delta_* A_{i+j}$. Then gr$A$ is a Lie^{ch} algebra in the obvious way.

If A is a chiral algebra then, unless it is stated explicitely otherwise, we assume that our filtration is *unital*, i.e., $1_A \in A_0$. Then gr.A is a chiral algebra. The filtration is *commutative* if gr.A is a commutative chiral algebra. Then gr.A^ℓ is a coisson algebra: the coisson bracket $\{\ \ \} \in P_2^*(\{\mathrm{gr}_i A, \mathrm{gr}_j A\}, A_{i+j-1})$ comes from the Lie* bracket on A.

[122] Look in 3.3.3.

Set $A_t := \oplus A_i$. This is a chiral $k[t]$-algebra in the obvious way[123] which is $k[t]$-flat. One has $A_{t=1} = A$, $A_{t=0} = \mathrm{gr}A$. If our filtration is commutative then we are in the situation of 3.3.10, and the above coisson bracket equals the one from 3.3.10.

§4 Local associative algebras

We assign to every chiral algebra A certain sheaf of topological associative algebras A^{as} equipped with a connection which governs the category of A-modules (and uniquely determines A itself).

3.4.1. Let A be a chiral algebra on X. Now, as promised in the beginning of this section, we are going to construct a sheaf of topological associative algebra A^{as} on X equipped with a connection ∇.[124] It yields a description of the category of A-modules similar to the one given in 3.3.8 in the commutative case.

Here is a short definition of A^{as}. Consider the forgetting of the A-action functor which assigns to every A-module M the same M considered as a plain $\mathcal{O}_Y$-module. Our A^{as} is the sheaf of endomorphisms of this functor. Unfortunately this definition says little about the structure of A^{as}. So we will give another "concrete" definition of A^{as} returning to the previous one later.

3.4.2. Let Y be a scheme[125] which we assume to be quasicompact and quasiseparated,[126] $x \in X(Y)$, $i_x : Y \hookrightarrow X \times Y$ be the graph of x, and $j_x : U_x \hookrightarrow X \times Y$ its complement. For a chiral algebra A consider the chiral algebras $A \boxtimes \mathcal{O}_Y$ and $j_{x*}j_x^*(\mathcal{A} \boxtimes \mathcal{O}_Y)$ on $X \times Y$ (see Example above). Denote by Ξ_x^{as} the set of chiral subalgebras $A_\xi \subset j_{x*}j_x^*(\mathcal{A} \boxtimes \mathcal{O}_Y)$ which coincide with $A \boxtimes \mathcal{O}_Y$ over U_x. Notice that if $A_\xi, A_{\xi'}$ are in Ξ_x^{as} then such is $A_\xi \cap A_{\xi'}$, so Ξ_x^{as} is a topology on $j_{x*}j_x^*(\mathcal{A} \boxtimes \mathcal{O}_Y)$ at x (we use terminology from 2.1.13 in version of 2.1.15).

[123] The chiral product on A_t is compatible with the grading and comes from the chiral product on A; multiplication by t is the sum of embeddings $A_i \hookrightarrow A_{i+1}$.

[124] A physicist would probably call A^{as} "the sheaf of observables".

[125] The reader can assume that Y is a point; the general case brings purely notational complications.

[126] The assumption is needed to assure that topological $\mathcal{O}_Y$-modules have local nature, i.e., proobjects in the category of quasicoherent $\mathcal{O}$-modules form a sheaf of categories on Y_{Zar}, and our constructions are compatible with localization of Y.

The fibers $A^\ell_{\xi x} := i_x^* A^\ell_\xi = i_x^!(j_{x*}j_x^*(\mathcal{A}\boxtimes \mathcal{O}_Y)/A_\xi)$ form a Ξ_x^{as}-projective system of $\mathcal{O}_Y$-modules connected by surjective morphisms. We denote by A_x^{as} its projective limit. Equivalently (see 2.1.15), A_x^{as} is the completion of $i_x^{\cdot}h(j_{x*}j_x^*(\mathcal{A}\boxtimes\mathcal{O}_Y))$ with respect to the Ξ_x^{as}-topology. So every $A^\ell_{\xi x}$ is a quotient of A_x^{as} modulo an open submodule I_ξ.

We leave it to the reader to check (cf. 2.1.15(i)) that the formation A^{as}_{xY} is compatible with the base change (i.e., for every $f: Y'\to Y$ the pro-$\mathcal{O}_{Y'}$-modules $f^*A^{as}_{xY}$ and $A^{as}_{xf\,Y'}$ coincide). In the universal case $Y=X$, $x=id_X$ (so $i_x=\Delta$, $U_x=U$) the corresponding topological $\mathcal{O}_X$-module is denoted simply by A^{as}.

Any automorphism τ of Y such that $id_X\times\tau$ preserves U_x acts on our picture, hence acts on the A_x^{as}. Since our construction is compatible with base change we see that any infinitesimal automorphism of Y acts canonically on A_x^{as}. Therefore if Y is a smooth variety then A_x^{as} carries a canonical flat connection (i.e., it is a topological $\mathcal{D}_Y$-module). In particular the topological $\mathcal{O}_X$-module A^{as} carries a canonical connection ∇.

We are going to define on A_x^{as} a canonical structure of associative algebra such that I_ξ become right ideals. To do this we need an auxiliary lemma. Let B be any chiral algebra on $X\times Y$. Denote by $\mathcal{M}(X\times Y, B)_x$ the category of B-modules on $X\times Y$ supported set-theoretically at (the graph of) x. Let $1_x\in i_x^*B^\ell = i_x^!(j_{x*}j_x^*B/B)$ be the value at x of the unit section of B^ℓ. We may consider 1_x as a section of $j_{x*}j_x^*B/B$ supported scheme-theoretically at x; it is obviously B-central.

3.4.3 Lemma. (i) The obvious functor $\mathcal{M}(X\times Y, j_{x*}j_x^*B)_x\to\mathcal{M}(X\times Y,B)_x$ is an equivalence of categories.

(ii)[127] For every $M\in\mathcal{M}(X\times Y,B)$ the map $\varphi\mapsto\varphi(1_x)$ identifies $\mathrm{Hom}(j_{x*}j_x^*B/B, M)$ with the space of B-central sections of M supported scheme-theoretically at x.

Proof. (i) Let M be a $\mathcal{D}_{X\times Y/Y}$-module supported at the graph of x. We want to show that every B-action on M extends uniquely to a $j_{x*}j_x^*B$-action, and this extension is compatible with morphisms of M's. Notice that for every finite family of $\mathcal{D}_{X\times Y/Y}$-modules

[127] cf. Exercise in 3.3.6.

B_i, $i \in I$, the obvious morphism of $\mathcal{D}_{X^{\tilde{I}}\times Y}$-modules[128]

$$j_*^{(\tilde{I})}j^{(\tilde{I})*}((\underset{I}{\boxtimes} B_i)\boxtimes M) \to j_*^{(\tilde{I})}j^{(\tilde{I})*}((\underset{I}{\boxtimes} j_{x*}j_x^*B_i)\boxtimes M)$$

is an isomorphism. Thus $P_{\tilde{I}}^{ch}(\{j_{x*}j_x^*B_i, M\}, M) \xrightarrow{\sim} P_{\tilde{I}}^{ch}(\{B_i, M\}, M)$. This implies the desired statement (take $B_i = B$).

(ii) Since 1_x generates $j_{x*}j_x^*B/B$ our map is injective. The inverse map is constructed as follows. Assume we have a section m of M as in (ii); let us define the corresponding morphism ψ_m. We may assume that M is generated by m, hence M is supported at x. Thus, by (i), M is a $j_{x*}j_x^*B$-module. Consider the chiral action morphism $j_{x*}j_x^*B \boxtimes M \to \Delta_* M$. Pulling it back by $(id_X \times i_x)^!$ we get a morphism of $\mathcal{D}_{X\times Y/Y}$-modules $\psi : j_{x*}j_x^*B \underset{\mathcal{O}_Y}{\otimes} i_x^! M \to M$. Since m is B-central the morphism $j_{x*}j_x^*B \to M$, $b \mapsto \psi(b\otimes m)$, vanishes on B, so we have $\psi_m : j_{x*}j_x^*B/B \to M$. This is the morphism we want. □

3.4.4. Applying 3.4.3(i) to our situation we see that for a $\mathcal{D}_{X\times Y/Y}$-module supported at x an A_ξ-action on it amounts to a $j_{x*}j_x^*(\mathcal{A}\boxtimes\mathcal{O}_Y)$-action. Thus the categories of A_ξ-modules supported at x are the same for all $A_\xi \in \Xi_x^{as}$, so we denote them simply $\mathcal{M}(X\times Y, A)_x$.

Now for every $\xi \in \Xi_x^{as}$ we have the A_ξ-module $\Phi_\xi := j_{x*}j_x^*(\mathcal{A}\boxtimes\mathcal{O}_Y)/A_\xi$ supported at x. For $A_{\xi'} \subset A_\xi$ the canonical projection $\Phi_{\xi'} \twoheadrightarrow \Phi_\xi$ is a morphism of $A_{\xi'}$-modules. So, according to 3.4.3(i), we have a Ξ_x^{as}-projective system $\Phi = \{\Phi_\xi\}$ in $\mathcal{M}(X\times Y, A)_x$.

One has $i_x^!\Phi = A_x^{as}$. The elements $1_x \in A_{\xi x}^\ell$ form a section 1_x of A_x^{as}.

Take any $M \in \mathcal{M}(X\times Y, A)_x$. For every section γ of $i^! M$ its centralizer (see 3.3.6; here we consider γ as a section of M) belongs to Ξ_x^{as}. Therefore, by 3.4.3(ii), we have

$$\text{(3.4.4.1)} \qquad \mathrm{Hom}(\Phi, M) \xrightarrow{\sim} \Gamma(Y, i_x^! M), \quad \psi \mapsto \psi(1_x).$$

In particular, $\Gamma(Y, A_x^{as}) = \mathrm{End}\Phi$. We define on $\Gamma(Y, A_x^{as})$ the structure of an associative algebra so that this identification becomes an antiisomorphism of algebras (so Φ is a right $\Gamma(Y, A_x^{as})$-module). The section 1_x is the unit in $\Gamma(Y, A_x^{as})$. We see that every $\Gamma(Y, i_x^! M)$ is a left unital $\Gamma(Y, A_x^{as})$-module. By construction the product and the action

[128] Here $\tilde{I} := I \sqcup \cdot$.

are continuous.[129] Localizing this picture with respect to Y we see that A_x^{as} is a sheaf of associative topological $\mathcal{O}_Y$-algebras and $i_x^! M$ is a discrete left A_x^{as}-module.

Exercise. Recall that the Lie* bracket on A yields a Lie bracket on $h_x(j_{x*}j_x^*A \boxtimes \mathcal{O}_Y) = i_x^{\cdot} h(j_{x*}j_x^*A \boxtimes \mathcal{O}_Y)$. Show that the obvious morphism[130] with dense image

$$h_x(j_{x*}j_x^*A \boxtimes \mathcal{O}_Y) \to A_x^{as} \tag{3.4.4.2}$$

is a morphism of Lie $\mathcal{O}_Y$-algebras.

Remark. Assume that Y is a point, i.e., x is a closed point of X and $j_x : U_x \hookrightarrow X$ is its complement. Then $A_x^{as} = (j_{x*}j_x^*A)_x^{as}$, so our topological associative algebra depends only on the restriction of A to U_x. If B is a chiral algebra on U_x we set $B_x^{as} := (j_{x*}B)_x^{as}$.

Denote by A_x^{as}mod the category of discrete left unital A_x^{as}-modules quasicoherent as $\mathcal{O}_Y$-modules.

3.4.5 Lemma. The functor $i_x^! : \mathcal{M}(X \times Y, A)_x \to A_x^{as}\mathrm{mod}$ is an equivalence of categories.

Proof. Let us define the inverse functor. Recall that by Kashiwara's lemma (see 2.1.3, 2.1.15) the functor $i_x^!$ is an equivalence between the category of right $\mathcal{D}_{X\times Y/Y}$-modules supported at x and that of quasicoherent $\mathcal{O}_Y$-modules. The inverse functor sends an $\mathcal{O}_Y$-module V to $i_{x*}V := (j_{x*}j_x^*\omega_X \boxtimes V)/\omega_X \boxtimes V$.

Now assume that V is a discrete A_x^{as}-module. Then $i_{x*}V$ is a chiral $A \boxtimes \mathcal{O}_Y$-module. Indeed, by Kashiwara's lemma, one has $i_{x*}A_x^{as} = \Phi$. So, since i_{x*} commutes with direct limits, we have[131] $i_{x*}V = \Phi \underset{A_x^{as}}{\otimes} p_Y^{\cdot} V$ which yields the promised $A \boxtimes \mathcal{O}_Y$-module structure. The functor $i_{x*} : A_x^{as}\mathrm{mod} \to \mathcal{M}(X \times Y, A)_x$ is obviously inverse to $i_x^!$. □

3.4.6. Let us consider the universal case of the topological associative algebra A^{as} on X (so we have $Y = X$ and $i_x = \Delta$, see 3.4.2). The canonical connection ∇ on A^{as} is obviously compatible with the product.

[129] With respect to the topology on A_x^{as} defined in 3.4.2 and the discrete topology on $i_x^! M$.

[130] It comes from the maps $j_{x*}j_x^*A \boxtimes \mathcal{O}_Y \to \Phi_\xi$ together with identifications $i_x^!\Phi_\xi = h_x\Phi_\xi$.

[131] Here $p_Y : X \times Y \to Y$ is the projection.

Let M be an A-module. Let us define on M^ℓ a canonical A^{as}-action compatible with connections. Consider the $A \boxtimes \mathcal{O}_X$-module $M \boxtimes \mathcal{O}_X$ equipped with the obvious connection along the second variable. The module $(j_* j^* M \boxtimes \mathcal{O}_X)/M \boxtimes \mathcal{O}_X$ supported at the diagonal carries the same structure. Therefore $i^!_x((j_* j^* M \boxtimes \mathcal{O}_X)/M \boxtimes \mathcal{O}_X)$ is an A^{as}-module equipped with a connection, and the A^{as}-action is obviously compatible with connections. This module equals M^ℓ as a plain left $\mathcal{D}_X$-module, and we are done.

Our canonical connection ∇ defines on $A^{as}[\mathcal{D}_X] := A^{as} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ the structure of an associative algebra.[132] A *discrete* $A^{as}[\mathcal{D}_X]$-*module* is a left unital $A^{as}[\mathcal{D}_X]$-module which is discrete as an A^{as}-module and quasicoherent as an $\mathcal{O}_X$-module. Equivalently, this is a left discrete A^{as}-module, quasicoherent as an $\mathcal{O}_X$-module, equipped with a connection such that the A^{as}-action morphism is horizontal, or a $\mathcal{D}_X$-module equipped with a left discrete horizontal A^{as}-action. We denote the category of discrete $A^{as}[\mathcal{D}_X]$-modules by $\mathcal{M}^\ell(X, A^{as})$.

3.4.7 Proposition. The functor $\mathcal{M}(X, A) \to \mathcal{M}^\ell(X, A^{as})$, $M \mapsto M^\ell$, is an equivalence of categories.

Proof. The equivalence from 3.4.5 identifies discrete $A^{as}[\mathcal{D}_X]$-modules with diagonal supported chiral $A \boxtimes \mathcal{O}_X$-modules equipped with a connection along the second variable compatible with the chiral action. By Kashiwara's lemma these are the same as A-modules on X. The resulting equivalence is inverse to the one of the proposition. □

Now it is clear that our A^{as} coincides with A^{as} defined back in 3.4.1.

3.4.8. For a $\mathcal{D}_X$-module M the above 1-1 correspondence between horizontal discrete A^{as}-actions and chiral A-actions on M can be written explicitly as follows.

The non-quasicoherent $\mathcal{O}_X$-module $A^{as} \otimes M$ carries a topology whose opens are subsheaves V that satisfy the following property: for every local section $m \in M$ there exist an open ideal $I_\xi \subset A^{as}$ such that $I_\xi \otimes m \subset V$. It is clear that any open contains a smaller open V such that it is an $\mathcal{O}$-submodule and $A^{as} \otimes M/V$ is a quasicoherent $\mathcal{O}_X$-module.

[132] It is characterized by properties that the maps $A^{as}, \mathcal{D}_X \to A^{as}[\mathcal{D}_X]$, $a \mapsto a \otimes 1, \partial \mapsto 1 \otimes \partial$, are morphisms of algebras, and for $a \in A^{as}, \tau \in \Theta_X$ one has $\tau a - a\tau = \tau(a)$.

Denote the completion by $A^{as}\hat{\otimes}M$. The $\mathcal{D}_X$-action on $A^{as}\otimes M$ extends by continuity to our completion.

The sheaf $j_*j^*A\boxtimes M$ carries a topology whose opens are subsheaves W that satisfy the following property: for every local section $m\in M$ there exists $A_\xi\subset j_*j^*(A\boxtimes \mathcal{O}_X)$ in Ξ^{as}_Δ such that $A_\xi\otimes m\subset W$. Every open contains a smaller open W such that W is a quasicoherent $\mathcal{D}_X\boxtimes\mathcal{O}_X$-module. The completion $(j_*j^*A\boxtimes M)\hat{}$ is a topological $\mathcal{D}_{X\times X}$-module.

The identifications of $\mathcal{D}_X\boxtimes\mathcal{O}_X$-modules $j_*j^*A\boxtimes\mathcal{O}_X/A_\xi=\Delta_*A^\ell_{\xi\Delta}=\Delta_*(A^{as}/I_\xi)$ (see 3.4.2) yield a canonical isomorphism of topological $\mathcal{D}_{X\times X}$-modules

$$(j_*j^*A\boxtimes M)\hat{}\ \xrightarrow{\sim}\ \Delta_*(A^{as}\hat{\otimes}M). \tag{3.4.8.1}$$

Now every chiral A-action on M is automatically continuous in the above topology, so it can be considered as a morphism $(j_*j^*A\boxtimes M)\hat{}\to\Delta_*M$. Similarly, every discrete A^{as}-action on M can be cosidered as a morphism $A^{as}\hat{\otimes}M\to M$. A chiral A-action μ_M is identified with an A^{as}-action $\cdot_M$ if (3.4.8.1) identifies $\Delta_*(\cdot_M)$ with μ_M.

3.4.9. Let us lift the equivalence of 3.4.7 to the pseudo-tensor level. Let M_i^ℓ, N^ℓ be discrete $A^{as}[\mathcal{D}_X]$-modules. A chiral operation $\varphi:\ j_*^{(I)}j^{(I)*}\boxtimes M_i\to\Delta_*^{(I)}N$ is A^{as}-*polylinear* if it satisfies the following condition. For $i\in I$ consider the (non-quasicoherent) associative $\mathcal{D}_{X^I}$-algebra $p_i^*A^{as}$ (here $p_i:X^I\to X$ is the projection). Then $j_*^{(I)}j^{(I)*}\boxtimes M_i$ and $\Delta_*^{(I)}N$ are $p_i^*A^{as}$-modules in the obvious way.[133] We demand that φ is a morphism of $p_i^*A^{as}$-modules for every $i\in I$.

A^{as}-polylinear chiral operations are stable with respect to compositions, so discrete $A^{as}[\mathcal{D}_X]$-modules form a pseudo-tensor category $\mathcal{M}^\ell(X,A^{as})^{ch}$.

Remark.[134] In fact, by Remark in 3.5.2, $\Delta_*^{(I)}N$ is a $\hat{\Delta}_*^{(I)}A^{as}$-module, and the above $p_i^*A^{as}$-action on it comes from the obvious morphism $p_i^*A^{as}\to\hat{\Delta}_*^{(I)}A^{as}$ (see 3.5.5).

[133] The $p_i^*A^{as}$-action on $\Delta_*^{(I)}N$ comes from the A^{as}-action on N via (2.1.3.1) (for $i=\Delta^{(I)}$, $L=A^{as}$, $M=N$).

[134] We use notation from 3.5.2, 3.5.5.

3.4.10 Lemma. A chiral operation φ is A^{as}-polylinear if and only if it satisfies the Leibnitz property with respect to the chiral A-actions on M_i, N. So we have the equivalence of pseudo-tensor categories $\mathcal{M}(X,A)^{ch} \xrightarrow{\sim} \mathcal{M}^\ell(X,A^{as})^{ch}$.

Proof. For $i \in I$ let $\Delta^i : X^I \to X \times X^I$ be the diagonal $x = x_i$. Denote by $\tilde{A}$ the chiral algebra $A \boxtimes \mathcal{O}_{X^I}$ localized at all the diagonals Δ^i, $i \in I$. Let Ξ^i be the topology on $\tilde{A}$ formed by chiral subalgebras A^i_ξ that coincide with $\tilde{A}$ outside of the diagonal Δ^i and such that the multiplication on A^i_ξ by an equation of every diagonal $\Delta^{i'}$, $i' \neq i$, is invertible.

The topologies Ξ^i on $\tilde{A}$ are independent, i.e., every I-tuple $\{A^i_\xi\}_{i\in I}$ is self-transversal: for every i one has $A^i_\xi + \bigcap_{i'\neq i} A^{i'}_\xi = \tilde{A}$.

For a section $m \in j^{(I)}_* j^{(I)*} \boxtimes M_i$ consider the morphism[135] $\mu^i_m : \tilde{A} \to \Delta^i_* j^{(I)}_* j^{(I)*} \boxtimes M_i$ of the chiral A-action on m along the ith variable. It is continuous with respect to the Ξ^i-topology, i.e., μ^i_m vanishes on some $A^i_\xi(m) \in \Xi^i$.

Our φ satisfies the Leibnitz rule if for every m the sum of morphisms μ^i_m equals to the chiral action of A on $\varphi(m)$.

By transversality it suffices to check this condition on every $B_i(m) := \bigcap_{i'\neq i} A^{i'}_\xi(m)$. Here this condition amounts to A^{as}-linearity of φ along the ith variable, and we are done. $\square$

3.4.11. Assume now that our A is commutative. Then the algebras A^{as}_x, A^{as} are also commutative. We already met A^{as}_x in situation when x is a single point, $x \in X(k)$ (see 2.4.8). The "geometric" description of A^{as}_x from 2.4.9 generalizes immediately to the general situation. Namely, for $x \in X(Y)$ as in 3.4.2 and a commutative quasicoherent $\mathcal{O}_Y$-algebra F we define a *non-quasicoherent* $\mathcal{O}_{X\times Y}$-algebra $O_x \hat{\otimes} F$ as the formal completion of $\mathcal{O}_X \boxtimes F$ at (the graph) of x; let $K_x \hat{\otimes} F$ be its localization with respect to an equation of x. Our $O_x \hat{\otimes} F \subset K_x \hat{\otimes} F$ are $\mathcal{D}_{X\times Y/Y}$-algebras, i.e., they carry an obvious connection along X. Now one has a canonical identification

$$(3.4.11.1) \qquad \mathrm{Hom}(A^{as}_x, F) = \mathrm{Hom}(A^\ell \boxtimes \mathcal{O}_Y, K_x \hat{\otimes} F) = \mathrm{Hom}(j_{x*} j^*_x (A^\ell \boxtimes \mathcal{O}_Y), K_x \hat{\otimes} F)$$

where the first Hom means continuous morphisms of $\mathcal{O}_Y$-algebras and the next ones mean

[135] Here $\tilde{I} := I \sqcup \cdot$.

morphisms of $\mathcal{D}_{X\times Y/Y}$-algebras. For the construction see the proof of 2.4.9. As we mentioned in Remark of 2.4.9, (3.4.11.1) is the subset of algebra morphisms in (3.5.4.1).

3.4.12 Lemma. If $\varphi : A^\ell \to B^\ell$ is a formally smooth morphism of commutative $\mathcal{D}_X$-algebras then the corresponding morphism of topological $\mathcal{O}_Y$-algebras $A_x^{as} \to B_x^{as}$ is formally smooth. Same is true if we replace "formally smooth" by "formally étale".[136]

Proof. Assume that φ is formally smooth. Let F be a commutative k-algebra, $I \subset F$ an ideal such that $I^2 = 0$. Let $f : \mathrm{Spec}F \to \mathrm{Spec}A_x^{as}$, $g : \mathrm{Spec}F/I \to \mathrm{Spec}B_x^{as}$ be φ-compatible morphisms.[137] We want to find a lifting $h : \mathrm{Spec}F \to \mathrm{Spec}B_x^{as}$ of f that extends g.

According to (3.4.11.1) we may rewrite f, g as morphisms of $\mathcal{D}_{X\times Y/Y}$-algebras $f' : A^\ell \boxtimes \mathcal{O}_Y \to K_x \hat{\otimes} F$, $g' : B^\ell \boxtimes \mathcal{O}_Y \to K_x \hat{\otimes} F/I$. Here $K_x \hat{\otimes} F$, $K_x \hat{\otimes} F/I$ are non-quasicoherent sheaves of $\mathcal{D}_{X\times Y/Y}$-algebras.

Our problem is local (see [RG]) so we can assume that X and Y are affine. Denote by Γ the functor of global sections over $X \times Y$. So we have commutative $\mathcal{O}(X)$-algebras $\Gamma(K_x \hat{\otimes} F)$, $\Gamma(K_x \hat{\otimes} F/I)$ equipped with a connection along X. The second algebra is a quotient of the first one modulo an ideal of square 0.

Our f', g' amount to maps of commutative $\mathcal{O}(X)$-algebras $\Gamma f' : A^\ell(X) \to \Gamma(K_x \hat{\otimes} F)$, $\Gamma g' : B^\ell(X) \to \Gamma(K_x \hat{\otimes} F/I)$ compatible with connections along X. Since φ is a formally smooth morphism of $\mathcal{D}_X$-algebras we can find a lifting $\Gamma h' : B^\ell(X) \to \Gamma(K_x \hat{\otimes} F)$ of $\Gamma f'$ that extends $\Gamma g'$. As above, $\Gamma h'$ is the same as a morphism $h : \mathrm{Spec}F \to \mathrm{Spec}B_x^{as}$ which is the desired lifting.

The case of formally étale φ is left to the reader. □

§5 Chiral algebras as factorization algebras.

In this section we give another definition of chiral algebras in terms of $\mathcal{O}$-modules on the space of finite subsets of X which satisfy certain factorization property for disjoint

[136] For the definition of formally smooth and formally étale morphisms of commutative $\mathcal{D}_X$-algebras (or, more generally, algebraic $\mathcal{D}_X$-spaces) see 2.3.17.

[137] $\mathrm{Spec}A_x^{as}, \mathrm{Spec}B_x^{as}$ are ind-schemes.

union of subsets. This approach is technically very convenient (see §§7, 11). Also some important chiral algebras of geomeric nature do arise in this manner.

3.5.1 A topological digression. This subsection plays an important but purely motivational role.

Let X be a topological space. Its *exponential* $\mathcal{E}xp(X)$ is the set of all its finite subsets; for $S \subset X$ the corresponding point of $\mathcal{E}xp(X)$ is denoted by $[S]$. For any finite index set I we have an obvious map $r_I : X^I \to \mathcal{E}xp(X)$. We equip $\mathcal{E}xp(X)$ with the strongest topology such that all maps r_I are continuous. Thus $\mathcal{E}xp(X)$ is the disjoint union of the base point $[\emptyset]$ and the subspace $\mathcal{R}(X)$ called *Ran's space*[138] associated to X.

Here are some properties of $\mathcal{E}xp$ and $\mathcal{R}$:

(i) The subspaces $\mathcal{E}xp(X)_n$ that consists of $[S]$ with $|S| \leq n$ form an increasing filtration on $\mathcal{E}xp(X)$; the projection $r_n : X^n \to \mathcal{E}xp(X)_n$ is an open map. Same for $\mathcal{E}xp$ replaced by $\mathcal{R}$. One has $\mathcal{R}(X)_0 = \emptyset$, $\mathcal{R}(X)_1 = X$, $\mathcal{R}(X)_2 = \mathrm{Sym}^2 X$. A stratum $\mathcal{E}xp(X)^o_n := \mathcal{E}xp(X)_n \smallsetminus \mathcal{E}xp(X)_{n-1}$ (the space of configurations of n points in X) equals the complement to diagonals in $\mathrm{Sym}^n X$.

(ii) For any surjection $J \twoheadrightarrow I$ we have the corresponding diagonal embedding $\Delta^{(J/I)} : X^I \to X^J$. One has $r_J \Delta^{(J/I)} = r_I$, and $\mathcal{E}xp(X)$ is the inductive limit of the topological spaces X^I with respect to all diagonal embeddings. Same for $\mathcal{R}(X)$ (add the condition $I \neq \emptyset$).

(iii) $\mathcal{E}xp(X)$ is a commutative monoid, and $\mathcal{R}(X)$ is a commutative semigroup with respect to the operation $[S] \circ [S'] := [S \cup S']$. The map $\mathcal{E}xp(X_1) \times \mathcal{E}xp(X_2) \to \mathcal{E}xp(X_1 \sqcup X_2)$, $([S_1], [S_2]) \mapsto [S_1] \circ [S_2] = [S_1 \sqcup S_2]$, is a homeomorphism.

(iv) **Proposition.** All the homotopy groups of $\mathcal{E}xp(X)$ vanish. So if X is a connected CW complex then $\mathcal{R}(X)$ is contractible.

Remark. This result (in fact, a stronger one) is due to Curtis and Nhu (see Lemmas 3.3 and 3.6 from [CN]); simple-connectedness of $\mathcal{R}(X)_3$ was proved by Bott [Bo] who showed that $\mathcal{R}(S^1)_3 = S^3$. An algebraic proof, similar to given below, was communicated to us by

[138] $\mathcal{R}(X)$ was introduced by Borsuk and Ulam [BU]; Ziv Ran [R] was, probably, first to consider it in algebraic geometry.

Jacob Mostovoy.

Proof of the proposition. Below we write the group law on homotopy groups π_i, $i \geq 1$, multiplicatively. Let $v \in \mathcal{E}xp(X)$ be a base point. Consider the maps

$$\mathcal{E}xp(X) \xrightarrow{\delta} \mathcal{E}xp(X) \times \mathcal{E}xp(X) \xrightarrow{\circ} \mathcal{E}xp(X) \tag{3.5.1.1}$$

where δ is the diagonal embedding; one has $\circ\delta = id_{\mathcal{E}xp(X)}$. For $\gamma \in \pi_i(\mathcal{E}xp(X), v)$ one has $\delta(\gamma) = i_1(\gamma) \cdot i_2(\gamma) \in \pi_i(\mathcal{E}xp(X) \times \mathcal{E}xp(X), (v, v))$ where $i_1, i_2 : \mathcal{E}xp(X) \hookrightarrow \mathcal{E}xp(X) \times \mathcal{E}xp(X)$ are the embeddings $[S] \mapsto ([S], v)$, $(v, [S])$. Therefore $\gamma = \circ\delta(\gamma) = (v \circ \gamma)^2$. We see that $\gamma = 1$ if $v \circ \gamma = 1$. Notice that $v \circ v \circ \gamma = v \circ \gamma$, so, replacing in the previous formula γ by $v \circ \gamma$, we get $v \circ \gamma = (v \circ \gamma)^2$. Thus $v \circ \gamma = 1$, so $\gamma = 1$, and we are done. $\square$

Remark. For $v = [S] \in \mathcal{E}xp(X)$ let $\mathcal{E}xp(X, v) \subset \mathcal{E}xp(X)$ be the subspace whose points are finite subsets of X that contain S. The same argument as above shows that all homotopy groups of $\mathcal{E}xp(X, v)$ vanish. This also follows from the existence of a retraction $\rho : \mathcal{E}xp(X) \to \mathcal{E}xp(X, v)$ (put $\rho(u) = u \circ v$).

A *factorization algebra* is a sheaf B of vector spaces on $\mathcal{E}xp(X)$ (or, may be, an object of the appropriate derived category) together with natural identifications $B_{[S_1]} \otimes B_{[S_2]} \xrightarrow{\sim} B_{[S_1]\circ[S_2]}$ defined for *disjoint* S_1, S_2, which are associative and commutative in the obvious manner. Denote by B_X the restriction of B to $X \subset \mathcal{E}xp(X)$. The restriction of B to $\mathcal{E}xp(X)^o_i$ equals to the restriction of $\mathrm{Sym}^i(B_X)$ to $\mathrm{Sym}^i X \smallsetminus$(diagonals). A morphism between factorization algebras is completely determined by its restriction to B_X's, so one may consider B as a sheaf B_X on X equipped with certain *factorization* structure. This structure has X-local nature. So one may speak of factorization algebras *on* X.

3.5.2. We want to play with factorization algebras in the algebro-geometric setting of quasicoherent sheaves. From now on X is our curve. Notice that $\mathcal{R}(X)_i$ are *not* algebraic varieties for $i \geq 3$.[139] Nevertheless, we will consider $\mathcal{O}$- and $\mathcal{D}$-modules on $\mathcal{R}(X)$; they will be introduced directly using 3.5.1(ii).

[139] Exercise: show that every germ of a holomorphic function on $\mathcal{R}(X)_3$ at a point $x \in X \subset \mathcal{R}(X)_3$ is constant on X.

An $\mathcal{O}$-*module on* $\mathcal{R}(X)$, or an $\mathcal{O}_{\mathcal{R}(X)}$-*module*, is a rule F that assigns to every finite non-empty I a quasicoherent $\mathcal{O}_{X^I}$-module F_{X^I} and to every $\pi : J \twoheadrightarrow I$ an identification $\nu^{(\pi)} : \Delta^{(\pi)*}F_{X^J} \xrightarrow{\sim} F_{X^I}$ compatible with composition of π's. We demand that F_{X^I} have no non-zero local sections supported at the diagonal divisor.

$\mathcal{O}_{\mathcal{R}(X)}$-modules form a k-category $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$. This category is exact in the sense of [QK].[140] It is also a tensor k-category in the obvious way. The unit object is $\mathcal{O}_{\mathcal{R}(X)}$ (:= the $\mathcal{O}$-module with components $\mathcal{O}_{X^I}$).

Replacing in the above definition $\mathcal{O}$-modules by left $\mathcal{D}$-modules we get the notion of left $\mathcal{D}$-module on $\mathcal{R}(X)$. These objects form an exact tensor k-category $\mathcal{M}^\ell(\mathcal{R}(X))$ with unit object $\mathcal{O}_{\mathcal{R}(X)}$.

3.5.3 Lemma. Let F be an $\mathcal{O}_{\mathcal{R}(X)}$-module, J a finite non-empty set.

(i) F_{X^J} is flat along every diagonal $X^I \hookrightarrow X^J$, i.e., $\mathcal{T}or_i(F_{X^J}, \mathcal{O}_{X^I}) = 0$ for $i > 0$.

(ii) Let $\ell : V \hookrightarrow X^J$ be the complement to the diagonal strata of codimension ≥ 2. Then the map $F_{X^J} \to \ell_* \ell^* F_{X^J}$ is an isomorphism.[141]

Proof. (i) Our statement amounts to to the claim that for every $\pi : J \twoheadrightarrow I$ the projection $L\Delta^{(\pi)*}F_{X^J} \to \Delta^{(\pi)*}F_{X^J}$ is an isomorphism. Let $\tilde{\nu}^{(\pi)} : L\Delta^{(\pi)*}F_{X^J} \to F_{X^I}$ be its composition with the structure isomorphism $\nu^{(\pi)}$. Our $\tilde{\nu}^{(\pi)}$'s are compatible with composition of π's, so (since every diagonal embedding is composition of diagonal embeddings of codimension 1) it suffices to treat the codimension 1 case. Here our statement amounts to the property that F_{X^J} has no sections supported on X^I, and we are done.

(ii) We know that our map is injective. For every diagonal $X^I \subset X^J$ of codimension ≥ 2 one has $\mathcal{E}xt^1(\mathcal{O}_{X^I}, F_{X^J}) = 0$ by (i) (indeed, $\mathcal{E}xt^j(\mathcal{O}_{X^I}, F_{X^J})$ is locally isomorphic to $\mathcal{T}or_{|J|-|I|-j}(\mathcal{O}_{X^I}, F_{X^J})$[142]). This implies surjectivity of our map. Namely, take the smallest m such that $C := \mathrm{Coker}(F_{X^J} \to \ell_*\ell^*F_{X^J})$ vanishes outside of diagonals of dimension $\leq m$. Then for a diagonal $X^I \subset X^J$ of dimension m our C is non-zero on $X^I \smallsetminus$ (diagonals), hence $\mathcal{H}om(\mathcal{O}_{X^I}, C) \neq 0$. The latter sheaf equals $\mathcal{E}xt^1(\mathcal{O}_{X^I}, F_{X^J})$, and we are done. $\square$

[140] A short sequence is exact if and only if all the corresponding sequences of $\mathcal{O}_{X^n}$-modules are exact. We do not know if $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$ is an abelian category.

[141] Here ℓ_* is the naive sheaf-theoretic direct image (no higher derived functors are considered).

[142] Compute them using a Koszul resolution of $\mathcal{O}_{X^I}$.

3.5.4. For $\pi : J \twoheadrightarrow I$ let $j^{[J/I]} : U^{[J/I]} \hookrightarrow X^J$ be the complement to all the diagonals that are transversal to $\Delta^{(J/I)} : X^I \hookrightarrow X^J$. So $U^{[J/I]} = \{(x_j) : x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2)\}$.

Let B be an $\mathcal{O}$-module on $\mathcal{R}(X)$. A *factorization structure* on B is a rule that assigns to every $J \twoheadrightarrow I$ an isomorphism of $\mathcal{O}_{U^{[J/I]}}$-modules

$$\kappa_{[J/I]} : j^{[J/I]*}(\underset{I}{\boxtimes} B_{X^{J_i}}) \xrightarrow{\sim} j^{[J/I]*} B_{X^J}. \tag{3.5.4.1}$$

We demand that κ's are mutually compatible: for $K \twoheadrightarrow J$ the isomorphism $\kappa_{[K/J]}$ coincides with the composition $\kappa_{[K/I]}(\boxtimes \kappa_{[K_i/J_i]})$.

Notice that $\kappa_{[J/J]}$ identifies $j^{(J)*}B_{X^J}$ with $j^{(J)*}B_X^{\boxtimes J}$, so we have a canonical embedding

$$B_{X^J} \hookrightarrow j^{(J)*}B_X^{\boxtimes J}. \tag{3.5.4.2}$$

$\mathcal{O}$-modules on $\mathcal{R}(X)$ equipped with factorization structure form a tensor category in the obvious manner.[143]

As follows from 3.5.4.2 the functor $B \mapsto B_X$ is faithful. Therefore we can consider an $\mathcal{O}$-module B on $\mathcal{R}(X)$ equipped with a factorization structure as an $\mathcal{O}$-module B_X on X equipped with certain structure which we also call factorization structure. Notice that the factorization structure has X-local nature, so $\mathcal{O}$-modules with factorization structure form a sheaf of tensor categories on the étale topology of X.

In the above discussion one may replace everywhere $\mathcal{O}$-modules by left $\mathcal{D}$-modules.

Remark. If an $\mathcal{O}_{\mathcal{R}(X)}$-module B admits a factorization structure then B is flat if (and only if) B_X is a flat $\mathcal{O}_X$-module.[144]

3.5.5 Definition. A *factorization algebra* B on X is an $\mathcal{O}$-module equiped with a factorization structure which satisfies the following condition:

There exists a global section 1 of B_X such that for every $f \in B_X$ one has $1 \boxtimes f \in B_{X^2} \subset j_*j^*B_X^{\boxtimes 2}$ and $\Delta^*(1 \boxtimes f) = f$.

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[143] This category is not additive.

[144] Proof: Assume that B_X is flat. We want to show that every B_{X^n} is flat. It amounts to vanishing of $Tor_{>0}(B_{X^n}, K)$ for all morphisms $\phi : \mathrm{Spec}K \to X^n$, K is a field. Using induction by n and 3.5.3(i) we can assume that $\phi(\mathrm{Spec}K) \subset U^{(n)}$, and $B_{U^{(n)}}$ is flat by factorization.

Notice that such section 1 is uniquely defined; it is called *the unit* in B.

Factorization algebras form a tensor category[145] which we denote by $\mathcal{FA}(X)$. They are objects of X-local nature, so we have a sheaf of categories $\mathcal{FA}(X_{\text{ét}})$ on the étale topology of X.

The unit object of our tensor category is the "trivial" factorization algebra $\mathcal{O}$. For any $B \in \mathcal{FA}(X)$ there is a unique morphism $\mathcal{O} \to B$. Therefore for a collection B_α of factorization algebras and $B := \otimes B_\alpha$ one has canonical morphisms $\nu_\alpha : B_\alpha \to B$.[146]

Remark. In the category $\mathcal{FA}(X)$ infinite tensor products are well-defined.[147] Namely, if B_α is any collection of factorization algebras then $\otimes B_\alpha$ is the inductive limit of $\underset{\alpha\in S}{\otimes} B_\alpha$ where S runs the set of finite subsets of indices α. Here for $S \subset S'$ the corresponding arrow $\underset{\alpha\in S}{\otimes} B_\alpha \to \underset{\alpha\in S'}{\otimes} B_\alpha$ is $\otimes b_\alpha \mapsto (\otimes b_\alpha) \otimes (\underset{\alpha'\in S'\smallsetminus S}{\otimes} 1'_\alpha)$.

3.5.6. Here is another description of factorization algebras. The reader may skip it and pass directly to 3.5.8.

For a scheme Z let $\mathcal{C}(X)_Z$ be the set of relative effective Cartier divisors in $X \times Z$ over Z. This is a commutative monoid (a submonoid in the group of all Cartier divisors). The usual ordering of Cartier divisors makes $\mathcal{C}(X)_Z$ a category. This is a tensor category with respect to the usual addition of divisors. Our $\mathcal{C}(X)_Z$ form a category $\mathcal{C}(X)$ fibered over the category of schemes Z; this is a sheaf of categories with respect to the flat topology.

Equivalently, objects of $\mathcal{C}(X)_Z$ are subschemes $S \subset X \times Z$ finite, flat, and of finite presentation over Z (see [KM], 1.2.3); the morphisms are embeddings of S's.

Remark. For any $n \geq 0$ the subcategory $\mathcal{C}(X,n)_Z \subset \mathcal{C}(X)_Z$ of divisors of degree n is discrete. The obvious divisor $\in C(X)_{\mathrm{Sym}^n X}$ is universal (the functor $Z \mapsto \mathcal{C}(X,n)_Z$ is representable by $\mathrm{Sym}^n X$).

Consider a pair (B, κ) where:

(i) B is a Morphism from the fibered category $\mathcal{C}(X)$ to that of $\mathcal{O}$-modules. Thus B is a rule that assigns to every $S \in \mathcal{C}(X)$ a quasicoherent $\mathcal{O}_Z$-module $B(S) = B(S)_Z$; for $S \subset T$

[145] We demand that the morphisms preserve units.
[146] Defined as tensor product of id_{B_α} and the canonical morphisms $\mathcal{O} \to B_{\alpha'}$, $\alpha' \neq \alpha$.
[147] Just as in the usual category of *unital* associative algebras.

we have a canonical morphism $B(S) \to B(T)$; everything is compatible with base change. In particular, the universal divisors yield $\mathcal{O}$-modules $B_{\mathrm{Sym}^n X}$ on $\mathrm{Sym}^n X$.

(ii) κ is a rule that assigns to every pair of *mutually disjoint* divisors $S_1, S_2 \in \mathcal{C}(X)_Z$ an identification $\kappa_{S_1,S_2} : B(S_1) \otimes B(S_2) \xrightarrow{\sim} B(S_1 + S_2)$. We demand that these identifications are commutative and associative in the obvious manner, and are compatible with the natural morphisms from (i). So for any finite family of mutually disjoint divisors S_α there is a canonical identification

$$\kappa_{\{S_\alpha\}} : \otimes B(S_\alpha) \xrightarrow{\sim} B(\Sigma S_\alpha). \tag{3.5.6.1}$$

Assume that the following conditions hold:

(a) For every $S, S' \in \mathcal{C}(X)_Z$ such that $S \subset S'$ and $S_{red} = S'_{red}$ the canonical morphism $B(S) \to B(S')$ is an isomorphism.

(b) The $\mathcal{O}$-modules $B_{\mathrm{Sym}^n X}$ on $\mathrm{Sym}^n X$ have no non-zero local sections supported at the diagonal divisor.

(c) One has $B(\emptyset) \neq 0$.[148]

We are going to show that such (B, κ) is the same as a factorization algebra. For the moment we call such object a factorization algebra in the sense of 3.5.6. They form a category which we denote by $\mathcal{FA}(X)'$. This is a tensor category in the obvious manner (one has $(\otimes B_i)(S) = \otimes(B_i(S))$).

Let (B, κ) be a factorization algebra in the sense of 3.5.6. Let us define the corresponding factorization algebra in the sense of 3.5.5 which we denote again by B. For any $I \in \mathcal{S}$ the points $x = x_i$, $i \in I$, form a Cartier divisor of degree $|I|$ in $X \times X^I / X^I$. So we have the corresponding $\mathcal{O}$-module B_{X^I} on X^I (it equals to the pull-back of $B_{\mathrm{Sym}^{|I|} X}$ by the projection $p_{|I|} : X^I \to \mathrm{Sym}^{|I|} X$). These $\mathcal{O}$-modules form an $\mathcal{O}$-module on $\mathcal{R}(X)$ in the sense of 3.5.2: indeed, the pull-back compatibility of B together with (a) provide the isomorphisms $\nu^{(\pi)}$, and (b) assures that B_{X^I} have no sections supported on the diagonal divisors.

[148]As follows from 3.5.6.1 this property just excludes B identically equal to 0.

The $\mathcal{O}$-module on $\mathcal{R}(X)$ we have defined carries a factorization structure (see 3.5.4) defined by isomorphisms 3.5.6.1. It has a unit section 1 (see 3.5.5). To define 1 look at the canonical isomorphism $\kappa_\emptyset : B(\emptyset)^{\otimes 2} \xrightarrow{\sim} B(\emptyset)$. Then $B(\emptyset)$ is a one dimensional vector space[149] generated by an element 1 such that $\kappa_\emptyset(1 \otimes 1) = 1$. Since $\emptyset$ lies in any divisor our 1 yields a section of every $B(S)$, in particular, of B_{X^I}. The property from 3.5.5 is immediate.

Our construction is obviously functorial and compatible with tensor products, so we have defined a canonical tensor functor

$$\mathcal{FA}(X)' \to \mathcal{FA}(X). \tag{3.5.6.2}$$

3.5.7 Proposition. This functor is an equivalence of categories.

Proof. We will construct the inverse functor. Let B be a factorization algebra in the sense of 3.5.5. We want to define the corresponding factorization algebra (B, κ) in the sense of 3.5.6.

First let us construct the $\mathcal{O}$-modules $B_{\mathrm{Sym}^n X}$. One has $\mathrm{Sym}^0 X = \mathrm{Spec} k$ and $B(\emptyset) = k$. For $n > 0$ we consider the projection $p_n : X^n \to \mathrm{Sym}^n X$. The symmetric group Σ_n acts on X^n and on B_{X^n}. Set $B_{\mathrm{Sym}^n X} := (p_{n*} B_{X^n})^{\Sigma_n}$. By 3.5.2 it satisfies condition 3.5.6(b).

Let us define $B(S)_Z$ for every $S \in \mathcal{C}(X)_Z$. Our problem is Z-local, so we may assume that the degree n of S over Z is constant. Such S amounts to a morphism $Z \to \mathrm{Sym}^n X$, and $B(S)_Z$ is the pull-back of $B_{\mathrm{Sym}^n X}$.

Lemma. The canonical morphism $\psi : p_n^* B_{\mathrm{Sym}^n X} \to B_{X^n}$ is an isomorphism.

Proof of the lemma. (a) Let $\ell : V \hookrightarrow X^n$ be the complement to the diagonals of codimension ≥ 2 and $\bar{\ell} : \bar{V} \hookrightarrow \mathrm{Sym}^n X$ be its p_n-image. By 3.5.3(ii) we know that $B_{X^n} = \ell_* \ell^* B_{X^n}$, thus $B_{\mathrm{Sym}^n X} = \bar{\ell}_* \bar{\ell}^* B_{\mathrm{Sym}^n X}$. Since p_n is flat we conclude that it suffices to check that ψ is an isomorphism over V. By factorization it suffices to consider the case $n = 2$.

(b) By étale descent our ψ is an isomorphism over the complement to the diagonal divisor. Therefore ψ is injective (since p_n is flat).

[149] Our $\kappa_\emptyset$ is commutative. It is also injective, so for any $a, b \in B(\emptyset)$ one has $a \otimes b = b \otimes a \in B(\emptyset)^{\otimes 2}$. Together with (c) this implies that $\dim B(\emptyset) = 1$. Exercise: deduce the same fact from associativity of $\kappa_\emptyset$.

(c) It remains to prove that $\psi : p_2^* B_{\mathrm{Sym}^2 X} \to B_{X^2}$ is surjective. Consider its cokernel C. This is a Σ_2-equivariant quotient of B_{X^2} such that $(p_{2*}C)^{\Sigma_2} = 0$. We know that C is supported (set-theoretically) on the diagonal divisor X, so one has $C^{\Sigma_2} = 0$.

Let f be a (local) equation of the diagonal $X \subset X \times X$ which is Σ_2-antiinvariant. Since Σ_2 acts on C by the non-trivial character sgn multiplication by f kills C, i.e., C is supported on the diagonal scheme-theoretically. Thus C is a quotient of the pull-back of B_{X^2} to X, i.e., of B_X. But Σ_2 acts on B_X trivially. So $C = 0$, and we are done. $\square$

Remark. Let $G \subset \Sigma_n$ be a subgroup, so we have the projections $X^n \xrightarrow{\pi_1} G \diagdown X^n \xrightarrow{\pi_2} \mathrm{Sym}^n X$. The lemma implies that $\pi_2^* B_{Sym^n X} = (\pi_{1*} B_{X^n})^G$.

To get the structure from 3.5.6(i) it remains to define for every $S, T \in \mathcal{C}(X)_Z$ such that $S \subset T$ a canonical morphism $B(S) \to B(T)$. We have $T = S + S'$ for an effective Cartier divisor S'. It suffices to define our morphism in the universal situation when $Z = \mathrm{Sym}^n X \times \mathrm{Sym}^m X$ and S, S' are the universal divisors of order n and m respectively. So let $q^{(n)} : \mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^n X$ be the projection, $p : \mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^{n+m} X$ the addition map. We want to define a canonical morphism $q^{(n)*} B_{Sym^n X} \to p^* B_{\mathrm{Sym}^{n+m} X}$. By the above lemma and remark this amounts to a $\Sigma_n \times \Sigma_m$ -equivariant morphism $B_{X^n} \boxtimes \mathcal{O}_{X^m} \to B_{X^{n+m}}$. By definition, our morphism sends $b \boxtimes f \in B_{X^n} \boxtimes \mathcal{O}_{X^m}$ to the section $\kappa(b \boxtimes f 1_B^{\boxtimes m})$. Here κ is the factorization isomorphism (3.5.4.1), and our section belongs to $B_{X^{n+m}}$ (see (3.5.4.2)) by 3.5.3(ii) and the definition of the unit 1_B.

The canonical morphism $B(S) \to B(T)$ so defined is compatible with composition of embeddings $S \subset T$. So we have defined the structure 3.5.6(i). As for the factorization isomorphism 3.5.6(ii), it is defined by the factorization structure (3.5.4.1) in the obvious manner (consider the universal situation and use the above remark).

We leave it to the reader to check the condition (a) of 3.5.6. So we have defined the factorization algebra in the sense of 3.5.6, i.e., we have a functor $\mathcal{FA}(X) \to \mathcal{FA}(X)'$. It follows from the construction (use the lemma) that it is inverse to (3.5.6.2). $\square$

3.5.8 Proposition. Let B be a factorization algebra. Then the $\mathcal{O}$-module B on $\mathcal{R}(X)$ admits a left $\mathcal{D}$-module structure compatible with the factorization structure and such that the section 1 of B_X is horizontal. Such $\mathcal{D}$-module structure is unique.

We call it *the canonical* $\mathcal{D}$-module structure on B.

Proof. Let Y be any smooth variety. For a finite non-empty set I denote by $Y^{<I>}$ the formal completion of Y^I at the diagonal $Y \subset Y^I$. A map $\pi : J \to I$ yields the morphism $\Delta^{(\pi)} : Y^I \to Y^J$, $(y_i) \mapsto (y_j)$, $y_j = y_{\pi(j)}$, hence the morphism of formal schemes $\Delta^{<\pi>} : Y^{<I>} \to Y^{<J>}$. According to Grothendieck, a left $\mathcal{D}$-module on Y amounts to a rule which assigns to every I an $\mathcal{O}$-module $F_{Y^{<I>}}$ on $Y^{<I>}$ and to every $\pi : J \to I$ an isomorphism $\nu^{<\pi>} : \Delta^{<\pi>*}F_{Y^{<J>}} \xrightarrow{\sim} F_{Y^{<I>}}$ compatible with composition of π's.[150] Namely, there is a unique way to define on each $F_{Y^{<I>}}$ a left action of $\mathcal{D}_{Y^I}$ so that all $\nu^{<\pi>}$ become isomorphisms of $\mathcal{D}$-modules. The corresponding $\mathcal{D}$-module on Y is F_Y equipped with this $\mathcal{D}$-module structure.

We define our left $\mathcal{D}$-module structure on B_X using this language. Key point: every map $\pi : J \to I$ yields a canonical morphism

$$\nu^{(\pi)} : \Delta^{(\pi)*}B_{X^J} \to B_{X^I}. \tag{3.5.8.1}$$

Namely, if we use the interpretation of factorization algebras from 3.5.6 then $\nu^{(\pi)}$ is just the structure morphism from 3.5.6(i). If the reader prefers to avoid 3.5.6, 3.5.7 then he may define the $\nu^{(\pi)}$'s directly: this is the unique family of morphisms compatible with composition of π's such that for π surjective $\nu^{(\pi)}$ is the structure isomorphism from 3.5.2, and for π injective our $\nu^{(\pi)} : \Delta^{(\pi)*}B_{X^J} = B_{X^J} \boxtimes \mathcal{O}_{X^{I\smallsetminus J}} \to B_{X^I}$ is $b \boxtimes f \mapsto \kappa(b \boxtimes f 1^{\boxtimes I \smallsetminus J})$. Here κ is the factorization isomorphism (3.5.4.1); to see that the latter section belongs to B_{X^I} (see (3.5.4.2)) use 3.5.3(ii) and the definition of the unit in B (see 3.5.5).

Now we are ready to define the canonical left $\mathcal{D}$-module structure on B. Let $B_{X^{<I>}}$ be the pull-back of B_{X^I} to $X^{<I>}$ (i.e., the completion of B_{X^I} at the diagonal $X \subset X^I$), $\nu^{<\pi>}$ be the pull-back of the morphism 3.5.8.1 to $X^{<I>}$. Our $\nu^{<\pi>}$ is actually an isomorphism: this follows from 3.5.3(i) and the fact that the pull-back of $\nu^{(\pi)}$ to the diagonal $X \subset X^I$ is an isomorphism (equal to id_{B_X}).

[150] We assume that F_Y is a quasicoherent $\mathcal{O}_Y$-module.

We have defined the $\mathcal{D}$-module structure on B_X. One defines a $\mathcal{D}$-module structure on each B_{X^S} in the similar way taking for $B_{(X^S)^{<I>}}$ the pull-back of $B_{X^{S\times I}}$ to the formal neighbourhood of $X^S \subset (X^S)^I = X^{S\times I}$, and for $\nu^{<\pi>}$ the pull-back of $\nu^{(id_S\times\pi)}$.

One checks immediately that the morphisms $\nu^{(\pi)}$ are compatible with the $\mathcal{D}$-module structure, as well as the morphisms κ from 3.5.4.1, so we have defined a left $\mathcal{D}$-module structure on B. The section 1 is obviously horizontal, as required.

Let us prove the uniqueness statement. Assume we have another $\mathcal{D}$-module structure on B such that 1 is horizontal. This property implies that every $\nu^{(\pi)}$ is compatible with our $\mathcal{D}$-module structure. Pulling it back to the formal neighbourhood of the diagonals we get a $\mathcal{D}$-module structure on every $B_{X^{<I>}}$ (and $B_{(X^S)^{<I>}}$) such that the $\nu^{<\pi>}$'s are morphisms of $\mathcal{D}$-modules. According to the uniqueness statement in the first paragraph of our proof, this datum of $\mathcal{D}$-module structures is uniquely defined. Hence the $\mathcal{D}$-module structures on B_{X^S} are also uniquely defined (set $I = \cdot$), and we are done. $\square$

Remarks. (i) Here is a down-to-earth definition of the canonical $\mathcal{D}$-module structure (which follows directly from the proof of 3.5.8). Let t be a local coordinate on X. Then for $b \in B$ one has

$$1\boxtimes b - b\boxtimes 1 = (\partial_t b)\boxtimes 1(t_2-t_1)\mathrm{mod}(t_2-t_1)^2 B_{X\times X}. \tag{3.5.8.2}$$

In fact, the restriction of $1\boxtimes b$ to the formal neighbourhood of the diagonal $X\subset X\times X$ equals $\sum \frac{(t_2-t_1)^i}{i!}(\partial_t^i b)\boxtimes 1$. This defines our $\mathcal{D}$-module structure on B_X. And B_{X^I} is a $\mathcal{D}$-submodule of $j_*^{(I)} j^{(I)*} B_X^{\boxtimes I}$.

(ii) In the setting of 3.5.6 $\mathcal{O}$-modules $B_Z(S)$ (for smooth Z) have a canonical left $\mathcal{D}$-module structure uniquely determined by the property that all the structure morphisms from 3.5.6(i),(ii) are compatible with the $\mathcal{D}$-module structure. The proof and construction essentially coincides with the one from the above proposition, so we skip it. In particular, we see that the canonical $\mathcal{D}$-module structure on B_{X^n} comes from the one on $B_{\mathrm{Sym}^n X}$.

3.5.9. Let $B^r = B_X\omega_X$ be the right $\mathcal{D}_X$-module that corresponds to the left $\mathcal{D}_X$-module B_X. Let us show that it carries a canonical structure of chiral algebra.

For every $J \twoheadrightarrow I$ we have a canonical isomorphism of left $\mathcal{D}$-modules $\Delta^{(J/I)*}B_{X^J} \xrightarrow{\sim} B_{X^I}$. According to (2.1.3.1) we may rewrite it as an isomorphism of right $\mathcal{D}_{X^J}$-modules

$$B_{X^J} \otimes \Delta_*^{(J/I)}\omega_X^{\boxtimes I} \xrightarrow{\sim} \Delta_*^{(J/I)}(B_{X^I} \otimes \omega_X^{\boxtimes I}). \tag{3.5.9.1}$$

In particular, we get a canonical map

$$P^{ch}(\omega_X)_J \longrightarrow P^{ch}(B^r_X)_J \tag{3.5.9.2}$$

which sends an operation $\varphi : j_*^{(J)}j^{(J)*}\omega_X^{\boxtimes J} \longrightarrow \Delta_*^{(J)}\omega_X$ to the composition

$$j_*^{(J)}j^{(J)*}B^{r\boxtimes J} = B_{X^J} \otimes j_*^{(J)}j^{(J)*}\omega_X^{\boxtimes J} \longrightarrow B_{X^J} \otimes \Delta_*^{(J)}\omega_X = \Delta_*^{(J)}B^r. \tag{3.5.9.3}$$

Here the first equality is the factorization isomorphism tensored by $id_{\omega_X^{\boxtimes J}}$, the arrow is φ tensored by $id_{B_{X^J}}$, and the last equality is 3.5.9.1 (for $I = \cdot$).

We leave it to the reader to check that 3.5.9.2 is compatible with composition of chiral operations. According to 3.1.4 we may rewrite 3.5.9.2 as a canonical morphism of operads $\mathcal{L}ie \to P^{ch}(B^r)$ which is the same as a Lie^{ch} algebra structure on B^r_X. It is easy to see that the section 1 of B_X is a unit in the Lie^{ch} algebra B^r. So B^r is a chiral algebra.

The folowing theorem is the principal result of this section:

3.5.10 Theorem. The functor $\mathcal{FA}(X) \to \mathcal{CA}(X)$, $\quad B \mapsto B^r$, is an equivalence of categories.

Proof. We will construct the inverse functor. So for every chiral algebra A we want to define on the left $\mathcal{D}_X$-module A^ℓ a canonical structure of factorization algebra. The construction makes use of certain canonical complexes associated to A. These complexes will also be used in 3.10.

3.5.11. For a finite non-empty I consider the following complex $C(A)_{X^I}$ of right $\mathcal{D}$-modules on X^I. Let us forget about the differential for a moment, so we consider a plain $\mathbb{Z}$-graded $\mathcal{D}$-module $C_I(A)^{\cdot}$. One has

$$C(A)^{\cdot}_{X^I} := \underset{T\in Q(I)}{\oplus} \Delta_*^{(I/T)}j_*^{(T)}j^{(T)*}(A[1])^{\boxtimes T}. \tag{3.5.11.1}$$

In other words, $C(A)^{\cdot}_{X^I}$ is a $Q(I)$-graded module, and for $T \in Q(I)$ its T-component is $\Delta_*^{(I/T)} j_*^{(T)} j^{(T)*} A^{\boxtimes T} \otimes \lambda_T$ sitting in degree $-|T|$. Let us define now the differential. Its (T,T')-component $d_{T,T'} : \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}(A[1])^{\boxtimes T} \to \Delta_*^{(I/T')} j_*^{(T')} j^{(T')*}(A[1])^{\boxtimes T'}$ may be non-zero only for $T' \in Q(T, |T|-1)$. Then $T = T'' \sqcup \{\alpha', \alpha''\}$, $T' = T'' \sqcup \{\alpha\}$ and $d_{T,T'}$ is the exterior tensor product of the chiral product map $\mu_A[1] : j_* j^*(A_{\alpha'}[1] \boxtimes A_{\alpha''}[1]) \to \Delta_* A_\alpha[1]$ and the identity map for $A^{\boxtimes T''}$ (localized at the diagonals transversal to $X^{T'}$). The Jacobi identity for μ_A implies that $d^2 = 0$.

Notice that if $v : V \hookrightarrow X^I$ is an open embedding such that $X^I \smallsetminus V$ is a union of diagonal strata then the canonical map $C(A)_{X^I} \to v_* v^* C(A)_{X^I}$ admits an obvious canonical section

$$s : v_* v^* C(A)^{\cdot}_{X^I} \to C(A)^{\cdot}_{X^I} \tag{3.5.11.2}$$

which preserves the $Q(I)$-grading but does not commute with the differentials.

The complexes $C(A)_{X^I}$ enjoy the following two functorial properties:

(i) For every $\pi : J \twoheadrightarrow I$ there is an obvious identification of complexes

$$\Delta^{(\pi)!} C(A)_{X^J} = C(A)_{X^I}, \tag{3.5.11.3}$$

and this plain $\Delta^{(\pi)!}$ pull-back equals to the derived functor one; these identifications are compatible with composition of π's. In other words, one has a canonical short exact sequence of complexes

$$0 \to \Delta_*^{(\pi)} C(A)_{X^I} \to C(A)_{X^J} \to v_* v^* C(A)_{X^J} \to 0. \tag{3.5.11.4}$$

Here $v : X^J \smallsetminus \Delta^{(\pi)}(X^I) \hookrightarrow X^J$ is the open embedding complementary to $\Delta^{(\pi)}$; the naive v_* coincides here with the derived one.

(ii) (factorization) In the setting of 3.5.4 for $J \twoheadrightarrow I$ there is an obvious canonical isomorphism of complexes

$$\kappa_{[J/I]} : j^{[J/I]*}(\underset{I}{\boxtimes} C(A)_{X^{J_i}}) \xrightarrow{\sim} j^{[J/I]*} C(A)_{X^J}. \tag{3.5.11.5}$$

The identifications κ are mutually compatible: for $K \twoheadrightarrow J$ the isomorphism $\kappa_{[K/J]}$ coincides with the composition $\kappa_{[K/I]}(\boxtimes \kappa_{[K_i/J_i]})$.

3.5.12 Lemma. The complex $C(A)_{X^I}$ is acyclic outside degree $-|I|$.

Proof. The statement is clear if $|I|=1$ (one has $C(A)_X = A[1]$). For $|I|>1$ we use induction on $|I|$. The proof goes in 3 steps:

(i) The module $H^a C(A)_{X^I}$ for $a \neq -|I|$ is supported on the diagonal $X \subset X^I$. This follows from 3.5.11.5 and the induction assumption.

Let $i : Y = X^{I'} \hookrightarrow X^I$ be a diagonal of codimension 1, $v : X^I \smallsetminus Y \hookrightarrow X^I$ the complementary embedding. Consider the canonical embedding $\xi : i_* C(A)_Y \hookrightarrow C(A)_{X^I}$ (see 3.5.11.4).

(ii) The morphism $H^a\xi : i_* H^a C(A)_Y = H^a i_* C(A)_Y \to H^a C(A)_{X^I}$ is surjective for $a \neq -|I|$. Indeed, the complex $v_* v^* C(A)_{X^I}$ is acyclic outside degree $-|I|$ by (i) (since v is affine), so our claim follows from the exact sequence $0 \to i_* C(A)_Y \to C(A)_{X^I} \to v_* v^* C(A)_{X^I} \to 0$ (see 3.5.11.4).

(iii) It remains to show that $H^a\xi = 0$. If $a \neq 1-|I|$ this is clear because in this case $H^a C(A)_Y = 0$ by the induction assumption.

Now let $a = 1-|I|$. Choose a section $I' \to I$ of $I \twoheadrightarrow I'$, then we have the decomposition $X^I = X \times X^{I'} = X \times Y$ and the embedding $i : Y \hookrightarrow X \times Y$. We want to show that $\xi(i_* Z^a_Y) \subset d(C^{a-1}(A)_{X^I})$ where $Z^a_Y := \mathrm{Ker}(d : C^a(A)_Y \to C^{a+1}(A)_Y)$. Let s be a local section of $i_* Z^a_Y = v_* v^*(\omega_X \boxtimes Z^a_Y)/\omega_X \boxtimes Z^a_Y$. Lift it to a local section $\tilde{s}$ of $v_* v^*(\omega_X \boxtimes Z^a_Y)$. Then $\xi(s) = d(h(\tilde{s}))$ where h is the composition

$$v_* v^*(\omega_X \boxtimes Z^a_Y) \xrightarrow{f} v_* v^*(A \boxtimes C^a(A)_Y) \hookrightarrow C^{a-1}(A)_{X^I}$$

and f comes from $1_A : \omega_X \to A$. □

3.5.13. *End of the proof of* 3.5.10. Now we are ready to define the factorization algebra structure on A^ℓ. Set $A^\ell_{X^I} := H^{-|I|} C(A)^\ell_{X^I} = \mathrm{Ker}(d : C(A)^{-|I|}_{X^I} \to C(A)^{-|I|+1}_{X^I})^\ell$. By 3.5.12 we may rewrite 3.5.11.3 as a canonical identification $\Delta^{(I/S)*} A^\ell_{X^I} = A^\ell_{X^S}$ (see 2.1.2). These identifications are obviously transitive, so the $A^\ell_{X^I}$'s form a left $\mathcal{D}$-module, hence an $\mathcal{O}$-module, on $\mathcal{R}(X)$. So[151] $A^\ell_{X^I} \subset C(A)^{0\ell}_{X^I} = j^{(I)}_* j^{(I)*} A^{\ell \boxtimes I}$. We leave it to the reader to

[151] We use the identification $\omega_{X^I} = \omega_X^{\boxtimes I} \otimes \lambda_I$, see 3.1.3.

check that this embedding defines a factorization algebra structure on our $\mathcal{O}_{\mathcal{R}(X)}$-module (with the unit equal to the unit in A). Notice that our factorization algebra came together with a $\mathcal{D}$-module structure, which obviously coincides with the canonical one (see 3.5.8). One sees immediately that the functor $\mathcal{FA}(X) \to \mathcal{CA}(X)$ we have defined is inverse to that of 3.5.10. We are done. □

3.5.14 Remark. For $n \geq 1$ an *n-truncated factorization algebra* is defined exactly as usual factorization algebra but we take in consideration only the X^I's with $|I| \leq n$. Denote the corresponding categories by $\mathcal{FA}^{\leq n}(X)$. As follows from 3.5.10, for $n \geq 3$ the obvious functors $\mathcal{FA}(X) \to \mathcal{FA}^{\leq n}(X)$, $\mathcal{FA}^{\leq n+1}(X) \to \mathcal{FA}^{\leq n}(X)$ are equivalences of categories.[152]

3.5.15. We know that $\mathcal{FA}(X)$ is a tensor category (see 3.5.4, 3.5.5), hence such is $\mathcal{CA}(X)$. Let us explain the meaning of tensor product of chiral algebras in purely chiral terms.

Let A_α be a family of chiral algebras. Let $A := \otimes A_\alpha \in \mathcal{CA}(X)$ be the corresponding tensor product, so $A^\ell = \otimes A^\ell_\alpha$. By 3.5.5 we have canonical morphisms of chiral algebras $\nu_\alpha : A_\alpha \to A$, $a \mapsto a \otimes (\underset{\alpha' \neq \alpha}{\otimes} 1_{A_{\alpha'}})$.

Let $\varphi_\alpha : A_\alpha \to C$ be morphisms of chiral algebras. We say that φ_α *mutually commute* if $[\varphi_{\alpha'}, \varphi_{\alpha"}]_C \in P^*_2(\{A_{\alpha'}, A_{\alpha"}\}, C)$ vanishes for every $\alpha' \neq \alpha"$. Denote by $\mathrm{Hom}^{com}_I(\{A_\alpha\}, C)$ the set of all mutually commuting (φ_α)'s.

Remark. Let $M_\alpha \subset A_\alpha$ be a sub-$\mathcal{D}$-module that generates A_α as a chiral algebra. Then $\varphi_{\alpha'}$, $\varphi_{\alpha''}$ mutually commute if (and only if) their restrictions to M's commute, i.e., the image of $[\varphi_{\alpha'}, \varphi_{\alpha"}]_C$ in $P^*_2(\{M_{\alpha'}, M_{\alpha"}\}, C)$ vanishes.

3.5.16 Proposition. $\{\nu_\alpha\}$ is a universal family of mutually commuting morphisms: for every $C \in \mathcal{CA}(X)$ the map $\mathrm{Hom}(\otimes A_\alpha, C) \to \mathrm{Hom}^{com}_I(\{A_\alpha\}, C)$, $\psi \mapsto (\psi\nu_\alpha)$, is bijective.

Proof. Let us translate 3.5.16 back in the setting of factorization algebras. Assume for simplicity of notation that we play with two algebras A_1, A_2. Notice that φ_1, φ_2 as above mutually commute if and only if for every $a_1 \in A^\ell_1, a_2 \in A^\ell_2$ the section $\varphi_1(a_1) \boxtimes \varphi_2(a_2)$ of $j_*j^*C^\ell_{X\times X}$ lies in $C^\ell_{X\times X}$. One has $\nu_1(a_1) \boxtimes \nu_2(a_2) = (a_1 \boxtimes 1_{A_1}) \otimes (1_{A_2} \boxtimes a_2) \in A^\ell_{1X\times X} \otimes A^\ell_{2X\times X} = (A^\ell_1 \otimes A^\ell_2)_{X\times X}$, so our $\nu_{1,2}$ mutually commute. For $\varphi_{1,2} \in \mathrm{Hom}^{com}_I(\{A_1, A_2\}, C)$

[152] They are fully faithful for $n = 2$ and faithful for $n = 1$.

the I-th component of the corresponding morphism $\psi : A_1^\ell \otimes A_2^\ell \to C'^\ell$ sends $m_1 \otimes m_2$, where $m_1 \in A^\ell_{1X^I}, m_2 \in A^\ell_{2X^I}$, to the pull-back of $\varphi_1(m_1) \boxtimes \varphi_2(m_2)$[153] by the diagonal map $X^I \hookrightarrow X^I \times X^I$. The details are left to the reader. $\square$

Let us show that the functor $A \mapsto A^{as}$ (see 3.4.2) commutes with tensor products:

3.5.17 Lemma. There is a canonical isomorphism $\hat{\otimes} A^{as}_\alpha \xrightarrow{\sim} (\otimes A_\alpha)^{as}$ of topological associative algebras on X equipped with a connection.

Proof. Recall that A^{as} equals the projective limit of $\mathcal{O}_X$-modules $\Delta^* A^\ell_\xi$ where $A_\xi \in \Xi^{as}(A)$, i.e., $A_\xi \subset j_* j^*(A \boxtimes \mathcal{O}_X)$ is a chiral subalgebra along the first variable equal to $A \boxtimes \mathcal{O}_X$ on U (see 3.4.2). It is clear that any subalgebra $(\otimes A_\alpha)_\xi$ contains a subalgebra of type $\otimes(A_{\alpha\xi_\alpha})$.[154] This yields a canonical isomorphism of topological $\mathcal{O}_X$-modules $\hat{\otimes} A^{as}_\alpha := \varprojlim \otimes \Delta^* A^\ell_{\alpha\xi_\alpha} \xrightarrow{\sim} (\otimes A_\alpha)^{as}$. It follows from definitions that our isomorphism is compatible with products and connections. $\square$

3.5.18. A *Hopf chiral algebra* is a chiral algebra A equipped with a coassociative morphism of chiral algebras $\delta : A \to A \otimes A$. If A is Hopf algebra then, according to 3.5.17, A^{as} is a topological Hopf algebra equipped with a connection.

Remark. Contrary to the usual situation there is no natural notion of action of a Hopf chiral algebra on another chiral algebra.

3.5.19. Let us describe modules over chiral algebras in factorization terms. Below for a finite set I we set $\widetilde{I} := I \sqcup \cdot$, so $X^{\widetilde{I}} = X \times X^I$.

Let B be any factorization algebra. A *factorization B-module* is a pair $(M, \tilde{\kappa})$ where:

(i) M is a rule that assigns to every finite set I a left $\mathcal{D}$-module $M^\ell_{X^{\widetilde{I}}}$ on $X \times X^I$, and to every $\pi : \widetilde{J} \twoheadrightarrow \widetilde{I}$ preserving $\cdot$'s an identification $\nu^{(\pi)} : \Delta^{(\pi)*} M^\ell_{X^{\widetilde{J}}} \xrightarrow{\sim} M^\ell_{X^{\widetilde{I}}}$. We demand that $\nu^{(\pi)}$'s are compatible with composition of π's, and $M^\ell_{X^{\widetilde{I}}}$ is flat along the diagonals.

(ii) Consider a surjection $\widetilde{J} \twoheadrightarrow \widetilde{I}$ preserving $\cdot$'s. Then $\widetilde{J}$ is disjoint union of the preimage subsets $\widetilde{J}_\cdot$ and J_i, $i \in I$. Our $\tilde{\kappa}$ assigns to such surjection an isomorphism of $\mathcal{D}_{U[\widetilde{J}/\widetilde{I}]}$-

[153] To see that this section belongs to $C^\ell_{X^I \times X^I}$ use 3.5.5 and the fact that $\varphi_{1,2}$ mutually commute.

[154] Take for $A_{\alpha\xi_\alpha}$ the preimage of $(\otimes A_\alpha)_\xi$ by the morphism $A_\alpha \to \otimes A_\alpha$.

modules[155]

$$(3.5.19.1) \qquad \tilde{\kappa}_{[\tilde{J}/\tilde{I}]} : j^{[\tilde{J}/\tilde{I}]*}(M^{\ell}_{X^{\tilde{J}.}} \boxtimes (\underset{I}{\boxtimes} B_{X^{J_i}})) \xrightarrow{\sim} j^{[\tilde{J}/\tilde{I}]*} B_{X^J}.$$

We demand that $\tilde{\kappa}$'s are mutually compatible: for $\tilde{K} \twoheadrightarrow \tilde{J}$ the isomorphism $\tilde{\kappa}_{[\tilde{K}/\tilde{J}]}$ coincides with the composition $\tilde{\kappa}_{[\tilde{K}/\tilde{I}]}(\boxtimes \tilde{\kappa}_{[\tilde{K}_i/\tilde{J}_i]})$.

We denote the category of factorization B-modules by $\mathcal{M}^{\ell}(X,B)$. This is an abelian k-category.

§6 Operator product expansions

In this section we give a third definition of chiral algebras. It shows that our chiral algebras are related to their namesake from mathematical physics, and (in the universal setting) are essentially the same as vertex operator algebras from references cited in 0.0. The results of this section will not be used in the rest of the work.

3.6.1. Let us begin with some preliminary considerations. For a smooth variety P a $\mathcal{D}_P$-*sheaf* is any (not necessary quasicoherent) sheaf of left $\mathcal{D}_P$-modules. We denote the category of $\mathcal{D}_P$-sheaves by $\bar{\mathcal{M}}^{\ell}(P)$, so $\mathcal{M}^{\ell}(P)$ is a full subcategory of $\bar{\mathcal{M}}^{\ell}(P)$. The $\mathcal{O}_P$-tensor product defines on $\bar{\mathcal{M}}^{\ell}(P)$ the structure of tensor category. Let $i : Z \hookrightarrow P$ be a closed smooth subvariety, $\mathcal{I} \subset \mathcal{O}_P$ the corresponding ideal. If F is a $\mathcal{D}_P$-sheaf then $i^*F := i^{\cdot}(F/\mathcal{I}F)$ is a $\mathcal{D}_Z$-sheaf in the usual manner. The functor i^* commutes with tensor products.

3.6.2 Lemma. (i) The functor $i^* : \bar{\mathcal{M}}^{\ell}(P) \to \bar{\mathcal{M}}^{\ell}(Z)$ admits a right adjoint functor $i_*^{\hat{}} : \bar{\mathcal{M}}^{\ell}(Z) \to \bar{\mathcal{M}}^{\ell}(P)$. the functor $i_*^{\hat{}}$ is exact and fully faithful. A $\mathcal{D}_P$-sheaf belongs to its image if and only if $F = \varprojlim F/\mathcal{I}^n F$.

(ii) For a $\mathcal{D}_Z$-sheaf G set $i_*G := (i_*\omega_Z)^{\ell} \otimes i_*^{\hat{}} G$.[156] The functor $i_* : \bar{\mathcal{M}}^{\ell}(Z) \to \bar{\mathcal{M}}^{\ell}(P)$ is exact and fully faithful. A $\mathcal{D}_P$-sheaf belongs to its image if and only if every its local section is killed by some power of $\mathcal{I}$.

[155] See 3.5.4 for notation.
[156] Here $(i_*\omega_Z)^{\ell} \in \mathcal{M}^{\ell}(P)$, see 2.1.1, 2.1.3.

(iii) The above i_* coincides on $\mathcal{M}^\ell(Z)$ with the direct image functor from 2.1.3: for $N \in \mathcal{M}(Z)$ one has $i_*(N^\ell) = (i_*N)^\ell$.

Proof. Left to the reader. $\square$

Remark. For $A, B \in \bar{\mathcal{M}}^\ell(Z)$ we have a canonical morphism $(i_*^\wedge A) \otimes (i_*^\wedge B) \to i_*^\wedge(A \otimes B)$ (defined by adjunction) and isomorphism $(i_*^\wedge A) \otimes (i_* B) \xrightarrow{\sim} i_*(A \otimes B)$. So if A is a $\mathcal{D}_Z$-algebra then $i_*^\wedge A$ is a $\mathcal{D}_P$-algebra, and if N is an A-module then i_*N is an $i_*^\wedge A$-module.

3.6.3. Assume that Z is a divisor; let $j : U := P \smallsetminus Z \hookrightarrow P$ be the complementary open embedding. For a $\mathcal{D}_Z$-sheaf G set

$$i_*^\sim G := (i_*^\wedge G) \otimes j_*\mathcal{O}_U \in \bar{\mathcal{M}}^\ell(P). \tag{3.6.3.1}$$

It is easy to see that the canonical map $i_*^\wedge G \to i_*^\sim G$ is injective. Since $(i_*\omega_Z)^\ell = j_*\mathcal{O}_U/\mathcal{O}_U$ we get a canonical exact sequence

$$0 \to i_*^\wedge G \to i_*^\sim G \to i_*G \to 0. \tag{3.6.3.2}$$

3.6.4 Lemma. For any $\mathcal{D}_P$-sheaf F the obvious arrows

$$\mathrm{Hom}(F, i_*^\sim G) \longleftarrow \mathrm{Hom}(F \otimes j_*\mathcal{O}_U, i_*^\sim G) \longrightarrow \mathrm{Hom}(F \otimes j_*\mathcal{O}_U, i_*G)$$

are isomorphisms.

Proof. For the left arrow the situation is clear. The inverse to the right arrow sends $\alpha \in \mathrm{Hom}(F \otimes j_*\mathcal{O}_U, i_*G)$ to $\tilde{\alpha} \in \mathrm{Hom}(F \otimes j_*\mathcal{O}_U, i_*^\sim G)$ defined by formula $\tilde{\alpha}(f) = \varprojlim q^n\alpha(q^{-n}f)$. Here $f \in F \otimes j_*\mathcal{O}_U$, $q \in \mathcal{O}_P$ an equation of Z, so $q^n\alpha(q^{-n}f) \in i_*^\sim G/\mathcal{I}^n i_*^\sim G$. $\square$

Remark. We also have canonical isomorphisms

$$\mathrm{Hom}(F, i_*^\sim G) \xleftarrow{\sim} \varinjlim \mathrm{Hom}(F_\xi, i_*^\wedge G) \xrightarrow{\sim} \varinjlim \mathrm{Hom}(i^*F_\xi, G). \tag{3.6.4.1}$$

where F_ξ runs the set of $\mathcal{D}_P$-subsheaves of $F \otimes j_*\mathcal{O}_U$ such that $F_\xi \cdot j_*\mathcal{O}_U = F \otimes j_*\mathcal{O}_U$. The first arrow assigns to $F_\xi \to i_*^\wedge G$ its tensor product with $j_*\mathcal{O}_U$, the second arrow is the adjunction isomorphism. The inverse map to the first arrow sends $\varphi : F \otimes j_*\mathcal{O}_U \to i_*^\sim G$ to $\varphi_\xi : F_\xi \to i_*^\wedge G$ where $F_\xi := \varphi^{-1}(i_*^\wedge G)$, φ_ξ is the restriction of φ to F_ξ.

Notice that if both F and G are quasicoherent then, as follows from the above lemma, we may assume in (3.6.4.1) that all F_ξ's are quasicoherent.

3.6.5. Let us return to our situation, so X is a curve. For a finite non-empty I and a $\mathcal{D}_X$-sheaf G set $\hat{\Delta}^{(I)}_* G := \Delta^{(I)}\hat{_*} G$ and

$$\tilde{\Delta}^{(I)}_* G := (\hat{\Delta}^{(I)}_* G) \otimes j^{(I)}_* \mathcal{O}_{U^{(I)}} \in \bar{\mathcal{M}}(X^I). \tag{3.6.5.1}$$

The obvious morphism $\hat{\Delta}^{(I)}_* G \to \tilde{\Delta}^{(I)}_* G$ is injective. Note that $\hat{\Delta}^{(I)}_* G$ has the following explicit description. For $i \in I$ consider a $\mathcal{D}_{X^I}$-sheaf $pr_i^* G := G \boxtimes \mathcal{O}_X^{\boxtimes I \setminus \{i\}}$. Then $\Delta^{(I)}\!\!\cdot pr_i^* G = G$, hence $\hat{\Delta}^{(I)}_* G = \varprojlim pr_i^* G / \mathcal{J}^n_\Delta pr_i^* G$ where $\mathcal{J}_\Delta \subset \mathcal{O}_{X^I}$ is the ideal of the diagonal $X \subset X^I$.

More generally, consider a surjective map $I \twoheadrightarrow S$ of finite sets. Let $\Delta^{(I/S)} : X^S \hookrightarrow X^I$ be the diagonal embedding, and $j^{(I/S)} : U^{(I/S)} \hookrightarrow X^I$ be the complement to the diagonals containing X^S. For a $\mathcal{D}_{X^S}$-sheaf F set

$$\tilde{\Delta}^{(I/S)}_* F := (\hat{\Delta}^{(I/S)}_* F) \otimes j^{(I/S)}_* \mathcal{O}_{U^{(I/S)}} \in \bar{\mathcal{M}}(X^I). \tag{3.6.5.2}$$

Note that if F is a $j^{(S)}_* \mathcal{O}_{U^{(S)}}$-module then $\tilde{\Delta}^{(I/S)}_* F$ is a $j^{(I)}_* \mathcal{O}_{U^{(I)}}$-module.

3.6.6. Consider the Cousin complex for ω_{X^I} with respect to the diagonal stratification (see the proof of 3.1.4). Tensoring it by $(\hat{\Delta}^{(I)}_* G)\omega^{-1}_{X^I}$ we get the complex $C_I(G)$ of $\mathcal{D}_{X^I}$-sheaves with terms $C_I(G)^m := \bigoplus_{T \in Q(I, |I|-m)} \Delta^{(I/T)}_* \tilde{\Delta}^{(T)}_* G$ (in particular, $C_I(G)^0 = \tilde{\Delta}^{(I)}_* G$) which is a resolution of $\hat{\Delta}^{(I)}_* G$.

3.6.7 Lemma. Assume that a $\mathcal{D}_{X^I}$-sheaf N is a $j^{(I)}_* \mathcal{O}_{U^{(I)}}$-module. Then the sequence

$$0 \to \mathrm{Hom}(N, C_I(G)^0) \to \mathrm{Hom}(N, C_I(G)^1) \to \mathrm{Hom}(N, C_I(G)^2)$$

is exact.

Proof. The statement is X-local, so let us choose an equation $q = 0$ of the diagonal divizor. Then $\tilde{\Delta}^{(I)}_* G = \varprojlim \tilde{\Delta}^{(I)}_* G / q^n \hat{\Delta}^{(I)}_* G$, and for any n the sequence $0 \to \tilde{\Delta}^{(I)}_* G / q^n \hat{\Delta}^{(I)}_* G \to C^0_I \xrightarrow{q^{-n} d^0} C_I(G)^1 \xrightarrow{d^1} C_I(G)^2$ is exact. We are done. $\square$

3.6.8. For an I-family of $\mathcal{D}_X$-sheaves set

$$O_I(\{F_i\}, G) := \mathrm{Hom}(\boxtimes F_i, \tilde{\Delta}^{(I)}_* G) = \mathrm{Hom}((\boxtimes F_i) \otimes j^{(I)}_* \mathcal{O}_{U^{(I)}}, \tilde{\Delta}^{(I)}_* G). \tag{3.6.8.1}$$

This is the vector space of *ope I-operations* where ope stands for "operator product expansion". they have local nature with respect to the étale topology of X.

The composition of ope operations need not be an ope operation: it belongs to a larger vector space. Namely, for a surjective map $J \twoheadrightarrow I$ set

$$\tilde{\Delta}_*^{(J,I)}G := \tilde{\Delta}_*^{(J/I)}\tilde{\Delta}_*^{(I)}G. \tag{3.6.8.2}$$

since $\hat{\Delta}_*^{(J)} = \hat{\Delta}_*^{(J/I)}\hat{\Delta}_*^{(I)}$ we have a canonical embedding $\tilde{\Delta}_*^{(J)}G \hookrightarrow \tilde{\Delta}_*^{(J,I)}G$.

Now for a J-family of $\mathcal{D}_X$-sheaves the *composition map*

$$O_I(\{F_i\},G)\otimes(\underset{I}{\otimes} O_{J_i}(\{E_j\},F_i)) \longrightarrow \mathrm{Hom}(\boxtimes E_j, \tilde{\Delta}_*^{(J,I)}G) \tag{3.6.8.3}$$

sends $\gamma\otimes(\otimes\delta_i)$ to $\gamma(\delta_i)$ defined as composition

$$\underset{J}{\boxtimes} E_j \xrightarrow{\boxtimes\delta_i} \underset{I}{\boxtimes}\tilde{\Delta}_*^{(J_i)}F_i \longrightarrow \hat{\Delta}_*^{(J/I)}(\boxtimes F_i)\otimes j_*^{(J)}\mathcal{O}_{U(J)} \longrightarrow \tilde{\Delta}_*^{(J,I)}G.$$

Here the second arrow comes from the obvious morphism $\boxtimes\hat{\Delta}_*^{(J_i)}F_i \to \hat{\Delta}_*^{(J/I)}(\boxtimes F_i)$, and the third one is $\hat{\Delta}_*^{(J/I)}(\gamma)$ tensored by identity morphism for $j_*^{(J)}\mathcal{O}_{U(J)}$.

We say that ope operations $(\gamma,\{\delta_i\})$ *compose nicely* if $\gamma(\delta_i)$ takes values in $\tilde{\Delta}_*^{(J)}G \subset \tilde{\Delta}_*^{(J/I)}G$. Then $\gamma(\delta_i)\in O_J(\{E_j\},G)$.

3.6.9. The composition of ope operations is associative in the following sense. Let $K \twoheadrightarrow J$ be another surjection. Set $\tilde{\Delta}_*^{(K,J,I)}G := \tilde{\Delta}_*^{(K,J)}\tilde{\Delta}_*^{(J,I)}G \in \bar{\mathcal{M}}^\ell(X^K)$. There is a commutative diagram of natural embeddings $\tilde{\Delta}_*^{(K)}G \subset \tilde{\Delta}_*^{(K,I)}G, \tilde{\Delta}_*^{(K,J)}G \subset \tilde{\Delta}_*^{(K,J,I)}G$.

Now for a K-family $\{C_k\}$ of $\mathcal{D}_X$-sheaves and operations $\varepsilon \in O_{K_j}(\{C_k\},E_j)$ the double composition morphisms $(\gamma(\delta_i))(\varepsilon_j), \gamma(\delta_i(\varepsilon_j)) \in \mathrm{Hom}(\boxtimes C_k, \tilde{\Delta}_*^{(K,J,I)}G)$ are defined, respectively, as compositions

$$\boxtimes C_k \xrightarrow{\boxtimes\varepsilon_j} \boxtimes\tilde{\Delta}_*^{(K_j)}E_j \longrightarrow \tilde{\Delta}_*^{(K/J)}(\boxtimes E_j) \xrightarrow{\tilde{\Delta}_*^{(K/J)}(\gamma(\delta_i))} \tilde{\Delta}_*^{(K,J,I)}G,$$

$$\boxtimes C_k \xrightarrow{\boxtimes\delta_i(\varepsilon_j)} \boxtimes\tilde{\Delta}_*^{(K_i,J_i)}F_i \longrightarrow \tilde{\Delta}_*^{(K/J)}\tilde{\Delta}_*^{(J/I)}(\boxtimes F_i) \longrightarrow \tilde{\Delta}_*^{(K,J,I)}G.$$

The associativity property says that they coincide.

3.6.10. Let $\circ \in O_2(\{G,G\},G)$ be a binary ope operation, $a\boxtimes b \mapsto a\circ b \in \tilde{\Delta}_* G$[157]. We say that $\circ$ is *commutative* if it is fixed by an obvious "transposition of coordinates" symmetry of $O_2(\{G,G\},G)$, and $\circ$ is *associative* if both $(\circ,\{\circ, id_G\})$, $(\circ,\{id_G,\circ\})$ compose nicely and the composition operations $\in O_3(\{G,G,G\},G)$ coincide.

We say that $\circ$ is an *ope algebra product*, or $(G,\circ)$ is an *ope algebra*, if $\circ$ is both commutative and associative. Then the triple product $\circ_3 := \circ(\{\circ, id_G\})$ is fixed by the action of symmetric group of 3 variables.

A *unit* in ope algebra G is a horizontal section $1\in G$ such that for every $a\in G$ one has $a\circ 1 = pr_1^* a \in \varprojlim pr_1^* G/\mathcal{J}_\Delta pr_1^* G = \hat{Delta}_* G \subset \tilde{\Delta}_* G$ (see 3.6.5). A unit is unique, if it exists.

3.6.11. Let L_1, L_2, M be right $\mathcal{D}_X$-modules. Then 3.6.2(iii), 3.6.4 yield a canonical isomorphism

$$P_2^{ch}(\{L_1,L_2\},M)\otimes\lambda_{\{1,2\}} \xrightarrow{\sim} O_2(\{L_1^\ell, L_2^\ell\}, M^\ell). \tag{3.6.11.1}$$

Here we use the identification $\omega_X \boxtimes \Omega_X = \omega_{X\times X}\otimes\lambda_{\{1,2\}}$, see 2.2.2.

Now we may state the main result of this section:

3.6.12 Proposition. For any $\mathcal{D}_X$-module A the above isomorphism

$$P_2^{ch}(\{A,A\},A)\otimes\lambda_{\{1,2\}} \xrightarrow{\sim} O_2(\{A^\ell,A^\ell\},A^\ell), \qquad \mu\mapsto \circ_\mu,$$

identifies the set of chiral Lie brackets with the set of ope algebra products. A horizontal section $1\in A^\ell$ is a unit for μ if and only if it is a unit for $\circ_\mu$.

Proof. We will prove the first statement; the second one is left to the reader.

Since (3.6.11.1) commutes with transposition of coordinates, skew-commutative μ correspond to commutative $\circ_\mu$. Take such μ. Let us show that the Jacobi identity for μ amounts to associativity of $\circ = \circ_\mu$.

Set $R := j_*^{(I)} j^{(I)*}(A^{\ell\boxtimes I})$. By 3.6.4 for every $S\in Q(I,|I|-1)$ we have isomorphisms

$$\mathrm{Hom}(A^{\ell\boxtimes I}, \tilde{\Delta}_*^{(I,S)}A^\ell) \xleftarrow{\sim} \mathrm{Hom}(R, \tilde{\Delta}_*^{(I/S)}\tilde{\Delta}_*^{(S)}A^\ell) \xrightarrow{\sim} \mathrm{Hom}(R, \Delta_*^{(I/S)}\tilde{\Delta}_*^{(S)}A^\ell).$$

[157] Here $\Delta := \Delta^{(\{1,2\})}$.

Since $\mathrm{Hom}(R, \tilde{\Delta}_*^{(I)}A^\ell = \mathrm{Hom}(A^{\ell\,\boxtimes I}, \tilde{\Delta}_*^{(I)}A^\ell)$, the exact sequence from 3.6.7 looks as

$$0 \to \mathrm{Hom}(A^{\ell\,\boxtimes I}, \tilde{\Delta}_*^{(I)}A^\ell) \xrightarrow{d^0} \bigoplus_{S\in Q(I,|I|-1)} \mathrm{Hom}(A^{\ell\,\boxtimes I}, \tilde{\Delta}_*^{(I,S)}A^\ell)$$
$$\xrightarrow{d^1} \bigoplus_{S\in Q(I,|I|-2)} \mathrm{Hom}(R, \Delta_*^{(I/S)}\tilde{\Delta}_*^{(S)}A^\ell).$$

The symmetric group of I indices acts on it in the obvious way.

Now assume that $I = \{1,2,3\}$. The third vector space in the above exact sequence is $\mathrm{Hom}(j_*^{(3)}j^{(3)*}(A^{\ell\,\boxtimes 3}), \Delta_*^{(3)}A^\ell) = P_3^{ch}(\{A,A,A\},A) \otimes \lambda_{\{1,2,3\}}$ (use 3.6.2(iii) and the identification $\omega_X^{\boxtimes 3} = \omega_{X^3} \otimes \lambda_{\{1,2,3\}}$). Our $\circ$ is commutative, so for every $S \in Q(I,2)$ we have the corresponding iterated product $\bar{\circ}_S \in \mathrm{Hom}(A^{\ell\,\boxtimes 3}, \tilde{\Delta}_*^{(I,S)}A^\ell)$. For example, if S identifies $1,2 \in I$ then $\bar{\circ}_S = \circ(\circ, id)$. The element $\bar{\circ}_3 := \Sigma\bar{\circ}_S$ of the middle term of the above exact sequence is invariant with respect to the action of the symmetric group. As follows from the construction, one has $d^1(\circ(\circ, id)) = \mu(\mu, id_A)$. Therefore $d^1(\bar{\circ}_3)$ is exactly the sum of terms of the Jacobi identity, so $d^1(\bar{\circ}_3) = 0$ if and only if μ is a Lie bracket. On the other hand, by the very definition, $\circ$ is associative if and only if $\bar{\circ}_3$ lies in the image of d^0. Since our sequence is exact, we are done. $\square$

3.6.13 Lemma. Let $(G, \circ)$ be an ope algebra. then for any finite non-empty I one has a canonical *I-fold ope product* operation $\circ_I \in O_I(\{G\}, G)$. These operations are uniquely determined by the following properties:

(i) If $|I| = 1$ then $\circ_I = id_G$; if $|I| = 2$ then $\circ_I = \circ$.

(ii) For every $J \twoheadrightarrow I$ the operations $(\circ_I, \{\circ_{J_i}\})$ compose nicely; the composition is $\circ_J$.

Proof. We define $\circ_I$ by induction. We know $\circ_I$ if $|I| \leq 3$. Now let I be a finite set of order $n > 3$, and assume that we know $\circ_{I'}$ for $|I'| < n$ so that (i), (ii) hold. Consider the exact sequence from the proof of 3.6.12. For $S \in Q(I, n-1)$ set $\bar{\circ}_S := \circ_S(\circ_{\{a,b\}}, id_{G_i})_{i\neq a,b} \in \mathrm{Hom}(G^{\boxtimes I}, \tilde{\Delta}^{(I,S)}G)$. Here $a, b \in I$ are the two distinct S-equivalent elements. Set $\bar{\circ}_I := \Sigma\bar{\circ}_S \in \bigoplus_{S\in Q(I,n-1)} \mathrm{Hom}(G^{\boxtimes I}, \tilde{\Delta}^{(I,S)}G)$. Associativity of composition 3.6.9 and the existence of $\circ_3$ imply that $d^1(\bar{\circ}_I) = 0$. Hence $\bar{\circ}_I = d^0(\circ_I)$ for certain $\circ_I$ uniquely determined by this condition. Property (ii) follows from associativity of composition. $\square$

3.6.14. Let us translate some of the definitions from 3.3 in the ope language. Let (A,μ) be a chiral algebra, and $(A^\ell,\circ)$ the corresponding ope algebra.

By definition, the Lie* bracket $[\ \ ]_\mu : A\boxtimes A\to\Delta_*A$ coincides with the composition $A^\ell\boxtimes A^\ell \xrightarrow{\circ} \tilde\Delta_*A^\ell \to \Delta_*A^\ell$ tensored by $id_{\omega_{X^2}}$, hence $[\ \ ]_\mu$ is the *singular part* of $\circ$. Therefore our chiral algebra is commutative if and only if $\circ$ takes values in $\hat\Delta_*A^\ell\subset\tilde\Delta_*A^\ell$. By adjunction property of $\hat\Delta_*$ (see 3.6.2(i)) one may rewrite such $\circ$ as a morphism $\cdot$: $\Delta^!(A^\ell\boxtimes A^\ell)=A^\ell\otimes A^\ell\to A^\ell$. Associativity and commutativity of $\circ$ amount to similar properties of $\cdot$, so we get another proof of 3.3.2.1, 3.3.3.1.

3.6.15. If $(G,\circ)$ is an ope algebra then a *G-module* is a $\mathcal{D}_X$-sheaf N together with an ope pairing $\circ_N\in O_2(\{G,N\},N)$ such that both $\circ_N(\circ,id_N)$ and $\circ_N(id_G,\circ_N)$ compose nicely and the compositions $\in O_3(\{G,G,N\},N)$ coincide. If G has unit, then one defines a unital G-module in the obvious manner.

Let A be a chiral algebra as above, M an A-module, so we have the chiral action $\mu_M\in P_2^{ch}(\{A,M\},M)$. Let $\circ_M\in O_2(\{A^\ell,M^\ell\},M^\ell)$ be the corresponding ope pairing (see 3.6.11). This is an $(A^\ell,\circ)$-module structure on M. This identifies the category of (A,μ)-modules with that of (quasicoherent) $(A^\ell,\circ)$-modules. This follows directly from 3.6.12 (use 3.3.5).

§7 Chiral enveloping algebras

Here is the key result of this section:

3.7.1 Theorem. The forgetful functor $\mathcal{CA}(X)\to\mathcal{L}ie^*(X)$, $A\mapsto A^{(*)}$, admits a left adjoint functor $U:\mathcal{L}ie^*(X)\to\mathcal{CA}(X)$.

For a Lie* algebra L we call $U(L)$ *the chiral enveloping algebra*, or simply *the chiral envelope* of L.

We give two different proofs of the above theorem – in the chiral (3.7.2–3.7.4) and factorization (3.7.5–3.7.11) settings respectively. The factorization picture has a definite advantage that it automatically yields immediately the Poincaré-Birkhoff-Witt theorem

3.7.14 (as well as computation of chiral homology of enveloping algebras, see 3.11.?).[158]

In the chiral setting we prove a more general result. To formulate it we need the following definition:

Definition. A k-operad $\mathcal{B}$ is *of Jacobi type* if it is generated by $\mathcal{B}_1$ and $\mathcal{B}_2$, and for every $\alpha, \alpha' \in \mathcal{B}_2$ there exist $\beta, \beta', \gamma, \gamma' \in \mathcal{B}_2$ such that $\alpha(x_1, \alpha'(x_2, x_3)) = \beta(x_3, \beta'(x_1, x_2)) + \gamma(x_2, \gamma'(x_3, x_1))$.

For example, the operads $\mathcal{L}ie$, $\mathcal{P}oiss$, $\mathcal{C}om$, $\mathcal{A}ss$ are of Jacobi type.

3.7.2 Theorem. For an operad $\mathcal{B}$ of Jacobi type the functor $\beta^{\mathcal{B}}$ of (3.3.1.1) admits a left adjoint functor $U_{\mathcal{B}} : \mathcal{B}^*(X) \to \mathcal{B}^{ch}(X)$.

Composing $U_{\mathcal{L}ie}$ with the "adding of unit" functor (see 3.3.3) we get the functor U of 3.7.1.

Proof of 3.7.2. Here is the idea. If the chiral pseudo-tensor structure were representable (see 1.1.3) then 3.7.2 would be immediate for arbitrary $\mathcal{B}$. Indeed, representability implies that for every L the free $\mathcal{B}^{ch}$ algebra generated by L is well-defined. So, if L is a $\mathcal{B}^*$ algebra, then $U_{\mathcal{B}}(L)$ is a quotient of the free $\mathcal{B}^{ch}$ algebra modulo the obvious relations. Now the *binary* chiral pseudo-tensor product is representable up to the non-representability of the $*$ pseudo-tensor product. This shows that the *quadratic* part of $U_{\mathcal{B}}(L)$ is well-defined. To get control of the whole $U_{\mathcal{B}}(L)$ one needs the Jacobi property of $\mathcal{B}$.

Let us turn to the actual proof. The first step (3.7.3 below) is to embed our pseudo-tensor category in a representable (actually, a tensor) one:

3.7.3. (i) A *right $\mathcal{D}$-module M on $X^{\mathcal{S}}$* is a rule that assigns to $I \in \mathcal{S}$ a right $\mathcal{D}$-module M_{X^I} on X^I and to $\pi : J \twoheadrightarrow I$ a morphism of $\mathcal{D}$-modules $\theta^{(\pi)} : \Delta_*^{(\pi)} M_{X^I} \to M_{X^J}$. We demand that $\theta^{(\pi)}$'s are compatible with composition of π's[159] and $\theta^{(id_I)} = id_{M_{X^I}}$.

These objects form an abelian k-category $\mathcal{M}(X^{\mathcal{S}})$. There is an exact fully faithful embedding $\mathcal{M}(X) \hookrightarrow \mathcal{M}(X^{\mathcal{S}})$ which identifies a $\mathcal{D}$-module M on X with the one on $X^{\mathcal{S}}$ given by $M_{X^I} := \Delta_*^{(I)} M$, $\theta^{(\pi)} = id_{\Delta_*^{(J)} M}$. This embedding is left adjoint to the projection functor $\mathcal{M}(X^{\mathcal{S}}) \to \mathcal{M}(X)$, $M \mapsto M_X$.

[158] In the chiral setting the proof of PBW takes more time, see 3.7.26–3.7.??.

[159] i.e., $\theta^{(\pi_1\pi_2)} = \theta^{(\pi_2)} \Delta_*^{(\pi_2)}(\theta^{(\pi_1)})$.

Let $\mathcal{M}(X^{\mathcal{S}})_\Delta \subset \mathcal{M}(X^{\mathcal{S}})$ be the full subcategory of those M that every M_{X^I} is supported on the diagonal $X \subset X^I$; it is closed under subquotients and extensions. $\mathcal{M}(X^{\mathcal{S}})_\Delta$ is equivalent to the category $\mathcal{M}(X)^{\mathcal{S}}$ of all functors $\mathcal{S}^\circ \to \mathcal{M}(X)$ (one identifies $M \in \mathcal{M}(X^{\mathcal{S}})_\Delta$ with the functor $I \mapsto \Delta^{(I)!}M_{X^I}$). Notice that $\mathcal{M}(X) \subset \mathcal{M}(X^{\mathcal{S}})_\Delta$, and this embedding admits a left adjoint functor $\epsilon : \mathcal{M}(X^{\mathcal{S}})_\Delta \to \mathcal{M}(X)$, $M \mapsto \varinjlim \Delta^{(I)!}M_{X^I}$.

Remark. For $n \geq 1$ let $\mathcal{S}_n \subset \mathcal{S}$ be the subcategory of sets of order $\leq n$. The above category has an obvious "truncated" version $\mathcal{M}(X^{\mathcal{S}_n})$. The "restriction" functor $\mathcal{M}(X^{\mathcal{S}}) \xrightarrow{\pi_n} \mathcal{M}(X^{\mathcal{S}_n})$ admits a left adjoint $a_n : \mathcal{M}(X^{\mathcal{S}_n}) \to \mathcal{M}(X^{\mathcal{S}})$, $a_n(N)_{X^I} = \varinjlim \Delta_*^{(I/S)} N_{X^S}$ where the inductive limit is taken over the ordered set $Q(I, \leq n)$. Since $\pi_n a_n$ is the identity functor we see that π_n identifies $\mathcal{M}(X^{\mathcal{S}_n})$ with the quotient category of $\mathcal{M}(X^{\mathcal{S}})$ modulo the full subcategory $\mathcal{M}(X^{\geq n+1})$ of such M's that $\pi_n M = 0$, i.e., $M_{X^I} = 0$ if $|I| \leq n$. In particular, $\mathrm{gr}_n \mathcal{M}(X^{\mathcal{S}}) := \mathcal{M}(X^{\geq n})/\mathcal{M}(X^{\geq n+1})$ equals the category of Σ_n-equivariant $\mathcal{D}$-modules on X^n.

(ii) $\mathcal{M}(X^{\mathcal{S}})$ carries two tensor products $\otimes^*$ and $\otimes^{ch}$ defined as follows. Let M_α, $\alpha \in A$ be a finite non-empty family of objects of $\mathcal{M}(X^{\mathcal{S}})$. One has

$$(\otimes_A^* M_\alpha)_{X^I} := \bigoplus_{I \twoheadrightarrow A} \boxtimes_A (M_{\alpha X^{I_\alpha}}), \tag{3.7.3.1}$$

the arrows $\theta^{(\pi)}$ are the obvious ones. For example, if $M_\alpha \in \mathcal{M}(X)$ then $(\otimes^* M_\alpha)_{X^I} = \bigoplus_{I \twoheadrightarrow A} \Delta_*^{(I/A)} \boxtimes_A M_\alpha$; in particular, $(\otimes_A^* M_\alpha)_{X^A} = \bigoplus_{\nu \in \mathrm{Aut}\, A} \nu^*(\boxtimes M_\alpha)$. Similarly, set

$$(\otimes_A^{ch} M_\alpha)_{X^I} := \bigoplus_{I \twoheadrightarrow A} j_*^{[J/A]} j^{[J/A]*} \boxtimes M_{\alpha X^{I_\alpha}}, \tag{3.7.3.2}$$

the arrows $\theta^{(\pi)}$ are obvious ones. Here $j^{[J/A]}$ was defined in 3.5.4.

Our tensor products are subject to obvious "associativity and commutativity constraints", so they define on $\mathcal{M}(X^{\mathcal{S}})$ tensor category structures which we denote by $\mathcal{M}(X^{\mathcal{S}})^*$, $\mathcal{M}(X^{\mathcal{S}})^{ch}$. We have an obvious canonical morphism

$$\otimes^* M_\alpha \to \otimes^{ch} M_\alpha, \tag{3.7.3.3}$$

so the identity functor for $\mathcal{M}(X^{\mathcal{S}})$ extends to a pseudo-tensor functor

$$\beta^{\mathcal{S}} : \mathcal{M}(X^{\mathcal{S}})^{ch} \to \mathcal{M}(X^{\mathcal{S}})^*. \tag{3.7.3.4}$$

(iii) The embedding $\mathcal{M}(X) \hookrightarrow \mathcal{M}(X^{\mathcal{S}})$ extends canonically to fully faithful pseudo-tensor embeddings

$$\mathcal{M}(X)^* \hookrightarrow \mathcal{M}(X^{\mathcal{S}})^*, \quad \mathcal{M}(X)^{ch} \hookrightarrow \mathcal{M}(X^{\mathcal{S}})^{ch}. \tag{3.7.3.5}$$

Namely, for $M_\alpha, N \in \mathcal{M}(X)$ the identification $\mathrm{Hom}(\otimes^*_A M_\alpha, N) \xrightarrow{\sim} P^*_A(\{M_\alpha, N)$ sends a morphism $\varphi : \otimes^*_A M_\alpha \to N$ to the restriction of φ_{X^A} to $\underset{A}{\boxtimes} M_\alpha \subset (\otimes^*_A M_\alpha)_{X^A}$. The identification $\mathrm{Hom}(\otimes^*_A M_\alpha, N) \xrightarrow{\sim} P^*_A(\{M_\alpha, N)$ is defined in the similar way. Notice that the restriction of $\beta_{\mathcal{S}}$ to $\mathcal{M}(X)^*$ is the pseudo-tensor functor β from (3.2.1.2).

The functor $\epsilon : \mathcal{M}(X^{\mathcal{S}})_\Delta \to \mathcal{M}(X)$ extends in the obvious way to pseudo-tensor functors

$$\mathcal{M}(X^{\mathcal{S}})^*_\Delta \to \mathcal{M}(X)^*, \quad \mathcal{M}(X^{\mathcal{S}})^{ch}_\Delta \to \mathcal{M}(X)^{ch} \tag{3.7.3.6}$$

which are left adjoint to the pseudo-tensor functors from (3.7.3.5).

(iv) For any k-operad $\mathcal{B}$ denote by $\mathcal{B}^*(X^{\mathcal{S}})$, $\mathcal{B}^{ch}(X^{\mathcal{S}})$ the categories of $\mathcal{B}$ algebras in $\mathcal{M}(X^{\mathcal{S}})^*$, $\mathcal{M}(X^{\mathcal{S}})^{ch}$ (we call them $\mathcal{B}^*$ and $\mathcal{B}^{ch}$ algebras on $X^{\mathcal{S}}$). Notice that by (3.7.3.4) every $\mathcal{B}$ algebra in $\mathcal{M}(X^{\mathcal{S}})^{ch}$ is automatically a $\mathcal{B}$ algebra in $\mathcal{M}(X^{\mathcal{S}})^*$, i.e., the pseudo-tensor functor $\beta^{\mathcal{S}}$ yields a functor $\mathcal{B}^{ch}(X^{\mathcal{S}}) \to \mathcal{B}^*(X^{\mathcal{S}})$. This functor admits a left adjoint functor

$$U^{\mathcal{S}}_{\mathcal{B}} : \mathcal{B}^*(X^{\mathcal{S}}) \to \mathcal{B}^{ch}(X^{\mathcal{S}}). \tag{3.7.3.7}$$

Indeed, for a $\mathcal{B}^*$ algebra L the corresponding $\mathcal{B}^{ch}$ algebra $U^{\mathcal{S}}_{\mathcal{B}}(L)$ is the quotient of the free $\mathcal{B}^{ch}$ algebra generated by L modulo the relations needed to assure that the canonical morphism $i^{\mathcal{S}} : L \to U^{\mathcal{S}}_{\mathcal{B}}(L)$ is a morphism of $\mathcal{B}^*$ algebras. Precisely, let Φ^*, Φ^{ch} be the free $\mathcal{B}^*$ and $\mathcal{B}^{ch}$ algebras generated by L. Since Φ^{ch} is a $\mathcal{B}^*$ algebra the canonical morphism $i^{ch} : L \to \Phi^{ch}$ extends to a morphism of $\mathcal{B}^*$ algebras $\alpha : \Phi^* \to \Phi^{ch}$. Since L is a $\mathcal{B}^*$ algebra the identity morphism id_L extends to a morphism of $\mathcal{B}^*$ algebras $\beta : \Phi^* \to L$. Now $U^{\mathcal{S}}_{\mathcal{B}}(L)$ is the quotient of Φ^{ch} modulo the $\mathcal{B}^{ch}$ ideal $\mathcal{J}$ generated by the image of $\alpha - i^{ch}\beta : \Phi^* \to \Phi^{ch}$.

3.7.4. We return to the proof of 3.7.2. Let L be a $\mathcal{B}^*$ algebra on X. We want to construct the corresponding universal $\mathcal{B}^{ch}$ algebra $U_{\mathcal{B}}(L)$ on X equipped with a morphism of $\mathcal{B}^*$ algebras $i : L \to U_{\mathcal{B}}(L)$.

Let us consider L as a $\mathcal{B}^*$ algebra on $X^{\mathcal{S}}$, so we have the enveloping algebra $U := U^{\mathcal{S}}_{\mathcal{B}}(L) \in \mathcal{B}^{ch}(X^{\mathcal{S}})$. In the next lemma we will show that $U \in \mathcal{M}(X^{\mathcal{S}})_{\Delta}$. Assuming this, set $U_{\mathcal{B}}(L) := \epsilon(U)$, $i := \epsilon(i^{\mathcal{S}}) : L \to U_{\mathcal{B}}(L)$. By (3.7.3.6) $U_{\mathcal{B}}(L)$ is a $\mathcal{B}$ algebra in $\mathcal{M}(X)^{ch}$ and i is a morphism of $\mathcal{B}^*$ algebras; by the adjunction property of ϵ the pair $(U_{\mathcal{B}}(L), i)$ satisfies the desired universal property. We are done.

It remains to prove the following lemma:

Lemma. If $\mathcal{B}$ is of Jacobi type then $U = U^{\mathcal{S}}_{\mathcal{B}}(L) \in \mathcal{M}(X^{\mathcal{S}})_{\Delta}$.

Proof of Lemma. (a) For a finite set I and $\alpha, \beta \in I$, $\alpha \neq \beta$, let $\mathcal{B}_I^{\alpha\beta} \subset \mathcal{B}_I$ be the image of the composition map $\mathcal{B}_{I/\{\alpha,\beta\}} \otimes \mathcal{B}_{\{\alpha,\beta\}} \to \mathcal{B}_I$. A simple induction shows that $\mathcal{B}$ is of Jacobi type if and only if it satisfies the following property:

For any finite non-empty sets I_1, I_2 the vector space $\mathcal{B}_{I_1 \sqcup I_2}$ is generated by the subspaces $\mathcal{B}^{\alpha\beta}_{I_1 \sqcup I_2}$, $\alpha \in I_1$, $\beta \in I_2$.

(b) Let Φ^{ch} be the free $\mathcal{B}^{ch}$ algebra on $X^{\mathcal{S}}$ generated by L as in 3.7.3(iv). It has a natural gradation $\Phi^{ch} = \underset{n\geq 1}{\oplus} \Phi^{ch}_n$ such that $\Phi^{ch}_n = (L^{\otimes^{ch} I} \otimes \mathcal{B}_I)_{\mathrm{Aut}\, I}$ where I is a set of order n. This gradation yields an increasing filtration on $U = \Phi^{ch}/\mathcal{J}$. We are going to show that $\mathrm{gr}U \in \mathcal{M}(X^{\mathcal{S}})_{\Delta}$; this will prove our lemma.

Clearly $\mathrm{Ker}(\Phi^{ch}_n \twoheadrightarrow \mathrm{gr}_n U) \supset N_{\mathrm{Aut}\, I}$ where $N := \sum\limits_{\alpha\neq\beta\in I} K_{\alpha\beta} \otimes \mathcal{B}^{\alpha\beta}_I \subset L^{\otimes^{ch} I} \otimes \mathcal{B}_I$ and $K_{\alpha\beta}$ is the image of $(L \otimes^* L) \otimes^{ch} L^{\otimes^{ch}(I \smallsetminus \{\alpha,\beta\})}$ in $L^{\otimes^{ch} I}$. So it is enough to show that $M := (L^{\otimes^{ch} I} \otimes \mathcal{B}_I)/N$ belongs to $\mathcal{M}(X^{\mathcal{S}})_{\Delta}$.

Set $U_{\alpha\beta} := \{(x_i) \in X^I : x_i \neq x_j \text{ if } i \neq j, \{i,j\} \neq \{\alpha,\beta\}\}$; let $j_{\alpha\beta} : U_{\alpha\beta} \hookrightarrow X^I$ be the embedding. Set $F := j^{(I)}_* \mathcal{O}_{U^{(I)}} \otimes \mathcal{B}_I$, $G := \sum\limits_{\alpha\neq\beta\in I} j_{\alpha\beta *} \mathcal{O}_{U_{\alpha\beta}} \otimes \mathcal{B}^{\alpha\beta}_I \subset F$, $H := F/G$. By definition, for every $J \in \mathcal{S}$ one has $M_{X^J} = \underset{J \twoheadrightarrow I}{\oplus} \Delta^{(J/I)}_* (L^{\boxtimes I} \otimes H)$. It remains to show that H is supported on the diagonal $X \subset X^I$.

(c) For $I = I_1 \sqcup I_2$ where $I_1, I_2 \neq \emptyset$ set $U_{I_1 I_2} := \{(x_i) \in X^I : x_i \neq x_j \text{ if } i \in I_1, j \in I_2\}$.

On $U_{I_1 I_2}$ the sheaves $j_{\alpha\beta *}\mathcal{O}_{U_{\alpha\beta}}$ coincide with $j_*^{(I)}\mathcal{O}_{U^{(I)}}$. So (a) above implies that the restriction of H to $U_{I_1 I_2}$ is zero. The union of all $U_{I_1 I_2}$'s is the complement to $X \subset X^I$, and we are done. □

3.7.5. Let us return to 3.7.1. We are going to construct the chiral envelope $U(L)$ of a Lie* algebra L explicitely as a factorization algebra (together with the canonical $\mathcal{D}$-module structure). Our problem is X-local, so we can assume that X is affine. To do this we first define in 3.7.6 an auxiliary Lie algebra $L^\natural$ in the tensor category of left $\mathcal{D}$-modules on $\mathcal{R}(X)$. The chiral envelope is constructed in 3.7.7 (it is denoted by V there). The universality property is checked in 3.7.11.

Remark. A slightly unpleasant point of our construction is that the auxiliary Lie algebra $L^\natural$ is a non-local object (while $U(L)$ is of course local). A remedy would be to consider in (3.7.2.1) instead of global cohomology the one of the formal neighbourhood of the diagonal. The price is to deal with non-quasicoherent modules; we prefer not to do this.

In 3.7.6–3.7.11 *we assume that X is affine.*

3.7.6. Let I be a finite set. Denote by p_I the projection $X \times X^I \to X^I$, and let $j_I : V = V_{\tilde{I}} \hookrightarrow X \times X^I$ be the open subset of those $(x, (x_i))$ that $x \in V_{(x_i)} := X \setminus \{x_i\}$. For $L \in \mathcal{M}^r(X)$[160] consider the de Rham complex $DR(L) = \mathcal{C}one(L \otimes \Theta_X \to L)$. Set

$$L^\natural_{X^I} := H^0(p_I j_I)_\cdot j_I^*(DR(L) \boxtimes \mathcal{O}_{X^I}). \tag{3.7.6.1}$$

This is an $\mathcal{O}_{X^I}$-module with fiber at $(x_i) \in X^I$ equal to $H^0_{DR}(V_{(x_i)}, L) = \Gamma(V_{(x_i)}, h(L))$. The obvious left $p_I^\cdot \mathcal{D}^I_X$-module structure on $DR(L) \boxtimes \mathcal{O}_{X^I}$ shows that $L^\natural_{X^I}$ is a left $\mathcal{D}_{X^I}$-module. In other words, $L^{\natural r}_{X^I} = H^0(p_I j_I)_* j_I^*(L \boxtimes \omega_{X^I})$.

Our $L^\natural$ carries a bunch of structures described in (i)-(iii) below that will be used in the construction of the chiral envelope:

(i) For every diagonal embedding $\Delta^{(I/S)} : X^S \hookrightarrow X^I$ we have an obvious identification of left $\mathcal{D}$-modules

$$\Delta^{(I/S)*} L^\natural_{X^I} \xrightarrow{\sim} L^\natural_{X^S}. \tag{3.7.6.2}$$

[160] We do not assume that L is a Lie* algebra at the moment.

So $L^\natural_{X^I}$ form a left $\mathcal{D}$-module on $\mathcal{R}(X)$ (see 3.5.2).

More generally, $L^\natural_{X^I}$ behave nicely with respect to *arbitrary* standard morphisms between X^I's. Namely, if $\pi : J \to I$ is any map of finite sets, then the pull-back of $V_{\tilde{J}}$ by $\Delta^{(\pi)} : X^I \to X^J$ contains $V_{\tilde{I}}$. So we have the restriction map

$$\Delta^{(\pi)*} L^\natural_{X^J} \to L^\natural_{X^I} \tag{3.7.6.3}$$

which coincides with (3.7.6.2) if π is surjective. These maps are injective morphisms of $\mathcal{D}$-modules; they are compatible with composition of π's.

For example, if $J = \emptyset$ then the corresponding vector space $L^\natural_0 := L^\natural_{X^\emptyset}$ is $\Gamma(X, h(L))$ (see 2.5.3). We get a canonical embedding of $\mathcal{D}_{X^I}$-modules

$$L^\natural_0 \otimes \mathcal{O}_{X^I} \hookrightarrow L^\natural_{X^I}. \tag{3.7.6.4}$$

(ii) Denote by $L^\ell_{X^I}$ the cokernel of (3.7.2.4). Since X is affine one has $L^\ell_{X^I} = H^0 p_{I\cdot} F_{\tilde{X^I}}$ where $F_{\tilde{X^I}} := (j_{I\cdot} j_I^* DR(L) \boxtimes \mathcal{O}_{X^I})/(DR(L) \boxtimes \mathcal{O}_{X^I})$. The corresponding right $\mathcal{D}_{X^I}$-module equals to $p_{I*}\Phi_{\tilde{X^I}}$ where $\Phi_{\tilde{X^I}} := j_{I*} j_I^*(L \boxtimes \omega_{X^I})/L \boxtimes \omega_{X^I}$. In particular, $L^\ell_X = L^\ell$.

Notice that $L^\ell_{X^I}$, unlike $L^\natural_{X^I}$, depends on L in a purely X-local way, i.e., $L^\ell_{(x_i)}$ depends only on restriction of L to a neighbourhood of $\{x_i\}$.

For $S \in Q(I)$ consider the open subset $j^{[I/S]} : U^{[I/S]} \hookrightarrow X^I$ (see 3.5.4); for $s \in S$ let $p_s : X^I \to X^{I_s}$ be the projection, and $\tilde{p_s} := id_X \times p_s$. Decomposing $F_{\tilde{X^I}}$ by connected components of its support we get $j^{[I/S]*} \underset{S}{\oplus} \tilde{p_s}^* F_{\tilde{X^{I_s}}} \xrightarrow{\sim} j^{[I/S]*} F_{\tilde{X^I}}$ hence the isomorphism

$$j^{[I/S]*} \underset{S}{\oplus} p_s^* L^\ell_{X^{I_s}} \xrightarrow{\sim} j^{[I/S]*} L^\ell_{X^I}. \tag{3.7.6.5}$$

In particular, one has $j^{(I)*} \oplus p_i^* L^\ell \xrightarrow{\sim} j^{(I)*} L^\ell_{X^I}$.

Here is another interpretation of (3.7.2.5). For $s \in S$ consider the projection $q_s : X^I \to X^{I \smallsetminus I_s}$ and the coresponding embedding (3.7.6.3) of $\mathcal{D}$-modules $q_s^* L^\natural_{X^{I \smallsetminus I_s}} \hookrightarrow L^\natural_{X^I}$. We have a commutative diagram of embeddings of $\mathcal{D}_{X^I}$-modules

$$\begin{array}{ccccc} \prod\limits_S p_s^* L^\natural_{X^{I_s}} & \longrightarrow & (L^\natural_{X^I})^S & \longleftarrow & L^\natural_{X^I} \\ \uparrow & & \uparrow & & \uparrow \\ (L^\natural_0 \otimes \mathcal{O}_{X^I})^S & \longrightarrow & \prod\limits_S q_s^* L^\natural_{X^{I \smallsetminus I_s}} & \longleftarrow & L^\natural_0 \otimes \mathcal{O}_{X^I} \end{array} \tag{3.7.6.6}$$

The morphisms between the cokernels of vertical embeddings $\prod_S (p_s^* L^\natural_{X^{I_s}}/L^\natural_0 \otimes \mathcal{O}_{X^{I_s}}) \to \prod_S L^\natural_{X^I}/L^\natural_{X^{I \smallsetminus I_s}} \leftarrow L^\natural_{X^I}/L^\natural_0 \otimes \mathcal{O}_{X^I}$ being restricted to $U^{[I/S]}$ become isomorphisms; the composition is (3.7.6.5).

(iii) Assume that L is a Lie* algebra. Then $L^\natural_{X^I}$ is a Lie algebra in the tensor category $\mathcal{M}^\ell(X^I)$. One defines the Lie bracket on $L^\natural_{X^I}$ in the same way we defined the bracket on $h(L)$. Namely, the $*$ bracket on L yields a morphism $DR(L) \boxtimes DR(L) \to DR(\Delta_* L)$ on $X \times X$. Take the exterior tensor product of this morphism with $\mathcal{O}_{X^I}$, restrict it to $V \underset{X^I}{\times} V \subset (X \times X) \times X^I$, and push forward to X^I. We get a morphism of complexes of of left $\mathcal{D}_{X^I}$-modules $((p_I j_I)_\cdot j_I^*(DR(L) \boxtimes \mathcal{O}_{X^I}))^{\otimes 2} \to (p_I j_I)_\cdot j_I^*(\Delta^\cdot DR(\Delta_* L) \boxtimes \mathcal{O}_{X^I})$. The second term is canonically quasiisomorphic to $(p_I j_I)_\cdot j_I^*(DR(L) \boxtimes \mathcal{O}_{X^I})$ (see 2.1.6(b)). Passing to H^0 we get a morphism $[\ \ ]^\natural : L^\natural_{X^I} \otimes L^\natural_{X^I} \to L^\natural_{X^I}$ which is the desired Lie bracket.

All the canonical morphisms in (i), (ii) above are compatible with our Lie algebra structure. In particular, $L^\natural$ is a Lie algebra in the tensor category of $\mathcal{D}$-modules on $\mathcal{R}(X)$.

3.7.7. Now we are ready to define our factorization algebra. Set

$$V_{X^I} := U(L^\natural_{X^I})/U(L^\natural_{X^I})L^\natural_0. \tag{3.7.7.1}$$

Here $U(L^\natural_{X^I})$ is the enveloping associative algebra of our Lie algebra $L^\natural_{X^I}$ in the abelian tensor k-category $\mathcal{M}^\ell(X^I)$. So V_{X^I} is *vacuum representation* - the $L^\natural_{X^I}$-module induced from the trivial $L^\natural_0 \otimes \mathcal{O}_{X^I}$-module.

The construction of enveloping algebra and induced module is compatible with pull-backs. Therefore $U(L^\natural_{X^I})$ and V_{X^I} form left $\mathcal{D}$-modules $U(L^\natural)$ and V on $\mathcal{R}(X)$.

Our V has a canonical factorization structure. To define factorization isomorphisms κ (see 3.5.4.1) look at (3.7.6.6) and consider the corresponding morphisms of induced modules

$$\underset{S}{\boxtimes} V_{X^{I_s}} \longrightarrow \underset{S}{\otimes} U(L^\natural_{X^I})/U(L^\natural_{X^I}) q_s^* L^\natural_{X^{I \smallsetminus I_s}} \longleftarrow V_{X^I}. \tag{3.7.7.2}$$

The following well-known lemma (together with 3.7.6(ii)) shows that over $U^{[I/S]}$ both arrows in (3.7.7.2) are isomorphisms. This provides the factorization isomorphisms. The compatibilities from 3.5.4 obviously hold.

3.7.8 Lemma. Let $\phi : L_1 \to L_2$ be a morphism of Lie algebras over a commutative ring R, $N_1 \subset L_1$, $N_2 \subset L_2$ be Lie subalgebras such that $\phi(N_1) \subset N_2$ and $L_1/N_1 \xrightarrow{\sim} L_2/N_2$. Then the morphism $U(L_1)/U(L_1)N_1 \to U(L_2)/U(L_2)N_2$ is an isomorphism.

We give two proofs of the lemma. The first one is elementary. The second proof is more "immediate" but it uses a machinery too heavy for the purpose.[161]

First Proof. Set $P_i := U(L_i)/N_i$; let $v_i \in P_i$ be the "vacuum vectors"; $\pi : P_1 \to P_2$ our morphism.

(i) π is surjective. Indeed, π is compatible with the standard filtrations on P_i's. Since $\mathrm{gr}.P_i$ is a quotient of $\mathrm{Sym}^{\cdot}(L_i/N_i)$, $\mathrm{gr}\pi$ is surjective.

(ii) So to prove our lemma it suffices to construct a left inverse $P_2 \to P_1$ of π. To do this we will define an L_2-module structure on P_1 such that:

(a) The obvious L_1-action on P_1 equals the one coming from the L_2-action via π.

(b) The action of $N_2 \subset L_2$ kills v_1.

As follows from (b) there is a unique morphism of L_2-modules $P_2 \to P_1$ which sends v_2 to v_1. By (a) it is left inverse to π.

(iii) It remains to construct the L_2-action. We use the following remark: for any L_1-modules Q, R one has $\mathrm{Hom}_{L_1}(Q \otimes P_1, R) = \mathrm{Hom}_{L_1}(P_1, \mathrm{Hom}(Q, R)) = \mathrm{Hom}_{N_1}(Q, R)$.

Taking $Q = L_2, R = P_1$ we see that there is a unique morphism of L_1-modules $a : L_2 \otimes P_1 \to P_1$ with property $a(l_2 \otimes v_1) = l_1 v_1$ for every $l_i \in L_i$ such that $l_1 \mathrm{mod} N_1 = l_2 \mathrm{mod} N_2$.

Let us show that a is a Lie algebra action. We need to check coincidence of the two maps $\Lambda^2 L_2 \otimes P_1 \to P_1$. Both maps are L_1-equivariant, so (take $Q = \Lambda^2 L_2, R = P_1$ in the above remark) it suffices to show that their restrictions to $\Lambda^2 L_2 \otimes v_1$ are equal. This is clear, since both maps vanish on $\Lambda^2 N_2 \otimes v_1$, and on the image of $L_1 \otimes L_2 \otimes v_1$ they coincide by the L_1-equivariance.

Property (a) follows from the case $Q = L_1, R = P_1$ in the above remark; property (b) holds by construction. We are done.

Second Proof. Consider the derived version of the induced module construction. Namely,

[161] It will be of use in the proof of the Poincaré-Birkhoff-Witt theorem 3.7.14 though.

for an emedding of Lie algebras $N \subset L$ one may find Lie DG algebras $\tilde{L}, \tilde{N}$ placed in degrees ≤ 0 whose components are flat (e.g, free) R-modules, together with projections $\tilde{L} \to L$, $\tilde{N} \to N$ which are quasiisomorphisms, and morphisms of Lie DG algebras $\tilde{N} \to \tilde{L}$ that lifts the embedding $N \hookrightarrow L$. Consider the enveloping associative DG algebra $U(\tilde{L})$ as an $(\tilde{L}, \tilde{N})$-bimodule (with respect to the left and right actions). Denote by C the homological Chevalley complex of $\tilde{N}$ with coefficients in $U(\tilde{L})$ (where $\tilde{N}$ acts on $U(\tilde{L})$ by right multiplication). This is a DG $\tilde{L}$-module ($\tilde{L}$ acts by left multiplication) which sits in degrees ≤ 0. There is an obvious isomorphism $H^0 C \xrightarrow{\sim} U(L)/U(L)N$.

As a graded module C equals $U(\tilde{L}) \otimes \mathrm{Sym}(\tilde{N}[1])$. It carries a filtration $C_0 \subset C_1 \subset ..$ equal to the tensor product of the standard filtration $U(\tilde{L})$. and the filtration $\mathrm{Sym}^{\leq \cdot}(\tilde{N}[1])$. This filtration is stable with respect to the differential, and there is a canonical isomorphism $\mathrm{gr}_i C = \mathrm{Sym}^i(\mathcal{C}one(\tilde{N} \to \tilde{L}))$. Therefore, as an object of the derived category of R-modules, $\mathrm{gr}_i C$ equals $\mathbb{L}\mathrm{Sym}^i(L/N)$.

Let us return to the situation of 3.7.8. Choose resolutions as above for both (L_1, N_1), (L_2, N_2) and morphisms of Lie DG algebras $\phi_{\tilde{L}} : \tilde{L_1} \to \tilde{L_2}$, $\phi_{\tilde{N}} : \tilde{N_1} \to \tilde{N_2}$ that lift ϕ and the diagram

$$\begin{array}{ccc} \tilde{L_1} & \xrightarrow{\phi_{\tilde{L}}} & \tilde{L_2} \\ \uparrow & & \uparrow \\ \tilde{N_1} & \xrightarrow{\phi_{\tilde{N}}} & \tilde{N_2} \end{array}$$

commutes. We get a morphism of filtered complexes $\phi_C : C_1 \to C_2$. Notice that $\mathrm{gr}_i(\phi_C)$ is the ith symmetric power of $\tilde{\phi} : \mathcal{C}one(\tilde{N_1} \to \tilde{L_1}) \to \mathcal{C}one(\tilde{N_2} \to \tilde{L_2})$. The condition of 3.7.4 assures that this $\tilde{\phi}$, hence $\mathrm{Sym}^{\cdot}\tilde{\phi}$, is a quasiisomorphism. Therefore ϕ_C is a quasiisomorphism. Passing to H^0 we get our lemma. $\square$

Remark. The above lemma also shows that V_X has étale local nature. Namely, let $\pi : X' \to X$ be an étale map, $L_{X'}$ the π-pull-back of $L = L_X$, $V_{X'}$ the corresponding vacuum module. One has an obvious morphism of Lie algebras $\pi^* L_X^{\natural} \to L_{X'}^{\natural}$. It yields the morphism of vacuum modules $\pi^* V_X \to V_{X'}$ which is an isomorphism.

3.7.9 Lemma. Our V is a factorization algebra; its $\mathcal{D}$-module structure coincides with the canonical one.

Proof. One needs only to check the flatness along codimension 1 diagonals. Indeed, the unit in V is the canonical generator 1 of the induced module V_X; since 1 is horizontal our $\mathcal{D}$-module structure is the canonical one.

Assume that $I = \{1,..,n\}$ and the codimension 1 diagonal $X^{n-1} \hookrightarrow X^n$ we consider is given by equation $x_{n-1} = x_n$. Let $j' : U' \hookrightarrow X^n$ be the complement to the union of diagonals $x_1 = x_n, .., x_{n-1} = x_n$, and projections $q : X^n \to X^{n-1}$, $p : X^n \to X$ are, respectively, $(x_1,..,x_n) \mapsto (x_1,..,x_{n-1})$, $(x_1,..,x_n) \mapsto x_n$.

The embeddings $q^* L^\natural_{X^{n-1}} \hookrightarrow L^\natural_{X^n} \hookleftarrow p^* L^\natural_X$ (see 3.7.6(i)) yield a short exact sequence

$$0 \to q^* L^\natural_{X^{n-1}} \to L^\natural_{X^n} \to j'_* j'^* p^* L^\ell_X \to 0. \tag{3.7.9.1}$$

Consider the canonical filtration on the enveloping algebra $U(L^\natural_{X^n})$. The Poincaré-Birkhoff-Witt theorem $\mathrm{gr} U = \mathrm{Sym}$ holds for Lie algebras in any abelian tensor $\mathbb{Q}$-category (see, e.g., [DM] 1.3.7), so $\mathrm{gr}_a U(L^\natural_{X^n}) = \mathrm{Sym}^a L^\natural_{X^n}$. Let us consider (3.7.9.1) as a two step increasing filtration on $L^\natural_{X^n}$. Notice that the sub- and quotient modules in our filtration are Tor-independent. Therefore it defines on every $\mathrm{Sym}^a L^\natural_{X^n}$ an increasing filtration with successive quotients $q^*(\mathrm{Sym}^{a-m} L^\natural_{X^{n-1}}) \otimes j'_* j'^* p^*(\mathrm{Sym}^m L^\ell_X)$. Set $Q := U(L^\natural_{X^n})/q^* U(L^\natural_{X^{n-1}})$; we see that $Q = j'_* j'^* Q$.

The Lie algebra $L^\natural_0 \subset q^* L^\natural_{X^{n-1}} \subset L^\natural_{X^n}$ acts on $U(L^\natural_{X^n})$ by right multiplications, and V_{X^n} is the $\mathcal{D}_{X^n}$-module of coinvariants $U(L^\natural_{X^n})_{L^\natural_0}$. The short exact sequence of $L^\natural_0$-modules $0 \to q^* U(L^\natural_{X^{n-1}}) \to U(L^\natural_{X^n}) \to Q \to 0$ yields the short exact sequence of coinvariants

$$0 \to q^* V_{X^{n-1}} \to V_{X^n} \to Q_{L^\natural_0} \to 0. \tag{3.7.9.2}$$

Indeed, one needs only check the exactness from the left which follows from the fact that every morphism from a $j'_* \mathcal{O}_{U'}$-module to $q^* V_{X^{n-1}}$ vanishes.

Both sub- and quotient modules in (3.7.9.2) are flat along our diagonal $X^{n-1} \subset X^n$ (recall that $Q_{L^\natural_0} = j'_* j'^* Q_{L^\natural_0}$), so same is true for the middle term. We are done. $\square$

3.7.10. Consider the chiral algebra V^r that corresponds to our factorization algebra V. So V^r is the right $\mathcal{D}$-module corresponding to the left $\mathcal{D}_X$-module $V_X = U(L^\natural_X)/U(L^\natural_X) L^\natural_0$.

The chiral product $\mu : j_*j^*V_X \boxtimes V_X \to \Delta_* V_X$ may be described explicitly as follows. The exact sequence $0 \to p_2^* L_X^\natural \to L_{X\times X}^\natural \to j_*j^*p_1^*L_X^\ell \to 0$ (see (3.7.9.1)) shows that the embedding $p_1^*L_X^\natural \hookrightarrow L_{X\times X}^\natural$ yields an isomorphism between the induced modules $j^*p_1^*V_X \xrightarrow{\sim} j^*U(L_{X\times X}^\natural)/U(L_{X\times X}^\natural)p_1^*L_X^\natural$. In particular, $L_{X\times X}^\natural$ acts on $j_*j^*p_1^*V_X$; same for $j_*j^*p_2^*V_X$. Therefore the tensor product $j_*j^*V_X \boxtimes V_X$ is an $L_{X\times X}^\natural$-module. There is a canonical isomorphism of $L_{X\times X}^\natural$-modules $j_*j^*V_{X\times X} \xrightarrow{\sim} j_*j^*V_X \boxtimes V_X$ which sends the generator $1_{X\times X}$ to $1_X \boxtimes 1_X$. Our chiral product μ is composition of the inverse to this isomorphism and projection $j_*j^*V_{X\times X} \to (j_*j^*V_{X\times X})/V_{X\times X} = \Delta_* V_X$ where the letter identification is the structure isomorphism $\Delta^* V_{X\times X} = V_X$. Notice that μ is a morphism of $L_{X\times X}^\natural$-modules.

The morphism $L_X^\natural \to V_X$, $l \mapsto l\cdot 1$, kills $L_0^\natural$, so it yields a morphism of right $\mathcal{D}_X$-modules $i : L \to V^r$. The next proposition finishes the proof of 3.7.1:

3.7.11 Proposition. The above $i : L \to V^r$ is a morphism of Lie* algebras. It satisfies the universality property: for every chiral algebra A the map $\mathrm{Hom}_{\mathcal{CA}(X)}(V^r, A) \to \mathrm{Hom}_{\mathcal{L}ie^*(X)}(L, A^{(*)})$, $\varphi \mapsto \varphi i$, is bijective.

Proof. We begin with a useful general construction.

Let A be a chiral algebra, M an A-module. Consider the Lie algebra[162] $A_X^\natural := A_X^{(*)\natural}$ in the tensor category $\mathcal{M}^\ell(X)$ (see 3.7.6). One has $H^0 p_{2*}j_*j^*A \boxtimes M = H^0(p_{2*}j_*j^*A \boxtimes \omega_X) \otimes^! M = A_X^\natural \otimes M$; here $p_2 : X\times X \to X$ is the second projection. Therefore we get a canonical morphism

$$(3.7.11.1) \qquad \bullet = \bullet_M := p_{2*}(\mu_M) : A_X^\natural \otimes M \to M.$$

Exercise: check that this is a Lie algebra action of $A_X^\natural$ on the $\mathcal{D}$-module M. On the Lie subalgebra $A_0^\natural = H^0_{DR}(X, A) = \Gamma(X, h(A)) \subset A_X^\natural$ this action coincides with the usual $h(A)$-action for the $A^{(*)}$-module structure on M (see 2.5.4).

Notice that $\bullet_M$ determines μ_M uniquely (since p_{2*} is right exact and for $\mathcal{D}_{X\times X}$-modules supported on the diagonal it is fully faithful).

[162] Specialists in vertex operator algebras sometimes call a similar object "the Lie algebra of local fields".

In particular, we have a canonical action of $A_X^\natural$ on A. The morphism $A_X^\natural \to A^\ell, a^\natural \mapsto a^\natural \bullet 1$, coincides with the standard projection $A_X^\natural \to A_X^\natural / A_0^\natural \otimes \mathcal{O}_X = A^\ell$ (see 3.7.6(ii)). A morphism of chiral algebras $A \to A'$ is the same as a morphism of $\mathcal{D}$-modules $\varphi : A \to A'$ which is compatible with $\bullet$ actions (i.e., $\varphi \bullet_A = \bullet_{A'}(\varphi^\natural \otimes \varphi)$) and sends 1_A to $1_{A'}$.

Let us return to our chiral algebra V^r. Let $\bullet_L : L_X^\natural \otimes V_X \to V_X$ be the morphism $\bullet_{V^r}$ composed with $i^\natural : L_X^\natural \to V_X^{r\natural}$.

Sublemma. $\bullet_L$ coincides with our old canonical $L^\natural$-action on V_X.

Proof of Sublemma. We know that the chiral product $\mu : j_*j^*V_X \boxtimes V_X \to \Delta_* V_X$ is a morphism of $L_{X\times X}^\natural$-modules (see 3.7.10). Restricting this action to $p_2^* L_X^\natural \subset L_{X\times X}^\natural$ and pushing μ forward by p_2 we see that $\bullet : V_X^{r\natural} \otimes V_X \to V_X$ is a morphism of $L_X^\natural$-modules. Here $L_X^\natural$ acts on $V_X^{r\natural} := H^0 p_{2*} j_* j^* p_1^* V_X$ via the embedding $p_2^* L_X^\natural \hookrightarrow L_{X\times X}^\natural$ and the $L_{X\times X}^\natural$-action on $j_*j^*p_1^*V_X = U(L_{X\times X}^\natural)/U(L_{X\times X}^\natural)p_2^*L_X^\natural$ (see 3.7.10), and on V_X in our old canonical way (as on the vacuum module). One checks easily that $i^\natural : L_X^\natural \to V_X^{r\natural}$ is a morphism of $L_X^\natural$-modules. So for $l, l' \in L_X^\natural$, $v \in V_X$ one has $l'(l \bullet_L v) = [l', l] \bullet_L v + l \bullet_L (l'v)$. Since $l \bullet_L 1 = l1$ and 1 generates V_X as an $L_X^\natural$-module, we are done. $\square$

End of proof of 3.7.11. Take $l \in L_0^\natural \subset L_X^\natural$, $l' \in L$. By sublemma, $l \bullet_L i(l') = i(ad_l l')$. This shows that i is a morphism of Lie* algebras (see 2.5.2). Our sublemma also implies that V^r is generated by $i(L)$ as a chiral algebra, so the map $\mathrm{Hom}_{\mathcal{CA}(X)}(V^r, A) \to \mathrm{Hom}_{\mathcal{L}ie^*(X)}(L, A^{(*)})$ is injective. Let us define its inverse. Let $\psi : L \to A^{(*)}$ be a morphism of Lie* algebras. It yields a morphism of Lie algebras $\psi^\natural : L_X^\natural \to A_X^\natural$, hence $\bullet_A$ defines an $L_X^\natural$-action on A. Since $L_0^\natural \subset L_X^\natural$ kills 1_A we get a morphism of $L_X^\natural$-modules $\tilde{\psi} : V_X \to A^\ell$ which sends 1 to 1_A. One has $\tilde{\psi} i = \psi$, so it remains to check that $\tilde{\psi}$ is a morphism of chiral algebras. Since it is a morphism of $L_X^\natural$-modules, we know that $\mu_A(\tilde{\psi}^{\boxtimes 2})$ equals $\tilde{\psi}\mu_{V^r}$ on the image of $j_*j^*L \boxtimes V^r \to j_*j^*V^r \boxtimes V^r$. Since the image of L generates V^r as a chiral algebra, we are done. $\square$

3.7.12. Let L_α be a family of Lie* algebras. Consider the Lie* algebra $\oplus L_\alpha$. It follows immediately from the universality property of $\otimes$ and U (see 3.5.16, 3.7.1) that one has a canonical isomorphism of chiral algebras

$$U(\oplus L_\alpha) \xrightarrow{\sim} \otimes U(L_\alpha). \tag{3.7.12.1}$$

In particular, for any Lie* algebra L the diagonal map $L \to L \times L$ yields a coproduct morphism $U(L) \to U(L) \otimes U(L)$ which makes $U(L)$ a chiral Hopf algebra (see 3.5.18).

3.7.13. We are going to prove a chiral version of the Poincaré-Birkhoff-Witt theorem.

Let A be a chiral algebra equipped with a filtration $A_0 \subset A_1 \subset .. \subset A$ (see 3.3.11). Notice that A_0 is a chiral subalgebra of A, and A_1 is an A_0-submodule. We say that our filtration $A.$ is *1-generated* if all the arrows $j_*j^*A_i \boxtimes A_j \to \Delta_* A_{i+j}$, $i, j \geq 0$, are surjective. If $R \subset A$ is a chiral subalgebra of A and $M \supset R$ an R-submodule of A that generates A then there is a unique 1-generated filtration $A.$ such that $A_0 = R$, $A_1 = M$. Namely, $A_{i+1} := \mu(j_*j^*A_1 \boxtimes A_i)$ for $i \geq 0$. We say that this filtration is generated by (R, M); if $R = \omega_X \cdot 1$, then we simply say that our filtration is generated by M. Assume that M is a Lie* subalgebra of A,[163] R is commutative, and M normalizes R. Then our filtration is commutative,[164] and the obvious surjective morphism of $\mathbb{Z}$-graded commutative $\mathcal{D}_X$-algebras $\mathrm{Sym}^{\cdot} M^\ell \to \mathrm{gr}.A^\ell$ is compatible with the coisson brackets.

Let us return to our situation. Denote by L^{tor} the $\mathcal{O}_X$-torsion submodule of L. Since any binary chiral operation on an $\mathcal{O}$-torsion $\mathcal{D}$-module vanishes the canonical morphism $L \to U(L)$ kills the commutator $[L^{tors}, L^{tors}]$ (which is a Lie ideal in L). Therefore $U(L) \xrightarrow{\sim} U(L/[L^{tor}, L^{tor}])$.

The chiral algebra $U(L)$ is generated by the image of L which is a Lie* subalgebra of $U(L)$. The corresponding filtration $U(L).$ is called *the standard*, or *Poincaré-Birkhoff-Witt, filtration*. We have a canonical surjective morphism of coisson algebras

$$\mathrm{Sym}^{\cdot} L^\ell \twoheadrightarrow \mathrm{gr}.U(L)^\ell \tag{3.7.13.1}$$

which factors through

$$\mathrm{Sym}^{\cdot}(L/[L^{tor}, L^{tor}])^\ell \twoheadrightarrow \mathrm{gr}.U(L)^\ell. \tag{3.7.13.2}$$

[163] The next conditions hold automatically if $R = \omega_X \cdot 1$.

[164] See 3.3.11.

3.7.14 Theorem. (3.7.13.2) is an isomorphism. In particular, if L is $\mathcal{O}_X$-flat then (3.7.13.1) is an isomorphism.

Proof. We can assume that X is affine. Consider the projection $U(L_X^\natural) \to V_X = U(L)^\ell$. As follows from the proof of 3.7.11 (see the Sublemma) our standard filtration equals to the image of the usual standard filtration on the enveloping algebra $U(L_X^\natural)$.

Consider the filtered $\mathcal{O}_X$-complex C from the second proof of 3.7.8 for the pair $L_0^\natural \subset L_X^\natural$. Then H^0C equals $U(L)^\ell$ (as a filtered module) and $\mathrm{gr}_i C = \mathbb{L}\mathrm{Sym}^i_{\mathcal{O}_X}(L^\ell)$. Therefore one has $H^0\mathrm{gr}_i C = \mathrm{Sym}^i L^\ell$, $H^{-1}\mathrm{gr}_i C = \mathrm{Sym}^{i-2}(L/L^{tor})^\ell \otimes \mathrm{Tor}_1^{\mathcal{O}_X}(L^{tor\ell}, L^{tor\ell})^{\Sigma_2} = \mathrm{Sym}^{i-2}(L/L^{tor})^\ell \otimes \Delta^!(L^{tor} \boxtimes L^{tor})^{\ell\, sgn}$. The differential $d_1^{-2,1} : H^{-1}\mathrm{gr}_2 C \to H^0\mathrm{gr}_1 C$ in the spectral sequence coincides with the Lie* bracket $\Delta^!(L^{tor}\boxtimes L^{tor})^{\ell\, sgn} \to L^{tor\ell} \subset L^\ell$.[165] Now recall that C is a module over some Lie algebra whose quotient is $L_X^\natural$. This implies that the image of the differential $d_1 : H^{-1}\mathrm{gr} C \to H^0 \mathrm{gr} C$ is the ideal generated by the image of $d_1^{-2,1}$, and all higher $d_r^{p,q}$'s for $p+q=-1$ vanish (since they all vanish on $\mathrm{Ker} d_1^{-2,1}$). We are done. □

We are going to describe $U(L)$-modules in more explicit terms (contrary to the usual situation $U(L)$-modules are *not* the same as L-modules).

3.7.15 Proposition. The obvious functor $\mathcal{M}(X, U(L)) \to \mathcal{M}(X, L)$ admits a left adjoint induction functor

$$\mathrm{Ind} = \mathrm{Ind}_L : \mathcal{M}(X, L) \to \mathcal{M}(X, U(L)).$$

Proof. For a $\mathcal{D}$-module M an L-action on it amounts to a Lie* algebra structure on $L \oplus M$ such that the bracket on M vanishes and $L \hookrightarrow L \oplus M$ is a morphism of Lie* algebras. So an L-module M yields a chiral algebra $U(L \oplus M)$. The $\mathbb{G}_m$-action on M by homotheties defines a $\mathbb{Z}$-grading on this chiral algebra. Its zero component is $U(L)$. The degree 1 component is $\mathrm{Ind} M$. To check the universality property use 3.3.5. □

Remark. The functor Ind is compatible with étale localization of X.

[165] To check this we can assume that $L = L^{tor}$ and it is supported at a single point $x \in X$, i.e., $L = i_{x*}F$ for some Lie algebra F. Then $L^\natural = j_{x*}(F \otimes \mathcal{O}_{U_x})$, $L_0^\natural = F \otimes \mathcal{O}_X$ and C is the Chevalley complex of F with coefficients in $U(F) \otimes j_{x*}\mathcal{O}_{U_x}$. Now the statement is clear.

Let us describe the induced module Ind$M$ explicitly. We can assume that X is affine. Consider the Lie algebra $L_X^\natural$ defined in 3.7.6(iii). The Lie* action of L on M yields an action on M of the Lie subalgebra $L_0^\natural \subset L_X^\natural$. Now one has a canonical isomorphism

$$(3.7.15.1) \qquad \mathrm{Ind}M \xrightarrow{\sim} U(L_X^\natural) \underset{U(L_0^\natural)}{\otimes} M.$$

3.7.16. Define a *chiral L-module* as a $\mathcal{D}_X$-module M equiped with a chiral pairing $\mu = \mu_{(L,M)} \in P_2^{ch}(\{L,M\},M)$ (called a *chiral action* of L on M) which satisfies the following "truncated" version of the Lie algebra action axiom. Let $j' : U' \hookrightarrow X \times X \times X$ be the complement to the diagonals $x_1 = x_3$ and $x_2 = x_3$. Consider a chiral operation $\mu_{(L_1,M)}(id_{L_1}, \mu_{(L_2,M)}) - \mu_{(L_2,M)}(id_{L_2}, \mu_{(L_1,M)}) \in P_3^{ch}(\{L,L,M\},M)$ and its restriction $\nu : j'_* j'^*(L \boxtimes L \boxtimes M) \to \Delta^{(3)} M$. We demand that ν equals to composition

$$(3.7.16.1) \qquad j'_* j'^*(L \boxtimes L \boxtimes M) \to \Delta_*^{1=2} j_* j^*(L \boxtimes M) \to \Delta_*^{(3)} M.$$

Here $\Delta^{1=2} : X \times X \hookrightarrow X \times X \times X$ is the embedding $(x,y) \mapsto (x,x,y)$, the first arrow is $[\quad]_L \boxtimes id_M$, the second one is $\Delta_*^{1=2}(\mu_{(L,M)})$.

Example. Assume that M is supported at a point $x \in X$. Then a chiral L-action on M is the same as a $*$ action of the Lie* algebra $j_{x*} j_x^* L$ on M. Here $j_x : U_x \hookrightarrow X$ is the complement to x.

Assume that X is affine. As in the beginning of the proof of 3.7.11, a chiral pairing $\mu_{(L,M)}$ amounts to a morphism $\bullet_\mu : L_X^\natural \otimes M \to M$ such that its restriction to $L_0^\natural \otimes M = \Gamma(X, h(L)) \otimes M$ comes from a $*$ operation $\in P_2^*(\{L,M\},M)$.

3.7.17 Lemma. $\mu_{(L,M)}$ is a chiral action if and only if $\bullet_\mu$ is an action of the Lie algebra $L_X^\natural$ on M.

Proof. Left to the reader. □

Denote by $\mathcal{M}(X, L^{ch})$ the category of chiral L-modules on X. A chiral $U(L)$-module structure defines in the obvious manner a chiral L-action on M. So we have a functor

$$(3.7.17.1) \qquad \mathcal{M}(X, U(L)) \to \mathcal{M}(X, L^{ch}).$$

3.7.18 Proposition. Functor (3.7.17.1) is an equivalence of categories.

Proof. Left to the reader. □

Let $x \in X$ be a point, $j_x : U_x \hookrightarrow X$ its complement. The topological associative algebra $U(L)_x^{as}$ (see 3.4.2–3.4.4) can be described as follows. This algebra depends only on restriction of $U(L)$ to U_x (see Remark in 3.4.4), so it is convenient to change notation for the moment and assume that L is a Lie* algebra on U_x. Consider the topological Lie algebra $L_{(x)} := \hat{h}_x^{Lie} j_{x*}L$ (see 2.5.12). Denote by $U(L_{(x)})$ the topological enveloping algebra of $L_{(x)}$ defined as completion of the plain enveloping algebra with respect to topology whose base of opens is formed by left ideals generated by an open subspace in $L_{(x)}$.

3.7.19 Proposition. There is a canonical isomorphism of topological associative algebras

$$U(L_{(x)}) \xrightarrow{\sim} U(L)_x^{as}.$$

Proof. $L_{(x)}$ and $U(L)_x^{as}$ are completions of, respectively, $h_x j_{x*}L$ and $h_x j_{x*}U(L)$ with respect to certain topologies. It follows from definitions that the canonical morphism $h_x j_{x*}L \to h_x j_{x*}U(L)$ is continuous with respect to these topologies, thus we have a continuous morphism $L_{(x)} \to U(L)_x^{as}$. This is a morphism of Lie algebras (as follows from Exercise in 3.4.4). So it yields a morphism of topological associative algebras $U(L_{(x)}) \to U(L)_x^{as}$. According to 2.5.15, 3.7.18, 3.4.5, and Example in 3.7.16 this morphism yields an equivalence between the categories of discrete modules. Since our algebras are complete and separated, and their topologies are defined by systems of left ideals, this shows that our morphism of topological algebras is an isomorphism. □

Let us describe some variants of the enveloping algebra construction.

3.7.20. Let $L^\flat$ be an (automatically central) extension of a Lie* algebra L by ω_X. Let $1^\flat$ be the corresponding horizontal section of $L^{\flat\ell}$ (so $L^\ell = L^{\flat\ell}/\mathcal{O}_X 1^\flat$). The $\flat$-*twisted* chiral enveloping algebra of L is the quotient $U(L)^\flat$ of $U(L^\flat)$ modulo the ideal generated by $1 - 1^\flat$. So for any chiral algebra A a morphism $U(L)^\flat \to A$ amounts to a morphism of Lie* algebras $L^\flat \to A^{(*)}$ which maps $1^\flat$ to 1_A.

As in 3.7.13, the image of $L^\flat$ in $U(L)^\flat$ defines the standard filtration on $U(L)^\flat$, and one has a canonical surjective map of coisson algebras

$$\mathrm{Sym}^{\cdot} L^\ell \twoheadrightarrow \mathrm{gr}.U(L)^\flat. \tag{3.7.20.1}$$

One has the twisted version of PBW:

3.7.21 Proposition. If L is $\mathcal{O}_X$-flat then (3.7.20.1) is an isomorphism.

Proof. Notice that $\varepsilon := 1 - 1^\flat$ is a central horizontal section of $U(L^\flat)^\ell$. It defines the endomorphism $\varepsilon\cdot$ of $\mathcal{D}$-module $U(L^\flat)$, and $U(L)^\flat$ is its cokernel. This endomorphism shifts the standard filtration of $U(L^\flat)$ by 1, and the corresponding endomorphism of $\mathrm{gr}U(L^\flat)^\ell = \mathrm{Sym}L^{\flat\ell}$ (see 3.7.14) is multiplication by $1^\flat \in L^\ell$. It is injective with cokernel equal to $\mathrm{Sym}L^\ell$. This implies our PBW. □

The morphism $L^\flat \to U(L)^\flat$ yields the obvious pseudo-tensor functor $\mathcal{M}(X, U(L)^\flat) \to \mathcal{M}(X, L)$ (since any $*$ action of ω_X on any module is trivial $L^\flat$-modules are the same as L-modules). It admits a left adjoint functor $\mathrm{Ind}^\flat = \mathrm{Ind}^\flat_L : \mathcal{M}(X, L) \to \mathcal{M}(X, U(L)^\flat)$ (see 3.7.15). For an L-module M we have $\mathrm{Ind}^\flat_L M = \mathrm{Ind}_{L^\flat} M/(1-1^\flat)\mathrm{Ind}_{L^\flat} M$. One defines the canonical filtration on $\mathrm{Ind}^\flat M$ as the image of the canonical filtration on $\mathrm{Ind}_{L^\flat} M$, so we have a canonical morphism of graded $\mathrm{Sym}L^\ell$-modules

$$\mathrm{Sym}^{\cdot} L^\ell \otimes M \twoheadrightarrow \mathrm{gr}.\mathrm{Ind}^\flat M. \tag{3.7.21.1}$$

3.7.22 Lemma. If L is $\mathcal{O}_X$-flat then (3.7.21.1) is an isomorphism.

Proof. Same as that of 3.7.21 (use ??). □

Here is the twisted version of compatibility 3.7.19. Assume that we are in the situation of 3.7.19, so L is a Lie* algebra on U_x and let $L^\flat$ be an extension of L by ω_{U_x}. Let us compute the topological associative algbera $U(L)^{\flat as}_x = (U(L)^\flat)^{as}_x$.

The morphism of topological Lie algebras $L^\flat_{(x)} \twoheadrightarrow L_{(x)}$ is continuous and open. Its kernel equals the image of $i_{(x)} : k = \hat{h}_x j_{x*}\omega_{U_x} \to L^\flat_{(x)}$. The twisted topological enveloping algebra $U(L_{(x)})^\flat$ is the quotient of $U(L^\flat_{(x)})$ modulo the closed ideal generated by the central element $1^\flat - 1$ where $1^\flat := i_{(x)}(1)$.

3.7.23 Proposition. There is a canonical isomorphism of topological associative algebras

$$U(L_{(x)})^\flat \xrightarrow{\sim} U(L)_x^{\flat as}.$$

Proof. The composition of the morphisms $U(L^\flat_{(x)}) \to U(L^\flat)_x^{as} \to U(L)_x^{\flat as}$ sends $1^\flat$ to 1, so it yields a morphism $U(L_{(x)})^\flat \to U(L)_x^{\flat as}$. As follows from 3.7.19 this morphism induces an equivalence between the categories of discrete modules, hence it is an isomorphism (use the fact that our algebras are complete and separated, and their topologies are defined by systems of left ideals). □

3.7.24. Let A be a chiral algebra and L a Lie* algebra that acts on A (see 3.3.3). For a chiral algebra B consider the set of pairs (i, ψ) where $i: A \to B$ is a morphism of chiral algebras and $\psi: L \to B$ a morphism of Lie* algebras, such that i is a morphism of L-modules (here L acts on B via ψ and the adjoint action of B). Such pairs depend on B functorially, and it is easy to see that there exists a universal pair (i, ψ); the corresponding chiral algebra B is denoted by $A \circledast U(L)$. Indeed, consider the L-action on A as an action of L on the Lie* algebra $A^{(*)}$,[166] so we have the semi-direct product of Lie* algebras L and $A^{(*)}$. Our $A \circledast U(L)$ is the quotient of its chiral enveloping algebra modulo the obvious relations.[167]

Here is another construction of $A \circledast U(L)$. Recall that Ind_L^{ch} is a pseudo-tensor functor, so $\mathrm{Ind}_L^{ch} A$ is a Lie^{ch} algebra. The image of 1_A is the unit in $\mathrm{Ind}_L^{ch} A$, so $\mathrm{Ind}_L^{ch} A$ is a chiral algebra. The canonical morphism $i: A \to \mathrm{Ind}_L^{ch} A$ and the morphism $\psi: L \to \mathrm{Ind}_L^{ch} A$ that comes from the chiral action of L on 1_A satisfy the above properties. One checks immediately the universality property, so $A \circledast U(L) = \mathrm{Ind}_L A$.

We leave it to the reader to check that the canonical filtration on $\mathrm{Ind}_L A$ is a chiral algebra filtration, and the canonical morphism

$$A \otimes \mathrm{Sym}^\cdot L^\ell \to \mathrm{gr}_\cdot(A \circledast U(L)) \tag{3.7.24.1}$$

is a morphism of chiral algebras. According to ?? if L is $\mathcal{O}_X$-flat then (3.7.24.1) is an isomorphism.

[166] We use notation of 3.3.2.

[167] Which say that the morphism $A \to A \circledast U(L)$ is a morphism of chiral algebras, not just of Lie* algebras.