(ii) The composition $\mathfrak{z}_{\mathfrak{g}}(K) \to \mathfrak{Z} \to \mathfrak{z}_{\mathfrak{g}}(O)$ is the morphism (97) for $A = \mathfrak{z}_{\mathfrak{g}}(O)$.

(iii) The morphism (100) is $\operatorname{Aut} K$-equivariant.

We will not prove this theorem. In fact, the only nontrivial statement is that (99) (or equivalently (34)) is a ring homomorphism; see ???for a proof.

The natural approach to the above theorem is based on the notion of VOA (i.e., vertex operator algebra) or its geometric version introduced in [BD] under the name of *chiral algebra*.[21] In the next subsection (which can be skipped by the reader) we outline the chiral algebra approach.

3.7.6. A chiral algebra on a smooth curve X is a (left) $\mathcal{D}_X$-module $\mathcal{A}$ equipped with a morphism

$$j_*j^!(\mathcal{A} \boxtimes \mathcal{A}) \to \Delta_*\mathcal{A} \tag{101}$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal, $j : (X \times X) \setminus \Delta(X) \hookrightarrow X$. The morphism (101) should satisfy certain axioms, which will not be stated here. A chiral algebra is said to be *commutative* if (101) maps $\mathcal{A} \boxtimes \mathcal{A}$ to 0. Then (101) induces a morphism $\Delta_*(\mathcal{A} \otimes \mathcal{A}) = j_*j^!(\mathcal{A} \boxtimes \mathcal{A})/\mathcal{A} \boxtimes \mathcal{A} \to \Delta_*\mathcal{A}$ or, which is the same, a morphism

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}\,. \tag{102}$$

In this case the chiral algebra axioms just mean that $\mathcal{A}$ equipped with the operation (102) is a commutative associative unital algebra. So a commutative chiral algebra is the same as a commutative associative unital $\mathcal{D}_X$-algebra in the sense of 2.6. On the other hand, the $\mathcal{D}_X$-module Vac'_X corresponding to the $\operatorname{Aut} O$-module Vac' by 2.6.5 has a natural structure of chiral algebra (see the Remark below). The map $\mathfrak{z}_{\mathfrak{g}}(O)_X \to Vac'_X$ induced by the embedding $\mathfrak{z}_{\mathfrak{g}}(O) \to Vac'$ is a chiral algebra morphism. Given a point $x \in X$ one defines a functor $\mathcal{A} \mapsto \mathcal{A}_{((x))}$ from chiral algebras to associative topological algebras. If $\mathcal{A} = A_X$ for some commutative $\operatorname{Aut} O$-algebra A

[21] In 2.9.4 – 2.9.5 we used some ideas of VOA theory (or chiral algebra theory).

then $\mathcal{A}_{((x))}$ is the algebra A_{K_x} from 3.7.3. If $\mathcal{A} = Vac'_X$ then $\mathcal{A}_{((x))}$ is the completed twisted universal enveloping algebra $\overline{U}' = \overline{U}'(\mathfrak{g} \otimes K)$. So by functoriality one gets a morphism $\mathfrak{z}_{\mathfrak{g}}(K) = \mathfrak{z}_{\mathfrak{g}}(O)_K \to \overline{U}'$. Its image is contained in $\mathfrak{Z}$ because $\mathfrak{z}_{\mathfrak{g}}(O)_X$ is the *center* of the chiral algebra Vac'_X.

Remark. Let us sketch a definition of the chiral algebra structure on Vac'_X. First of all, for every n one constructs a $\mathcal{D}$-module $Vac'_{\mathrm{Sym}^n X}$ on $\mathrm{Sym}^n X$ (for $n = 1$ one obtains Vac'_X). The fiber Vac'_D of $Vac'_{\mathrm{Sym}^n X}$ at $D \in \mathrm{Sym}^n X$ can be described as follows. Consider D as a closed subscheme of X of order n, denote by O_D the ring of functions on the formal completion of X along D, and define K_D by $\mathrm{Spec}\, K_D = (\mathrm{Spec}\, O_D) \setminus D$. One defines the central extension $\widetilde{\mathfrak{g} \otimes K_D}$ of $\mathfrak{g} \otimes K_D$ just as in the case $n = 1$. Vac'_D is the twisted vacuum module corresponding to the Harish-Chandra pair $(\widetilde{\mathfrak{g} \otimes K_D}, G(O_D))$ (see 1.2.5). Denote by $Vac'_{X\times X}$ the pullback of $Vac'_{\mathrm{Sym}^2 X}$ to $X \times X$. Then

$$j^! \, Vac'_{X\times X} = j^! (Vac'_X \boxtimes Vac'_X) \,, \tag{103}$$

$$\Delta^\dagger \, Vac'_{X\times X} = Vac'_X \tag{104}$$

where j and Δ have the same meaning as in (101) and $\Delta^\dagger$ denotes the naive pullback, i.e., $\Delta^\dagger = H^1 \Delta^!$. One defines (101) to be the composition

$$j_* j^! \, Vac'_X \boxtimes Vac'_X = j_* j^! \, Vac'_{X\times X} \to j_* j^! \, Vac'_{X\times X} / Vac'_{X\times X} = \Delta_* \, Vac'_X$$

where the last equality comes from (104).

3.7.7. *Theorem.* (i) The morphism (100) is a topological isomorphism.

(ii) The adjoint action of $G(K)$ on $\mathfrak{Z}$ is trivial.

The proof will be given in 3.7.10. It is based on the Feigin - Frenkel theorem, so it is essential that $\mathfrak{g}$ is semisimple and the central extension of $\mathfrak{g} \otimes K$ corresponds to the “critical” scalar product (18). This was not essential for Theorem 3.7.5.

We will also prove the following statements.

3.7.8. *Theorem.* The map $\operatorname{gr}\mathfrak{Z}\to\mathfrak{Z}^{cl}$ defined in 2.9.8 induces a topological isomorphism $\operatorname{gr}_i\mathfrak{Z}\xrightarrow{\sim}\mathfrak{Z}^{cl}_{(i)}:=\{$the space of homogeneous polynomials from $\mathfrak{Z}^{cl}$ of degree $i\}$.

3.7.9. *Theorem.* Denote by $\mathcal{I}_n$ the closed left ideal of $\overline{U}'$ topologically generated by $\mathfrak{g}\otimes t^nO$, $n\geq 0$. Then the ideal $I_n:=\mathcal{I}_n\cap\mathfrak{Z}\subset\mathfrak{Z}$ is topologically generated by the spaces $\mathfrak{Z}_i^m$, $m<i(1-n)$, where $\mathfrak{Z}_i^m:=\{z\in\mathfrak{Z}_i|L_0z=mz\}$, $\mathfrak{Z}_i$ is the standard filtration of $\mathfrak{Z}$, and $L_0:=-t\frac{d}{dt}\in\operatorname{Der}O$.

3.7.10. Let us prove the above theorems. The elements of the image of (100) are $G(K)$-invariant (see the Remark from 2.9.6). So 3.7.7(ii) follows from 3.7.7(i). Let us prove 3.7.7(i), 3.7.8, and 3.7.9.

By 2.5.2 $\operatorname{gr}\mathfrak{z}_{\mathfrak{g}}(O)=\mathfrak{z}^{cl}_{\mathfrak{g}}(O)$. According to 2.4.1 $\mathfrak{z}^{cl}_{\mathfrak{g}}(O)$ can be identified with the ring of $G(O)$-invariant polynomial functions on $\mathfrak{g}^*\otimes\omega_O$. Choose homogeneous generators $p_1,\dots,p_r$ of the algebra of G-invariant polynomials on $\mathfrak{g}^*$ and set $d_j:=\deg p_j$. Define $v_{jk}\in\mathfrak{z}^{cl}_{\mathfrak{g}}(O)$, $1\leq j\leq r$, $0\leq k<\infty$, by

$$p_j(\eta)=\sum_{k=0}^{\infty}v_{jk}(\eta)t^k(dt)^{d_j}\,,\quad \eta\in\mathfrak{g}^*\otimes\omega_O\,. \tag{105}$$

According to 2.4.1 the algebra $\mathfrak{z}^{cl}_{\mathfrak{g}}(O)$ is freely generated by v_{jk}. The action of $\operatorname{Der}O$ on $\mathfrak{z}^{cl}_{\mathfrak{g}}(O)$ is easily described. In particular $v_{jk}=(L_{-1})^kv_{j0}/k!$, $L_0v_{j0}=d_jv_{j0}$. Lift $v_{j0}\in\mathfrak{z}^{cl}_{\mathfrak{g}}(O)=\operatorname{gr}\mathfrak{z}_{\mathfrak{g}}(O)$ to an element $u_j\in\mathfrak{z}_{\mathfrak{g}}(O)$ so that $L_0u_j=d_ju_j$. Then the algebra $\mathfrak{z}_{\mathfrak{g}}(O)$ is freely generated by $u_{jk}:=(L_{-1})^ku_j/k!$, $1\leq j\leq r$, $0\leq k<\infty$. Just as in the example at the end of 3.7.3 we see that $\mathfrak{z}_{\mathfrak{g}}(O)_K=\mathbb{C}[[\dots,\widetilde{u}_{j,-1},\widetilde{u}_{j0},\widetilde{u}_{j1},\dots]$ and $L_0\widetilde{u}_{jk}=(d_j+k)\widetilde{u}_{jk}$.

Denote by $\overline{u}_{jk}$ the image of $\widetilde{u}_{jk}$ in $\mathfrak{Z}$. By 2.9.8 $\overline{u}_{jk}\in\mathfrak{Z}_{d_j}$ and the image of $\overline{u}_{jk}$ in $\mathfrak{Z}^{cl}_{(d_j)}$ is the function $\widetilde{v}_{jk}:\mathfrak{g}^*\otimes\omega_K\to\mathbb{C}$ defined by

$$p_j(\eta)=\sum_k\widetilde{v}_{jk}(\eta)t^k(dt)^{d_j},\quad \eta\in\mathfrak{g}^*\otimes\omega_K\,. \tag{106}$$

We have an isomorphism of topological algebras

$$\mathfrak{Z}^{cl}=\mathbb{C}[[\dots\widetilde{v}_{j,-1},\widetilde{v}_{j0},\widetilde{v}_{j1},\dots] \tag{107}$$

because

(108) the algebra of $G(O)$-invariant polynomial functions on $\mathfrak{g}^* \otimes t^{-n}\omega_O$ is freely generated by the restrictions of $\widetilde{v}_{jk}$ for $k \geq -nd_j$ while for $k < -nd_j$ the restriction of $\widetilde{v}_{jk}$ to $\mathfrak{g}^* \otimes t^{-n}\omega_O$ equals 0.

(This statement is immediately reduced to the case $n = 0$ considered in 2.4.1). Theorem 3.7.8 follows from (107).

Now consider the morphism $f_n : \mathfrak{z}_{\mathfrak{g}}(O)_K \to \mathfrak{Z}/I_n$ where I_n was defined in 3.7.9. We will show that

(109) f_n is surjective and its kernel is the ideal J_n topologically generated by u_{jk}, $k < d_j(1-n)$.

Theorems 3.7.7 and 3.7.9 follow from (109).

To prove (109) consider the composition $\overline{f}_n : \mathfrak{z}_{\mathfrak{g}}(O)_K \to \mathfrak{Z}/I_n \hookrightarrow (\overline{U}'/\mathcal{I}_n)^{G(O)}$. Equip $\overline{U}'/\mathcal{I}_n$ with the filtration induced by the standard one on $\overline{U}'$. The eigenvalues of L_0 on the i-th term of this filtration are $\geq i(1-n)$. So $\operatorname{Ker} \overline{f}_n \supset J_n$ where J_n was defined in (109). Now $\operatorname{gr}(\overline{U}'/\mathcal{I}_n)^{G(O)}$ is contained in $(\operatorname{gr} \overline{U}'/\mathcal{I}_n)^{G(O)}$, i.e., the algebra of $G(O)$-invariant polynomials on $\mathfrak{g}^* \otimes t^{-n}\omega_O$. Using (108) one easily shows that the map $\mathfrak{z}_{\mathfrak{g}}(O)_K/J_n \to (\overline{U}'/\mathcal{I}_n)^{G(O)}$ induced by $\overline{f}_n$ is an isomorphism. This implies (109). We have also shown that

(110) the map $\mathfrak{Z} \to (\overline{U}'/\mathcal{I}_n)^{G(O)}$ is surjective

and therefore

$$\mathfrak{Z} = (\overline{U}')^{G(O)}. \tag{111}$$

3.7.11. *Remarks*

(i) Here is another proof[22] of (111). Let $u \in (\overline{U}')^{G(O)}$. Take $h \in H(K)$ where $H \subset G$ is a fixed Cartan subgroup. Then $h^{-1}uh$ is invariant with respect to a certain Borel subgroup $B_h \subset G$. So $h^{-1}uh$ is G-invariant

[22]It is analogous to the proof of the fact that an integrable discrete representation of $\mathfrak{g} \otimes K$ is trivial. We are not able to use the fact itself because $\overline{U}'$ is not discrete.

(it is enough to prove this for the image of $h^{-1}uh$ in the *discrete* space $\overline{U}'/\mathcal{I}_n$ where $\mathcal{I}_n$ was defined in 3.7.9). Therefore u is invariant with respect to $h\mathfrak{g}h^{-1} \subset \mathfrak{g}\otimes K$ for any $h \in H(K)$. The Lie algebra $\mathfrak{g}\otimes K$ is generated by $\mathfrak{g}\otimes O$ and $h\mathfrak{g}h^{-1}$, $h \in H(K)$. So $u \in \mathfrak{Z}$.

(ii) In fact

$$\mathfrak{Z} = (\overline{U}')^{\mathfrak{a}} \quad \text{for any open } \mathfrak{a} \subset \mathfrak{g}\otimes K\,. \tag{112}$$

Indeed, one can modify the above proof as follows. First write u as an (infinite) sum of u_χ, $\chi \in \mathfrak{h}^* := (\operatorname{Lie} H)^*$, $[a, u_\chi] = \chi(a)u_\chi$ for $a \in \mathfrak{h}$. Then take an $h \in H(K)$ such that the image of h in $H(K)/H(O) = \{\text{the coweight lattice}\}$ is "very dominant" with respect to a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $\mathfrak{h}$, so that $h^{-1}\mathfrak{a}h \supset [\mathfrak{b}, \mathfrak{b}]$. We see that $u_\chi = 0$ unless χ is dominant, and $h^{-1}u_0h$ is $\mathfrak{g}$-invariant. Replacing h by h^{-1} we see that $u = u_0$, etc.

(iii) Here is another proof of 3.7.7(ii). Consider the canonical filtration $\overline{U}'_k$ of $\overline{U}'$. It follows from (109) that the union of the spaces $\overline{U}'_k \cap \mathfrak{Z}$, $k \in \mathbb{N}$, is dense in $\mathfrak{Z}$. So it suffices to show that the action of $G(K)$ on $\overline{U}'_k \cap \mathfrak{Z}$ is trivial for every k. The action of $G(K)$ on $\mathfrak{Z}^{cl}$ is trivial (see (107), (106)). So the action of $G(K)$ on $\operatorname{gr}\mathfrak{Z}$ is trivial. The action of $\mathfrak{g}\otimes K$ on $\widetilde{\mathfrak{g}\otimes K}$ corresponding to the action of $G(K)$ defined by (19) is the adjoint action, and the adjoint action of $\mathfrak{g}\otimes K$ on $\mathfrak{Z}$ is trivial. So the action of $G(K)$ on $\mathfrak{Z}$ factors through $\pi_0(G(K))$. The group $\pi_0(G(K))$ is finite (see 4.5.4), so we are done.

3.7.12. We are going to deduce Theorem 3.6.7 from [FF92]. Denote by $A_{^L\mathfrak{g}}(O)$ the coordinate ring of $\mathcal{O}\mathfrak{p}_{^L\mathfrak{g}}(O)$ (i.e., the scheme of ${}^L\mathfrak{g}$-opers on $\operatorname{Spec} O$). Let $\varphi_O : A_{^L\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}_{\mathfrak{g}}(O)$ be an isomorphism satisfying the conditions of 3.2.2. It induces an $\operatorname{Aut} K$-equivariant isomorphism $\varphi_K : A_{^L\mathfrak{g}}(K) \xrightarrow{\sim} \mathfrak{z}_{\mathfrak{g}}(K)$ where $A_{^L\mathfrak{g}}(K)$ is the algebra A_K from 3.7.3 corresponding to $A = A_{^L\mathfrak{g}}(O)$. Recall that A_K is the coordinate ring of the ind-scheme of horizontal sections of $\operatorname{Spec} A_{Y'}$, $Y' := \operatorname{Spec} K$. If

$A = A_{{}^L\mathfrak{g}}(O)$ then $\operatorname{Spec} A_{Y'}$ is the scheme of jets of ${}^L\mathfrak{g}$-opers on Y' and its horizontal sections are ${}^L\mathfrak{g}$-opers on Y' (cf. 3.3.3). So $A_{{}^L\mathfrak{g}}(K)$ is the coordinate ring of $\mathcal{O}\mathfrak{p}_{{}^L\mathfrak{g}}(K) :=$ the ind-scheme of ${}^L\mathfrak{g}$-opers on $\operatorname{Spec} K$. It is a Poisson algebra with respect to the *Gelfand - Dikii bracket* (we remind its definition in 3.7.14). The Gelfand - Dikii bracket depends on the choice of a non-degenerate invariant bilinear form on ${}^L\mathfrak{g}$. We define it to be *dual* to the form (18) on $\mathfrak{g}$ (i.e., its restriction to $\mathfrak{h}^* = {}^L\mathfrak{h} \subset {}^L\mathfrak{g}$ is dual to the restriction of (18) to $\mathfrak{h}$).

By 3.7.5 and 3.7.7 we have a canonical isomorphism $\mathfrak{z}_\mathfrak{g}(K) \xrightarrow{\sim} \mathfrak{Z}$, so φ_K can be considered as an $\operatorname{Aut} K$-equivariant isomorphism

$$A_{{}^L\mathfrak{g}}(K) \xrightarrow{\sim} \mathfrak{Z}\,. \tag{113}$$

$\mathfrak{Z}$ is a Poisson algebra with respect to the Hayashi bracket (see 3.6.2).

3.7.13. *Theorem.* [FF92]

There is an isomorphism

$$\varphi_O :\ A_{{}^L\mathfrak{g}}(O) \xrightarrow{\sim} \mathfrak{z}_\mathfrak{g}(O) \tag{114}$$

satisfying the conditions of 3.2.2 and such that the corresponding isomorphism (113) is compatible with the Poisson structures.

We will show (see 3.7.16) that an isomorphism (114) with the properties mentioned in the theorem satisfies the conditions of 3.6.7. So it is unique (see the Remark from 3.6.7).

Remark. As explained in 3.6.12, one can associate a Poisson bracket on $\mathfrak{Z}$ to *any* invariant bilinear form B on $\mathfrak{g}$ (the bracket from 3.6.2 corresponds to the form (18)). If B is non-degenerate one can consider the dual form on ${}^L\mathfrak{g}$ and the corresponding Gelfand - Dikii bracket on $A_{{}^L\mathfrak{g}}(K)$. The isomorphism (113) corresponding to (114) is compatible with these Poisson brackets.

3.7.14. Let us recall the definition of the Gelfand - Dikii bracket from [DS85]. This is a Poisson bracket on $\mathcal{O}\mathfrak{p}_\mathfrak{g}(K)$ (i.e., a Poisson bracket on its coordinate ring $A_{\mathfrak{g}}(K)$). It depends on the choice of a non-degenerate invariant bilinear form $(\ ,\ )$ on $\mathfrak{g}$.

Denote by $\widetilde{\mathfrak{g}\otimes K}$ the Kac–Moody central extension of $\mathfrak{g}\otimes K$ corresponding to $(\ ,\ )$. As a vector space $\widetilde{\mathfrak{g}\otimes K}$ is $(\mathfrak{g}\otimes K)\oplus\mathbb{C}$ and the commutator in $\widetilde{\mathfrak{g}\otimes K}$ is defined by the 2-cocycle $\operatorname{Res}(du,v)$, $u,v\in\mathfrak{g}\otimes K$. The topological dual space $(\widetilde{\mathfrak{g}\otimes K})^*$ is an ind-scheme. The algebra of regular functions on $(\widetilde{\mathfrak{g}\otimes K})^*$ is a Poisson algebra with respect to the Kirillov bracket[23] (i.e., the unique continuous Poisson bracket such that the natural map from $\widetilde{\mathfrak{g}\otimes K}$ to the algebra of regular functions on $(\widetilde{\mathfrak{g}\otimes K})^*$ is a Lie algebra morphism). So $(\widetilde{\mathfrak{g}\otimes K})^*$ is a Poisson "manifold". Denote by $(\widetilde{\mathfrak{g}\otimes K})^*_1$ the space of continuous linear functionals $l:\widetilde{\mathfrak{g}\otimes K}\to\mathbb{C}$ such that the restriction of l to the center $\mathbb{C}\subset\widetilde{\mathfrak{g}\otimes K}$ is the identity. $(\widetilde{\mathfrak{g}\otimes K})^*_1$ is a Poisson submanifold of $(\widetilde{\mathfrak{g}\otimes K})^*$.

We identify $(\widetilde{\mathfrak{g}\otimes K})^*_1$ with Conn := the ind-scheme of connections on the trivial G-bundle on $\operatorname{Spec} K$: to a connection $\nabla = d+\eta$, $\eta\in\mathfrak{g}\otimes\omega_K$, we associate $l\in(\widetilde{\mathfrak{g}\otimes K})^*_1$ such that for any $u\in\mathfrak{g}\otimes K\subset\widetilde{\mathfrak{g}\otimes K}$ one has $l(u)=\operatorname{Res}(u,\eta)$. It is easy to check that the gauge action of $\mathfrak{g}\otimes K$ on Conn corresponds to the coadjoint action of $\mathfrak{g}\otimes K$ on $(\widetilde{\mathfrak{g}\otimes K})^*_1$, and one defines the coadjoint action[24] of $G(K)$ on $(\widetilde{\mathfrak{g}\otimes K})^*$ so that its restriction to $(\widetilde{\mathfrak{g}\otimes K})^*_1$ corresponds to the gauge action of $G(K)$ on Conn. The action of $G(K)$ on the Poisson "manifold" $(\widetilde{\mathfrak{g}\otimes K})^*_1$ is not Hamiltonian in the literal sense, i.e., one cannot define the moment map $(\widetilde{\mathfrak{g}\otimes K})^*_1\to(\mathfrak{g}\otimes K)^*$. However one can define the moment map $(\widetilde{\mathfrak{g}\otimes K})^*_1\to(\widetilde{\mathfrak{g}\otimes K})^*$: this is the identity map.

The point is that $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ can be obtained from Conn $=(\widetilde{\mathfrak{g}\otimes K})^*_1$ by Hamiltonian reduction (such an interpretation of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ automatically defines a Poisson bracket on $A_{\mathfrak{g}}(K)$). Fix a Borel subgroup $B\subset G_{\mathrm{ad}}$. Let N be its unipotent radical, $\mathfrak{n} := \operatorname{Lie} N$. Since the restriction of

[23]As explained in [We83] the "Kirillov bracket" was invented by Sophus Lie and then rediscovered by several people including A.A. Kirillov.

[24]It is dual to the adjoint action of $G(K)$ on $\widetilde{\mathfrak{g}\otimes K}$ defined by (19) (of course in (19) c should be replaced by our bilinear form on $\mathfrak{g}$).

the Kac-Moody cocycle to $\mathfrak{n}\otimes K$ is trivial we have the obvious splitting $\mathfrak{n}\otimes K\to\widetilde{\mathfrak{g}\otimes K}$. It is $B(K)$-equivariant and this property characterizes it uniquely. The action of $N(K)$ on Conn is Hamiltonian: the moment map $\mu:\mathrm{Conn}=(\widetilde{\mathfrak{g}\otimes K})^*_1\to(\mathfrak{n}\otimes K)^*$ is induced by the above splitting. Let $\mathrm{Char}^\star\subset(\mathfrak{n}\otimes K)^*$ be the set of *non-degenerate characters*, i.e., the set of Lie algebra morphisms $l:\mathfrak{n}\otimes K\to\mathbb{C}$ such that for each simple root α the restriction of l to $\mathfrak{g}^\alpha\otimes K$ is nonzero. For every $l\in\mathrm{Char}^\star$ the action of $N(K)$ on $\mu^{-1}(l)$ is free and the quotient $N(K)\backslash\mu^{-1}(l)$ can be canonically identified with $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ (indeed, $\mu^{-1}(l)$ is the space of connections $\nabla=d+\eta\in\mathrm{Conn}$ such that $\eta=\sum\limits_{\alpha\in\Gamma}J_\alpha+q$ where $q\in\mathfrak{b}\otimes\omega_K$, Γ is the set of simple roots, and $J_\alpha=J_\alpha(l)$ is a fixed nonzero element of $\mathfrak{g}^{-\alpha}\otimes\omega_K$). So $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ is obtained from Conn by Hamiltonian reduction over l with respect to the action of $N(K)$, whence we get a Poisson bracket on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. It is called the *Gelfand - Dikii bracket*. It does not depend on l. Indeed, for $l,l'\in\mathrm{Char}^\star$ consider the isomorphism

$$N(K)\setminus\mu^{-1}(l)\xrightarrow{\sim}N(K)\setminus\mu^{-1}(l') \tag{115}$$

that comes from the identification of both sides of (115) with $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. The (co) adjoint action of $H(K)$ on $\mathrm{Conn}=(\widetilde{\mathfrak{g}\otimes K})^*_1$ preserves the relevant structures (i.e., the Poisson bracket on Conn, the action of $N(K)$ on Conn, and the moment map $\mu:\mathrm{Conn}\to(\mathfrak{n}\otimes K)^*$). There is a unique $h\in H(K)$ that transforms l to l' and (115) is induced by the action of this h. So (115) is a Poisson map.

Remarks

(i) If the bilinear form (,) on $\mathfrak{g}$ is multiplied by $c\in\mathbb{C}^*$ then the Poisson bracket on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ is multiplied by c^{-1}.

(ii) The Gelfand - Dikii bracket defined above is the “second Gelfand - Dikii bracket”. The definition of the first one and an explanation of the relation with the original works by Gelfand - Dikii ([GD76], [GD78]) can be found in [DS85] (see Sections 2.3, 3.6, 3.7, 6.5, and 8 from loc. cit).

3.7.15. Let $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$, i.e., $\mathfrak{F} = (\mathfrak{F}_B, \nabla)$ where $\mathfrak{F}_B$ is a B-bundle on $\operatorname{Spec} K$ and ∇ is a connection on the corresponding G-bundle satisfying the conditions of 3.1.3 (here G is the adjoint group corresponding to $\mathfrak{g}$ and $B \subset G$ is the Borel subgroup). We are going to describe the tangent space $T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ and the cotangent space $T^*_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. Then we will write an explicit formula for $\{\varphi, \psi\}(\mathfrak{F})$, $\varphi, \psi \in A_{\mathfrak{g}}(K)$.

Remark. Of course $\mathfrak{F}_B$ is always trivial, so we could consider $\mathfrak{F}$ as a connection ∇ in the trivial G-bundle (i.e., $\nabla = d + q$, $q \in \mathfrak{g} \otimes \omega_K$) modulo gauge transformations with respect to B.

To describe $T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ we must study infinitesimal deformations of $\mathfrak{F} = (\mathfrak{F}_B, \nabla)$. Since $\mathfrak{F}_B$ cannot be deformed all of them come from infinitesimal deformations of ∇, which have the form $\nabla(\varepsilon) = \nabla + \varepsilon q$, $q \in H^0(\operatorname{Spec} K, \mathfrak{g}^{-1}_{\mathfrak{F}} \otimes \omega_K)$ (see 3.1.1 for the definition of $\mathfrak{g}^{-1}$; $\mathfrak{g}^{-1}_{\mathfrak{F}} := \mathfrak{g}^{-1}_{\mathfrak{F}_B}$ is the $\mathfrak{F}_B$-twist of $\mathfrak{g}^{-1}$). Taking in account the infinitesimal automorphisms of $\mathfrak{F}_B$ we get:

$$T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K) = H^0(\operatorname{Spec} K, \operatorname{Coker}(\nabla : \mathfrak{b}_{\mathfrak{F}} \to \mathfrak{g}^{-1}_{\mathfrak{F}} \otimes \omega_K)) . \tag{116}$$

Here is a more convenient description of the tangent space:

$$T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K) = \operatorname{Coker}(\nabla : \mathfrak{n}^K_{\mathfrak{F}} \to \mathfrak{b}^K_{\mathfrak{F}} \otimes \omega_K) \tag{117}$$

where $\mathfrak{n}^K_{\mathfrak{F}} := H^0(\operatorname{Spec} K, \mathfrak{n}_{\mathfrak{F}})$, $\mathfrak{b}^K_{\mathfrak{F}} := H^0(\operatorname{Spec} K, \mathfrak{b}_{\mathfrak{F}})$ (the natural map from the r.h.s. of (117) to the r.h.s. of (116) is an isomorphism). Using the invariant scalar product (,) on $\mathfrak{g}$ we identify $\mathfrak{b}^*$, $\mathfrak{n}^*$ with $\mathfrak{g}/\mathfrak{n}$, $\mathfrak{g}/\mathfrak{b}$ and get the following description of the cotangent space:

$$T^*_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K) = \{u \in \mathfrak{g}^K_{\mathfrak{F}} | \nabla(u) \in \mathfrak{b}^K_{\mathfrak{F}} \otimes \omega_K\}/\mathfrak{n}^K_{\mathfrak{F}} . \tag{118}$$

Here is an explicit formula for the Gelfand - Dikii bracket:

$$\{\varphi, \psi\}(\mathfrak{F}) = \operatorname{Res}(\nabla(d_{\mathfrak{F}}\varphi), d_{\mathfrak{F}}\psi), \qquad \varphi, \psi \in A_{\mathfrak{g}}(K) . \tag{119}$$

In this formula the differentials $d_{\mathfrak{F}}\varphi$ and $d_{\mathfrak{F}}\psi$ are considered as elements of the r.h.s. of (118).

3.7.16. *Theorem.* [25]

(i) Set $I := \operatorname{Ker}(A_{\mathfrak{g}}(K) \to A_{\mathfrak{g}}(O))$. Then $\{I, I\} \subset I$ and therefore I/I^2 is a Lie algebroid over $A_{\mathfrak{g}}(O)$.

(ii) There is an $\operatorname{Aut} O$-equivariant topological isomorphism of Lie algebroids

$$I/I^2 \xrightarrow{\sim} \mathfrak{a}_{\mathfrak{g}} \tag{120}$$

(see 3.5.11, 3.5.15 for the definition of $\mathfrak{a}_{\mathfrak{g}}$).

(In this theorem I^2 denotes the *closure* of the subspace generated by ab, $a \in I$, $b \in I$).

Theorem 3.6.7 follows from 3.7.13 and 3.7.16.

Remark. The isomorphism (120) is unique (see 3.5.13 or 3.5.14).

3.7.17. Let us prove Theorem 3.7.16. We keep the notation of 3.7.15. Take $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$. Here is a description of $T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ similar to (117):

$$T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \operatorname{Coker}(\nabla : \mathfrak{n}_{\mathfrak{F}}^{O} \to \mathfrak{b}_{\mathfrak{F}}^{O} \otimes \omega_O) \tag{121}$$

where $\mathfrak{n}_{\mathfrak{F}}^{O} := H^0(\operatorname{Spec} O, \mathfrak{n}_{\mathfrak{F}})$. The fiber of I/I^2 over $\mathfrak{F}$ is the conormal space $T_{\mathfrak{F}}^{\perp}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \subset T_{\mathfrak{F}}^{*}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. According to (121) it has the following description in terms of (118):

$$T_{\mathfrak{F}}^{\perp}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) = \{u \in \mathfrak{g}_{\mathfrak{F}}^{O} \mid \nabla(u) \in \mathfrak{b}_{\mathfrak{F}}^{O} \otimes \omega_O\}/\mathfrak{n}_{\mathfrak{F}}^{O}\,. \tag{122}$$

Now it is clear that $\{I, I\} \subset I$: if $\varphi, \psi \in I$, $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ then $d_{\mathfrak{F}}\varphi$ and $d_{\mathfrak{F}}\psi$ belong to the r.h.s. of (122) and therefore the r.h.s. of (119) equals 0.

The map

$$I/I^2 \to \operatorname{Der} A_{\mathfrak{g}}(O)\,, \tag{123}$$

[25] Inspired by [Phys]

which is a part of the algebroid structure on I/I^2, is defined by $\varphi \mapsto \partial_\varphi$, $\partial_\varphi(\psi) := \{\varphi, \psi\}$, $\varphi \in I$, $\psi \in A_{\mathfrak{g}}(K)/I = A_{\mathfrak{g}}(O)$. So according to (119) the map

$$T_{\mathfrak{F}}^{\perp}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \to T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \tag{124}$$

induced by (123) is the operator

$$\nabla : \{u \in \mathfrak{g}_{\mathfrak{F}}^{O} \,|\, \nabla(u) \in \mathfrak{b}_{\mathfrak{F}}^{O} \otimes \omega_O\}/\mathfrak{n}_{\mathfrak{F}}^{O} \to (\mathfrak{b}_{\mathfrak{F}}^{O} \otimes \omega_O)/\nabla(\mathfrak{n}_{\mathfrak{F}}^{O})\,. \tag{125}$$

The algebroid structure on I/I^2 induces a Lie algebra structure on the kernel $\mathfrak{a}_{\mathfrak{F}}$ of the map (124). On the other hand, $\mathfrak{a}_{\mathfrak{F}}$ is the kernel of (125), i.e., $\mathfrak{a}_{\mathfrak{F}} = \{u \in \mathfrak{g}_{\mathfrak{F}}^{O} \,|\, \nabla(u) = 0\}/\{u \in \mathfrak{n}_{\mathfrak{F}}^{O} \,|\, \nabla(u) = 0\}$. Since $\{u \in \mathfrak{n}_{\mathfrak{F}}^{O} \,|\, \nabla(u) = 0\} = 0$ we have

$$\mathfrak{a}_{\mathfrak{F}} = \{u \in \mathfrak{g}_{\mathfrak{F}}^{O} \,|\, \nabla(u) = 0\}\,. \tag{126}$$

The r.h.s. of (126) is a Lie subalgebra of $\mathfrak{g}_{\mathfrak{F}}^{O}$.

Lemma. The Lie algebra structure on $\mathfrak{a}_{\mathfrak{F}}$ that comes from the algebroid structure on I/I^2 coincides with the one induced by (126).

Proof. It suffices to show that if $\varphi_1, \varphi_2 \in A_{\mathfrak{g}}(K)$ and $d_{\mathfrak{F}}\varphi_i \in \mathfrak{a}_{\mathfrak{F}}$ then

$$d_{\mathfrak{F}}\{\varphi_1, \varphi_2\} = [d_{\mathfrak{F}}\varphi_1, d_{\mathfrak{F}}\varphi_2] \tag{127}$$

(in the r.h.s. of (127) $d_{\mathfrak{F}}\varphi_i$ are considered as elements of $\mathfrak{g}_{\mathfrak{F}}^{O}$ via (126)). Consider a deformation $\mathfrak{F}(\varepsilon)$ of $\mathfrak{F}$, $\varepsilon^2 = 0$. Write $\mathfrak{F}$ as $(\mathfrak{F}_B, \nabla)$. Without loss of generality we can assume that $\mathfrak{F}(\varepsilon) = (\mathfrak{F}_B, \nabla + \varepsilon q)$, $q \in \mathfrak{b}_{\mathfrak{F}}^{K} \otimes \omega_K$. Write $d_{\mathfrak{F}(\varepsilon)}\varphi_i$ as $d_{\mathfrak{F}}\varphi_i + \varepsilon\mu_i$. Then

$$\{\varphi_1, \varphi_2\}(\mathfrak{F}(\varepsilon)) = \operatorname{Res}((\nabla + \varepsilon \operatorname{ad} q)(d_{\mathfrak{F}}\varphi_1 + \varepsilon\mu_1), d_{\mathfrak{F}}\varphi_2 + \varepsilon\mu_2) =$$

$$\varepsilon \operatorname{Res}([q, d_{\mathfrak{F}}\varphi_1], d_{\mathfrak{F}}\varphi_2) = \varepsilon \operatorname{Res}(q, [d_{\mathfrak{F}}\varphi_1, d_{\mathfrak{F}}\varphi_2])$$

(we have used that $\nabla(d_{\mathfrak{F}}\varphi_i) = 0$). The equality (127) follows. $\square$

According to the lemma the kernel $\mathfrak{a}_{\mathfrak{F}}$ of the map (124) coincides as a Lie algebra with $(\mathfrak{g}_{\mathrm{univ}})_{\mathfrak{F}}$, i.e., the fiber at $\mathfrak{F}$ of the Lie algebra $\mathfrak{g}_{\mathrm{univ}}$ from 3.5.11. The map (124)=(125) is surjective because $\nabla : \mathfrak{g}_{\mathfrak{F}}^{O} \to \mathfrak{g}_{\mathfrak{F}}^{O} \otimes \omega_O$ is surjective. It is easy to show that (121) and (122) are homeomorphisms and that the map (124) is open.

In a similar way one shows that the morphism (123) is surjective and open, and its kernel can be canonically identified with $\mathfrak{g}_{\mathrm{univ}}$ equipped with the discrete topology (the identification induces the above isomorphism $\mathfrak{a}_{\mathfrak{F}} \xrightarrow{\sim} (\mathfrak{g}_{\mathrm{univ}})_{\mathfrak{F}}$ for every $\mathfrak{F} \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$). Lemma 3.5.12 yields a continuous Lie algebroid morphism $f : I/I^2 \to \mathfrak{a}_{\mathfrak{g}}$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g}_{\mathrm{univ}} & \longrightarrow & I/I^2 & \longrightarrow & \operatorname{Der} A_{\mathfrak{g}}(O) & \longrightarrow & 0 \\
 & & {\scriptstyle\mathrm{id}}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle\mathrm{id}} & & \\
0 & \longrightarrow & \mathfrak{g}_{\mathrm{univ}} & \longrightarrow & \mathfrak{a}_{\mathfrak{g}} & \longrightarrow & \operatorname{Der} A_{\mathfrak{g}}(O) & \longrightarrow & 0
\end{array}$$

is commutative. Since the rows of the diagram are exact in the topological sense, f is a topological isomorphism. Clearly f is $\operatorname{Aut} O$-equivariant.

3.8. **Singularities of opers.**

3.8.1. Let U be an open dense subset of our curve X. We are going to represent the ind-scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(U)$ as a union of certain closed subschemes $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$ where D runs through the set of finite subschemes of X such that $D \cap U = \emptyset$.

According to 3.1.9 we have a canonical isomorphism $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(U) \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(U)$ where $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(U)$ is the $\Gamma(U, V_{\omega_X})$-torsor induced from the $\Gamma(U, \omega_X^{\otimes 2})$-torsor $\mathcal{O}\mathfrak{p}_{sl_2}(U)$ by a certain embedding $\Gamma(U, \omega_X^{\otimes 2}) \subset \Gamma(U, V_{\omega_X})$. The definition of this embedding and of $V = V_{\mathfrak{g}}$ can be found in 3.1.9. Let us remind that V is a vector space equipped with a $\mathbb{G}_m$-action (i.e., a grading) and V_{ω_X} denotes the twist of V by the $\mathbb{G}_m$-torsor ω_X. We have $\dim V = r := \operatorname{rank} \mathfrak{g}$ and the degrees of the graded components of V are equal to the degrees $d_1, \dots, d_r$ of "basic" invariant polynomials on $\mathfrak{g}$.

If D is a finite subscheme of X one has a canonical embedding $V_{\omega_X} \hookrightarrow V_{\omega_X(D)}$. Denote by $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g},D}(X)$ the $\Gamma(X, V_{\omega_X(D)})$-torsor induced by the $\Gamma(X, V_{\omega_X})$-torsor $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X)$. Clearly $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g},D}(X)$ is a closed subscheme of $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}}(X \setminus D)$. Denote by $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$ the image of $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g},D}(X)$ in $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X \setminus D)$. If $D \subset D'$ then $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X) \subset \mathcal{O}\mathfrak{p}_{\mathfrak{g},D'}(X)$. For any open dense $U \subset X$ we have $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(U) = \bigcup\limits_{D \cap U = \emptyset} \mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$.

In 3.8.23 we will give an "intrinsic" description of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$, which does not use the isomorphism $\underline{\mathcal{O}\mathfrak{p}}_{\mathfrak{g}} \xrightarrow{\sim} \mathcal{O}\mathfrak{p}_{\mathfrak{g}}$. The local version of this description is given in 3.8.7 – 3.8.10.

3.8.2. Now we can formulate the answer to the problem from 2.8.6:

$$N_\Delta(G) = \mathcal{O}\mathfrak{p}_{^L\mathfrak{g},\Delta}(X)\,. \tag{128}$$

$N_\Delta(G)$ is defined as a subscheme of an ind-scheme $N'_\Delta(G)$, which is canonically identified with $\mathcal{O}\mathfrak{p}_{^L\mathfrak{g}}(X \setminus \Delta)$ via the Feigin - Frenkel isomorphism. (128) is an equality of subschemes of $\mathcal{O}\mathfrak{p}_{^L\mathfrak{g}}(X \setminus \Delta)$.

We will not prove (128). A hint will be given in 3.8.6.

3.8.3. The definition of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$ from 3.8.1 makes sense in the following local situation: $X = \operatorname{Spec} O$, $O := \mathbb{C}[[t]]$, $D = \operatorname{Spec} O/t^n O$. In this case we write $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$ instead of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$. $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$ is a closed subscheme of the ind-scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. Of course $\mathcal{O}\mathfrak{p}_{\mathfrak{g},0}(O) = \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$, $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O) \subset \mathcal{O}\mathfrak{p}_{\mathfrak{g},n+1}(O)$, and $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ is the inductive limit of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$.

According to 3.7.12 $A_{\mathfrak{g}}(K)$ is the algebra of regular functions on $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. Denote by I_n the ideal of $A_{\mathfrak{g}}(K)$ corresponding to $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O) \subset \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$. Clearly $I_n \supset I_{n+1}$ and I_0 is the ideal I from 3.7.16 (i). The ideals I_n form a base of neighbourhoods of 0 in $A_{\mathfrak{g}}(K)$.

3.8.4. Here is an explicit description of $A_{\mathfrak{g}}(K)$ and I_n. We use the notation of 3.5.6, so $\mathfrak{g}$-opers on $\operatorname{Spec} K$ are in one-to-one correspondence with operators (64) such that $u_j(t) \in \mathbb{C}((t))$. Write $u_j(t)$ as $\sum\limits_k \widetilde{u}_{jk} t^k$. Then $A_{\mathfrak{g}}(K) = \mathbb{C}[[\ldots \widetilde{u}_{j,-1}, \widetilde{u}_{j0}, \widetilde{u}_{j1}, \ldots]$ (we use notation (98)). The ideal I_n is topologically generated by $\widetilde{u}_{jk}$, $k < -d_j n$. The u_{jk} from 3.5.6 are the images of $\widetilde{u}_{jk}$ in $A_{\mathfrak{g}}(O) = A_{\mathfrak{g}}(K)/I$.

It is easy to describe the action of $\operatorname{Der} K$ on $A_{\mathfrak{g}}(K)$. In particular

$$L_0 \widetilde{u}_{jk} = (d_j + k)\widetilde{u}_{jk}\,. \tag{129}$$

Just as in the global situation (see 3.1.12 – 3.1.14) the coordinate ring $A_{\mathfrak{g}}(K)$ of $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ carries a canonical filtration. Its i-th term consists of those "polynomials" in $\widetilde{u}_{jk}$ whose weighted degree is $\leq i$, it being understood that the weight of $\widetilde{u}_{jk}$ is d_j.

3.8.5. *Proposition.* The ideal $I_n \subset A_{\mathfrak{g}}(K)$ is topologically generated by the spaces A_i^m, $m < i(1-n)$, where A_i^m is the set of elements a from the i-th term of the filtration of the $A_{\mathfrak{g}}(K)$ such that $L_0 a = ma$. □

The isomorphism $A_{{}^L\mathfrak{g}}(K) \xrightarrow{\sim} \mathfrak{Z}$ (see (113)) preserves the filtrations and is $\operatorname{Aut} K$-equivariant. So Proposition 3.8.5 implies the following statement.

3.8.6. *Proposition.* The Feigin - Frenkel isomorphism $A_{{}^L\mathfrak{g}}(K) \xrightarrow{\sim} \mathfrak{Z}$ maps $I_n \subset A_{{}^L\mathfrak{g}}(K)$ onto the ideal I_n from 3.7.9.

This is one of the ingredients of the proof of (128).

3.8.7. We are going to describe $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$ in "natural" terms (without using the isomorphism (43)). Denote by $\mathfrak{g}^+$ the locally closed reduced subscheme of $\mathfrak{g}$ consisting of all $a \in \mathfrak{g}$ such that for positive roots α one has $a_{-\alpha} = 0$ if α is non-simple, $a_{-\alpha} \neq 0$ if α is simple ($a_{-\alpha}$ is the component of a from the root subspace $\mathfrak{g}^{-\alpha}$). Then for any $\mathbb{C}$-algebra R the set $\mathfrak{g}^+(R)$ consists of $a \in \mathfrak{g} \otimes R$ such that $a_{-\alpha} = 0$ for each non-simple $\alpha > 0$ and $a_{-\alpha}$ generates the R-module $\mathfrak{g}^{-\alpha} \otimes R$ for each simple α.

Recall that a $\mathfrak{g}$-oper over $\operatorname{Spec} K$ is a $B(K)$-conjugacy class of operators $\frac{d}{dt} + q(t)$, $q \in \mathfrak{g}^+(K)$. Here B is the Borel subgroup of the adjoint group G corresponding to $\mathfrak{g}$.

3.8.8. *Definition.* A ($\leq n$)-singular $\mathfrak{g}$-oper on $\operatorname{Spec} O$ is a $B(O)$-conjugacy class of operators $\frac{d}{dt} + t^{-n}q(t)$, $q \in \mathfrak{g}^+(O)$.

Remarks

(i) The action of $B(O)$ on the set of operators $\frac{d}{dt} + t^{-n}q(t)$, $q \in \mathfrak{g}^+(O)$, is free. Indeed, the action of $B(K)$ on $\{\frac{d}{dt} + q(t) | q \in \mathfrak{g}^+(K)\}$ is free (see 3.1.4).

(ii) For $n = 0$ one obtains the usual notion of $\mathfrak{g}$-oper on $\operatorname{Spec} O$.

3.8.9. *Proposition.* The map $\{(\leq n)\text{-singular } \mathfrak{g}\text{-opers on } \operatorname{Spec} O\} \to \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(K)$ is injective. Its image equals $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$.

Proof. We use the notation of 3.5.6. For every $v_1, \dots, v_r \in \mathbb{C}[[t]]$ the operator

$$\frac{d}{dt} + t^{-n}(i(f) + v_1(t)e_1 + \ldots + v_r(t)e_r) \tag{130}$$

defines a ($\leq n$)-singular $\mathfrak{g}$-oper on $\operatorname{Spec} O$. It is easy to show that this is a bijection between operators (130) and ($\leq n$)-singular $\mathfrak{g}$-opers on $\operatorname{Spec} O$. Now let us transform (130) to the "canonical form" (64) by $B(K)$-conjugation. Conjugating (130) by $t^{-n\check{\rho}}$ we obtain

$$\frac{d}{dt} + i(f) + n\check{\rho}t^{-1} + t^{-nd_1}v_1(t)e_1 + \ldots + t^{-nd_r}v_r(t)e_r\,. \tag{131}$$

To get rid of $n\check{\rho}t^{-1}$ we conjugate (131) by $\exp(-ne_1/2t)$ and obtain the operator (64) with

$$u_j(t) = t^{-nd_j}v_j(t) \quad \text{for} \quad j > 1\,,$$

$$u_1(t) = t^{-nd_1}v_1(t) + n(n-2)/4t^2\,, \quad d_1 = 2\,.$$

Clearly $v_j \in \mathbb{C}[[t]]$ if and only if $u_j \in t^{-nd_j}\mathbb{C}[[t]]$. □

3.8.10. If points of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O)$ are considered as ($\leq n$)-singular $\mathfrak{g}$-opers on $\operatorname{Spec} O$ then the canonical embedding $\mathcal{O}\mathfrak{p}_{\mathfrak{g},n}(O) \hookrightarrow \mathcal{O}\mathfrak{p}_{\mathfrak{g},n+1}(O)$ maps the $B(O)$-conjugacy class of $\frac{d}{dt} + t^{-n}q(t)$, $q \in \mathfrak{g}^+(O)$, to the $B(O)$-conjugacy class of $t^{\check{\rho}}(\frac{d}{dt} + t^{-n}q(t))t^{-\check{\rho}}$ (it is well-defined because $t^{\check{\rho}}B(O)t^{-\check{\rho}} \subset B(O)$).

3.8.11. Denote by $\mathrm{Inv}(\mathfrak{g})$ the algebra of G-invariant polynomials on $\mathfrak{g}$. There is a canonical morphism $\mathfrak{g} \to \operatorname{Spec}\mathrm{Inv}(\mathfrak{g}) = W \setminus \mathfrak{h}$ where W is the Weyl group.

Suppose one has a (≤ 1)-singular $\mathfrak{g}$-oper on $\operatorname{Spec} O$, i.e., a $B(O)$-conjugacy class of $\frac{d}{dt} + t^{-1}q(t)$, $q \in \mathfrak{g}^+(O)$. The image of $q(0) \in \mathfrak{g}$ in $\operatorname{Spec}\mathrm{Inv}(\mathfrak{g})$ is called the *residue* of the oper. So we have defined the *residue map*

$$\mathrm{Res}: \mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O) \to \operatorname{Spec}\mathrm{Inv}(\mathfrak{g}) = W \setminus \mathfrak{h}\,. \tag{132}$$

It is surjective. Therefore it induces an embedding

$$\mathrm{Inv}(\mathfrak{g}) \hookrightarrow A_{\mathfrak{g}}(K)/I_1 \tag{133}$$

(recall that $A_{\mathfrak{g}}(K)/I_1$ is the coordinate ring of $\mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$; see 3.8.3).

3.8.12. *Proposition.* $\mathrm{Res}(\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)) \subset W \setminus \mathfrak{h}$ consists of a single point, which is the image of $-\check{\rho} \in \mathfrak{h}$.

Remark. We prefer to forget that $-\check{\rho}$ and $\check{\rho}$ have the same image in $W \setminus \mathfrak{h}$.

Proof. We must compute the composition of the map $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O) \to \mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$ described in 3.8.10 and the map (132). If $q(t) \in \mathfrak{g}^+(O)$ then $t^{\check{\rho}}(\frac{d}{dt}+q(t))t^{-\check{\rho}} = \frac{d}{dt}+\frac{a-\check{\rho}}{t}+\{\text{something regular}\}$ where a belongs to the sum of the root spaces corresponding to simple negative roots. Now $a - \check{\rho}$ and $-\check{\rho}$ have the same image in $W \setminus \mathfrak{h}$. $\square$

3.8.13. *Proposition.* Let $f \in A_{\mathfrak{g}}(K)/I_1$, i.e., f is a regular function on $\mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$. Then the following conditions are equivalent:

(i) $f \in \mathrm{Inv}(\mathfrak{g})$, where $\mathrm{Inv}(\mathfrak{g})$ is identified with its image by (133);
(ii) f is $\mathrm{Aut}^0 O$-invariant;
(iii) $L_0 f = 0$.

Proof. Clearly (i)$\Rightarrow$(ii)$\Rightarrow$(iii). Let us deduce (i) from (iii). Consider a (≤ 1)-singular $\mathfrak{g}$-oper on $\operatorname{Spec} O$. This is the $B(O)$-conjugacy class of a connection $\frac{d}{dt} + t^{-1}q(t)$, $q \in \mathfrak{g}^+(O)$. If t is replaced by λt this connection is replaced by $\frac{d}{dt} + t^{-1}q(\lambda t)$. Since $L_0 f = 0$ the value of f on the connection

$\frac{d}{dt}+t^{-1}q(\lambda t)$ does not depend on λ, so it depends only on $q(0)\in\mathfrak{g}^+$ (because $\lim\limits_{\lambda\to 0} q(\lambda t)=q(0)$). It remains to use the fact that a B-invariant regular function on $\mathfrak{g}^+$ extends to a G-invariant polynomial on $\mathfrak{g}$ (see Theorem 0.10 from [Ko63]). □

3.8.14. *Remark.* According to 3.8.4 the algebra $A_{\mathfrak{g}}(K)/I_1$ is freely generated by $\overline{u}_{jk}$, $k\geq -d_j$, where $\overline{u}_{jk}\in A_{\mathfrak{g}}(K)/I_1$ is the image of $\widetilde{u}_{jk}\in A_{\mathfrak{g}}(K)$. By 3.8.13 and (129) $\operatorname{Inv}(\mathfrak{g})\subset A_{\mathfrak{g}}(K)/I_1$ is generated by $v_j:=\overline{u}_{j,-d_j}$. The isomorphism $\operatorname{Spec}\mathbb{C}[v_1,\dots,v_r]\xrightarrow{\sim}\operatorname{Spec}\operatorname{Inv}(\mathfrak{g})$ is the composition $\operatorname{Spec}\mathbb{C}[v_1,\dots,v_r]\to\mathfrak{g}\to\operatorname{Spec}\operatorname{Inv}(\mathfrak{g})$ where the first map equals $i(f)-\check{\rho}+v_1e_1+\ldots+v_re_r$ (we use the notation of 3.5.6).

3.8.15. We are going to prove Theorem 3.6.11. In 3.8.16 – 3.8.17 we will formulate a property of the Feigin - Frenkel isomorphism (113). This property reduces Theorem 3.6.11 to a certain statement (see 3.8.19), which involves only opers and the Gelfand - Dikii bracket. This statement will be proved in 3.8.20 – 3.8.22.

3.8.16. We will use the notation of 3.5.17. Besides, if $\operatorname{Der} O$ acts on a vector space V we set $V^0:=\{v\in V|L_0v=0\}$.

As explained in 3.6.9, the map π from 3.6.8 induces a morphism

$$(\mathfrak{z}/\mathfrak{z}\cdot\mathfrak{z}^{<0})^0=(\mathfrak{z}/\mathfrak{z}\cdot\mathfrak{z}^{<0})^{\leq 0}=\mathfrak{z}^{\leq 0}/(\mathfrak{z}\cdot\mathfrak{z}^{<0}\cap\mathfrak{z}^{\leq 0})\to C \tag{134}$$

where C is the center of $U\mathfrak{g}$. Now (113) induces an isomorphism

$$(\mathfrak{z}/\mathfrak{z}\cdot\mathfrak{z}^{<0})^0\xrightarrow{\sim}(A_{{}^L\mathfrak{g}}(K)/I_1)^0 \tag{135}$$

because by 3.8.5 $I_1=A_{{}^L\mathfrak{g}}(K)\cdot A_{{}^L\mathfrak{g}}(K)^{<0}$. By 3.8.13 the r.h.s. of (135) equals $\operatorname{Inv}({}^L\mathfrak{g})$. So (134) and (135) yield a morphism

$$\operatorname{Inv}({}^L\mathfrak{g})\to C\,. \tag{136}$$

Denote by $\operatorname{Inv}(\mathfrak{h}^*)$ the algebra of W-invariant polynomials on $\mathfrak{h}^*$. Since ${}^L\mathfrak{h}=\mathfrak{h}^*$ there is a canonical isomorphism $\operatorname{Inv}({}^L\mathfrak{g})\xrightarrow{\sim}\operatorname{Inv}(\mathfrak{h}^*)$. We also have the Harish-Chandra isomorphism $C \xrightarrow{\sim} \mathrm{Inv}(\mathfrak{h}^*)$. So (136) can be considered as a map

$$\mathrm{Inv}(\mathfrak{h}^*) \to \mathrm{Inv}(\mathfrak{h}^*)\,. \tag{137}$$

3.8.17. *Theorem.* (E. Frenkel, private communication)

The morphism (137) maps $f \in \mathrm{Inv}(\mathfrak{h}^*)$ to f^- where $f^-(\lambda) := f(-\lambda)$, $\lambda \in \mathfrak{h}^*$. $\square$

3.8.18. Using 3.8.17 we can replace the mysterious lower left corner of diagram (84) by its oper analog. Diagram (143) below is obtained essentially this way. Let us define the lower arrow of (143), which is the oper analog of the map (83) constructed in 3.6.9 – 3.6.10.

According to 3.8.5

$$I_1 = A_{\mathfrak{g}}(K) \cdot A_{\mathfrak{g}}(K)^{<0}\,. \tag{138}$$

By 3.8.13 we have a canonical isomorphism

$$(A_{\mathfrak{g}}(K)/I_1)^0 \xrightarrow{\sim} \mathrm{Inv}(\mathfrak{g})\,. \tag{139}$$

For $h \in \mathfrak{h}$ denote by m_h the maximal ideal of $\mathrm{Inv}(\mathfrak{g})$ consisting of polynomials vanishing at h. Set $m := m_{-\check{\rho}}$. By 3.8.12 the isomorphism (139) induces

$$(I/I_1)^0 \xrightarrow{\sim} m\,. \tag{140}$$

Now we obtain

$$(I/(I^2 + I_1))^0 \xrightarrow{\sim} m/m^2 \tag{141}$$

(to get (141) from (140) we use that

$$(I^2)^0 \subset (I^0)^2 + I \cdot I^{<0} \subset (I^0)^2 + A_{\mathfrak{g}}(K) \cdot A_{\mathfrak{g}}(K)^{<0} = (I^0)^2 + I_1\,;$$

see (138)).

For a regular $h \in \mathfrak{h}$ we identify m_h/m_h^2 with $\mathfrak{h}^*$ by assigning to a W-invariant polynomial on $\mathfrak{h}$ its differential at h. In particular for $m = m_{-\check{\rho}}$ we have $m/m^2 \xrightarrow{\sim} \mathfrak{h}^*$ (by the way, if we wrote m as $m_{\check{\rho}}$ we would obtain a different isomorphism $m/m^2 \xrightarrow{\sim} \mathfrak{h}^*$).

Finally, using (138) we rewrite the l.h.s. of (141) in terms of I/I^2 and get an isomorphism

$$(I/I^2)^{\leq 0}/(A_{\mathfrak{g}}(O)\cdot(I/I^2)^{<0}\cap(I/I^2)^{\leq 0})\xrightarrow{\sim}\mathfrak{h}^*\,. \tag{142}$$

3.8.19. *Proposition.* The diagram

$$\begin{array}{ccc}(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}/(A_{\mathfrak{g}}(O)\cdot\mathfrak{a}_{\mathfrak{g}}^{<0}\cap(\mathfrak{a}_{\mathfrak{g}})^{\leq 0}) & \xrightarrow{\sim} & \mathfrak{h}\\ \uparrow\wr & & \downarrow\wr\\ (I/I^2)^{\leq 0}/(A_{\mathfrak{g}}(O)\cdot(I/I^2)^{<0}\cap(I/I^2)^{\leq 0}) & \xrightarrow{\sim} & \mathfrak{h}^*\end{array} \tag{143}$$

commutes. Here the lower arrow is the isomorphism (142), the upper one is the isomorphism (78), the left one is induced by the isomorphism (120) (which comes from the Gelfand - Dikii bracket on $A_{\mathfrak{g}}(K)$), and the right one is induced by the invariant scalar product on $\mathfrak{g}$ used in the definition of the Gelfand - Dikii bracket.

The proposition will be proved in 3.8.20 – 3.8.22.

Theorem 3.6.11 follows from 3.8.17 and 3.8.19. The commutativity of (143) implies the *anti*commutativity of (84) because the following diagram is anticommutative:

$$\begin{array}{ccc}m_{\check{\rho}}/(m_{\check{\rho}})^2 & \xrightarrow{\sim} & m_{-\check{\rho}}/(m_{-\check{\rho}})^2\\ & \searrow\sim\quad\swarrow\sim & \\ & \mathfrak{h}^* & \end{array}$$

Here the upper arrow is induced by the map $f\mapsto f^-$ from 3.8.17.

3.8.20. We are going to formulate a lemma used in the proof of Proposition 3.8.19. Consider the composition

$$I/I^2\to I/(I^2+I_1)\xrightarrow{\sim}\mathfrak{a}_{\mathfrak{g}}/A_{\mathfrak{g}}(O)\cdot\mathfrak{a}_{\mathfrak{g}}^{<0}=\mathfrak{a}_{\mathfrak{g}}/\mathfrak{a}_{\mathfrak{n}}=\mathfrak{g}_{\mathrm{univ}}/\mathfrak{n}_{\mathrm{univ}}\,. \tag{144}$$

Here the second arrow comes from (120) and (138); $\mathfrak{a}_{\mathfrak{n}}$ and $\mathfrak{n}_{\mathrm{univ}}$ were defined in 3.5.16, $\mathfrak{a}_{\mathfrak{g}}$ was defined in 3.5.11; the equality $\mathfrak{a}_{\mathfrak{n}}=A_{\mathfrak{g}}(O)\cdot\mathfrak{a}_{\mathfrak{g}}^{<0}$ was proved in 3.5.18. The fiber of I/I^2 over $\mathfrak{F}=(\mathfrak{F}_B,\nabla)\in\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$ equals $\{u\in\mathfrak{g}_{\mathfrak{F}}^O|\nabla(u)\in\mathfrak{b}_{\mathfrak{F}}^O\otimes\omega_O\}/\mathfrak{n}_{\mathfrak{F}}^O$ (see (122)) and the fiber of $\mathfrak{g}_{\mathrm{univ}}/\mathfrak{n}_{\mathrm{univ}}$ over $\mathfrak{F}$

equals $(\mathfrak{g}_{\mathfrak{F}}/\mathfrak{n}_{\mathfrak{F}})_0$:=the fiber of $\mathfrak{g}_{\mathfrak{F}}/\mathfrak{n}_{\mathfrak{F}}$ at the origin $0 \in \operatorname{Spec} O$. Consider the maps

$$\varphi, \psi : \{u \in \mathfrak{g}_{\mathfrak{F}}^O | \nabla(u) \in \mathfrak{b}_{\mathfrak{F}}^O \otimes \omega_O\}/\mathfrak{n}_{\mathfrak{F}}^O \to (\mathfrak{g}_{\mathfrak{F}}/\mathfrak{n}_{\mathfrak{F}})_0$$

where φ is induced by (144) and ψ is evaluation at 0.

3.8.21. *Lemma.* $\varphi = \psi$.

Proof. It follows from 3.7.17 that the restrictions of φ and ψ to $\mathfrak{a}_{\mathfrak{F}} := \{u \in \mathfrak{g}_{\mathfrak{F}}^O | \nabla(u) = 0\}$ are equal. So it suffices to show that $\operatorname{Ker}\varphi \subset \operatorname{Ker}\psi$. Clearly $\operatorname{Ker}\varphi = T_{\mathfrak{F}}^{\perp}\mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$:= the conormal space to $\mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$ at $\mathfrak{F}$. For any $q \in \mathfrak{b}_{\mathfrak{F}}^O$ the oper $\mathfrak{F}_q := (\mathfrak{F}_B, \nabla + q \cdot \frac{dt}{t})$ is (≤ 1)-singular. So the image of $\mathfrak{b}_{\mathfrak{F}}^O \otimes t^{-1}\omega_O$ in the r.h.s. of (117) is contained in the tangent space $T_{\mathfrak{F}}\mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O)$. Therefore $T_{\mathfrak{F}}^{\perp}\mathcal{O}\mathfrak{p}_{\mathfrak{g},1}(O) \subset \operatorname{Ker}\psi$. □

3.8.22. Now let us prove 3.8.19. Since the l.h.s. of (142) equals the l.h.s. of (141) we can reformulate 3.8.19 as follows.

Let $f \in \operatorname{Inv}(\mathfrak{g})$, $f(-\check{\rho}) = 0$. Consider f as an element of $A_{\mathfrak{g}}(K)/I_1$ (see (133)). By 3.8.12 $f \in I/I_1$. The image of f in $I/(I^2 + I_1)$ can be considered as an element $\nu \in \mathfrak{g}_{\text{univ}}/\mathfrak{n}_{\text{univ}}$ (see (144)). On the other hand, let $\lambda \in \mathfrak{h}^*$ be the differential at $-\check{\rho}$ of the restriction of $f \in \operatorname{Inv}(\mathfrak{g})$ to $\mathfrak{h}$. To prove 3.8.19 we must show that ν equals the image of λ under the composition

$$\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h} \subset \mathfrak{h} \otimes A_{\mathfrak{g}}(O) = \mathfrak{b}_{\text{univ}}/\mathfrak{n}_{\text{univ}} \subset \mathfrak{g}_{\text{univ}}/\mathfrak{n}_{\text{univ}}\,.$$

By 3.8.21 this is equivalent to the following statement: let $\mathfrak{F} = (\mathfrak{F}_B, \nabla) \in \mathcal{O}\mathfrak{p}_{\mathfrak{g}}(O)$, $q \in \mathfrak{b}_{\mathfrak{F}}^O$, $\mathfrak{F}_{\varepsilon q} := (\mathfrak{F}_B, \nabla + \varepsilon q \frac{dt}{t})$, then

$$\frac{d}{d\varepsilon} f(\operatorname{Res}(\mathfrak{F}_{\varepsilon q}))|_{\varepsilon=0} = \lambda(q_{\mathfrak{h}}(0)) \tag{145}$$

where $q_{\mathfrak{h}}(t) \in \mathfrak{h}[[t]]$ is the image of q in $\mathfrak{b}_{\mathfrak{F}}^O/\mathfrak{n}_{\mathfrak{F}}^O = \mathfrak{h} \otimes O$. Just as in the proof of 3.8.12 one shows that $\operatorname{Res}(\mathfrak{F}_{\varepsilon q})$ equals the image of $-\check{\rho} + \varepsilon q_{\mathfrak{h}}(0)$ in $W \setminus \mathfrak{h}$. So (145) is clear.

3.8.23. In this subsection (which can certainly be skipped by the reader) we give an "intrinsic" description of the scheme $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$ from 3.8.1. It is obtained by a straightforward "globalization" of 3.8.7 – 3.8.10.

Denote by G the adjoint group corresponding to $\mathfrak{g}$. Suppose we are in the situation of 3.1.2. So we have a B-bundle $\mathfrak{F}_B$ on X, the induced G-bundle $\mathfrak{F}_G$, and the $\mathfrak{g}_{\mathfrak{F}} \otimes \omega_X$-torsor $\operatorname{Conn}(\mathfrak{F}_G)$. Let D be a finite subscheme of X. Denote by $\operatorname{Conn}_D(\mathfrak{F}_G)$ the $\mathfrak{g}_{\mathfrak{F}} \otimes \omega_X(D)$-torsor induced by $\operatorname{Conn}(\mathfrak{F}_G)$; so sections of $\operatorname{Conn}_D(\mathfrak{F}_G)$ are connections with $(\leq D)$-singularities. Just as in 3.1.2 one defines $c : \operatorname{Conn}_D(\mathfrak{F}_G) \to (\mathfrak{g}/\mathfrak{b})_{\mathfrak{F}} \otimes \omega_X(D)$. The notion of $(\leq D)$-*singular* $\mathfrak{g}$-*oper* on X is defined as follows: in Definition 3.1.3 replace Conn by Conn_D and ω_X by $\omega_X(D)$.

If X is complete then $(\leq D)$-singular $\mathfrak{g}$-opers on X form a scheme. Just as in 3.8.9 one shows that the natural morphism from this scheme to $\mathcal{O}\mathfrak{p}_{\mathfrak{g}}(X \backslash D)$ is a closed embedding and its image equals $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$. So one can consider $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X)$ as the moduli scheme of $(\leq D)$-singular $\mathfrak{g}$-opers on X.

If $D \subset D'$ then $\mathcal{O}\mathfrak{p}_{\mathfrak{g},D}(X) \subset \mathcal{O}\mathfrak{p}_{\mathfrak{g},D'}(X)$, so we should have a natural way to construct a $(\leq D')$-singular $\mathfrak{g}$-oper $(\mathfrak{F}'_B, \nabla')$ from a $(\leq D)$-singular $\mathfrak{g}$-oper $(\mathfrak{F}_B, \nabla)$. Of course $(\mathfrak{F}'_B, \nabla')$ should be equipped with an isomorphism $\alpha : (\mathfrak{F}'_B, \nabla')|_{X\backslash\Delta} \xrightarrow{\sim} (\mathfrak{F}_B, \nabla)|_{X\backslash\Delta}$ where $\Delta \subset X$ is the finite subscheme such that $D' = D + \Delta$ if D, D', Δ are considered as effective divisors. The connection ∇' is reconstructed from ∇ and α, while $(\mathfrak{F}'_B, \alpha)$ is defined by the following property (cf. 3.8.10): if $x \in \Delta$, f is a local equation of Δ at x and s is a local section of $\mathfrak{F}_B$ at x then there is a local section s' of $\mathfrak{F}'_B$ at x such that $\alpha(s') = \lambda(f)s$ where $\lambda : \mathbb{G}_m \to H$ is the morphism corresponding to $\check{\rho}$.

4. **Pfaffians and all that**

4.0. **Introduction.**

4.0.1. Consider the "normalized" canonical bundle

$$\omega^{\sharp}_{\mathrm{Bun}_G} := \omega_{\mathrm{Bun}_G} \otimes \omega_0^{\otimes -1} \tag{146}$$

where ω_0 is the fiber of ω_{Bun_G} over the point of Bun_G corresponding to the trivial G-bundle on X. In this section we will associate to an ${}^LG$-oper $\mathfrak{F}$ the invertible sheaf $\lambda_{\mathfrak{F}}$ on Bun_G mentioned in 0.2(d). $\lambda_{\mathfrak{F}}$ will be equipped with an isomorphism $\lambda_{\mathfrak{F}}^{\otimes 2n} \xrightarrow{\sim} (\omega^{\sharp}_{\mathrm{Bun}_G})^{\otimes n}$ for some $n \neq 0$. This isomorphism induces the twisted $\mathcal{D}$-module structure on $\lambda_{\mathfrak{F}}$ required in 0.2(d).

According to formula (57) from 3.4.3 $\mathcal{O}\mathfrak{p}_{{}^LG}(X) = \mathcal{O}\mathfrak{p}_{{}^L\mathfrak{g}}(X) \times Z\,\mathrm{tors}_\theta(X)$ where Z is the center of ${}^LG$. Actually $\lambda_{\mathfrak{F}}$ depends only on the image of $\mathfrak{F}$ in $Z\,\mathrm{tors}_\theta(X)$. So our goal is to construct a canonical functor

$$\lambda\colon\ Z\,\mathrm{tors}_\theta(X) \to \mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G) \tag{147}$$

where $\mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G)$ is the groupoid of line bundles $\mathcal{A}$ on Bun_G equipped with an isomorphism $\mathcal{A}^{\otimes 2n} \xrightarrow{\sim} (\omega^{\sharp}_{\mathrm{Bun}_G})^{\otimes n}$ for some $n \neq 0$.

4.0.2. The construction of (147) is quite simple if G is simply connected. In this case Z is trivial, so one just has to construct an object of $\mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G)$. Since G is simply connected Bun_G is connected and simply connected (interpret a G-bundle on X as a G-bundle on the C^∞ manifold corresponding to X equipped with a $\bar{\partial}$-connection). So the problem is to show the existence of a square root of $\omega^{\sharp}_{\mathrm{Bun}_G}$ (then $\mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G)$ has a unique object whose fiber over the point of Bun_G corresponding to the trivial G-bundle is trivialized). To solve this problem we use the notion of *Pfaffian.*

To any vector bundle $\mathcal{Q}$ equipped with a non-degenerate symmetric form $\mathcal{Q} \otimes \mathcal{Q} \to \omega_X$ Laszlo and Sorger associate in [La-So] its Pfaffian $\mathrm{Pf}(\mathcal{Q})$, which is a canonical square root of $\det R\Gamma(X, \mathcal{Q})$. In 4.2 we give another definition of Pfaffian presumably equivalent to the one from [La-So].

Fix $\mathcal{L} \in \omega^{1/2}(X)$ (i.e., $\mathcal{L}$ is a square root of ω_X). Then the line bundle on Bun_G whose fiber at $\mathcal{F} \in \mathrm{Bun}_G$ equals

$$\mathrm{Pf}(\mathfrak{g}_{\mathcal{F}} \otimes \mathcal{L}) \otimes \mathrm{Pf}(\mathfrak{g} \otimes \mathcal{L})^{\otimes -1} \tag{148}$$

is a square root of $\omega^{\sharp}_{\mathrm{Bun}_G}$ (see 4.3.1 for details).

So to understand the case where G is simply connected it is enough to look through 4.2 and 4.3.1. In the general case the construction of (147) is more complicated. The main point is that the square root of $\omega^{\sharp}_{\mathrm{Bun}_G}$ defined by (148) depends on $\mathcal{L} \in \omega^{1/2}(X)$.

4.0.3. Here is an outline of the construction of (148) for any semisimple G.

As explained in 3.4.6 $Z\,\mathrm{tors}_\theta(X)$ is a Torsor over the Picard category $Z\,\mathrm{tors}(X)$ and $\mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G)$ is a Torsor over the Picard category

$$\mu_\infty\,\mathrm{tors}(\mathrm{Bun}_G) := \varinjlim_n \mu_n\,\mathrm{tors}(\mathrm{Bun}_G) \tag{149}$$

The functor (147) we are going to construct is ℓ-affine in the sense of 3.4.6 for a certain Picard functor $\ell :\ Z\,\mathrm{tors}(X) \to \mu_\infty\,\mathrm{tors}(\mathrm{Bun}_G)$. We define ℓ in 4.1. The Torsor $Z\,\mathrm{tors}_\theta(X)$ is induced from $\omega^{1/2}(X)$ via a certain Picard functor $\mu_2\,\mathrm{tors}(X) \to Z\,\mathrm{tors}(X)$ (see 3.4.6). So to construct λ it is enough to construct an ℓ'-affine functor $\lambda'\colon\ \omega^{1/2}(X) \to \mu_\infty\,\mathrm{tors}_\theta(X)$ where ℓ' is the composition $\mu_2\,\mathrm{tors}(X) \to Z\,\mathrm{tors}(X) \xrightarrow{\ell} \mu_\infty\,\mathrm{tors}(\mathrm{Bun}_G)$. We define λ' by $\mathcal{L} \mapsto \lambda'_{\mathcal{L}}$ where $\lambda'_{\mathcal{L}}$ is the line bundle on Bun_G whose fiber at $\mathcal{F} \in \mathrm{Bun}_G$ equals (148). The fact that λ' is ℓ'-affine is deduced in 4.4 from 4.3.10, which is a general statement on SO_n-bundles[26]. Actually in subsections 4.2 and 4.3 devoted to Pfaffians the group G does not appear at all.

4.0.4. Each line bundle on Bun_G constructed in this section is equipped with the following extra structure: for every $x \in X$ a central extension of $G(K_x)$ acts on its pullback to the scheme $\mathrm{Bun}_{G,\underline{x}}$ from 2.3.1. This structure is used in 4.3. We will also need it in Chapter 5.

[26]In fact 4.3.10 is a refinement of Proposition 5.2 from [BLaSo].

4.1. μ_∞-torsors on Bun_G.

4.1.1. Let G be a connected affine algebraic group, Π a finite abelian group, $0 \to \Pi(1) \to \tilde{G} \to G \to 0$ an extension of G. Our goal is to construct a canonical Picard functor $\ell : \Pi^\vee \operatorname{tors}(X) \to \mu_\infty \operatorname{tors}(\mathrm{Bun}_G)$ where $\Pi^\vee := \mathrm{Hom}(\Pi, \mu_\infty)$.

Remark. If G is semisimple and $\tilde{G}$ is the universal covering of G then $\Pi = \pi_1(G)$ and $\Pi^\vee$ is canonically isomorphic to the center Z of ${}^L G$ (the isomorphism is induced by the duality between the Cartan tori of G and ${}^L G$). So in this case ℓ is a Picard functor $Z \operatorname{tors}(X) \to \mu_\infty \operatorname{tors}(\mathrm{Bun}_G)$, as promised in 4.0.3.

We construct ℓ in 4.1.2–4.1.4. We "explain" the construction in 4.1.5 and slightly reformulate it in 4.1.6. In 4.1.7–4.1.9 the action of a central extension of $G(K_x)$ is considered. In 4.1.10–4.1.11 we give a description of the fundamental groupoid of Bun_G, which clarifies the construction of torsors on Bun_G. The reader can skip 4.1.5 and 4.1.10–4.1.11.

4.1.2. For $\mathcal{F} \in \mathrm{Bun}_G$ denote by $\tilde{\mathcal{F}}$ the $\Pi(1)$-gerbe on X of $\tilde{G}$-liftings of $\mathcal{F}$. Its class $c(\mathcal{F})$ is the image of $cl(\mathcal{F})$ by the boundary map $H^1(X, G) \to H^2\big(X, \Pi(1)\big) = \Pi$. For a finite non-empty $S \subset X$ the gerbe $\tilde{\mathcal{F}}_{X \setminus S}$ is neutral. Therefore $|\tilde{\mathcal{F}}(X \setminus S)|$ (:= the set of isomorphism classes of objects of $\tilde{\mathcal{F}}(X \setminus S)$) is a non-empty $H^1\big(X \setminus S, \Pi(1)\big)$-torsor. Denote it by $\phi_{S,\mathcal{F}}$. When $\mathcal{F}$ varies $\phi_{S,\mathcal{F}}$ become fibers of an $H^1\big(X \setminus S, \Pi(1)\big)$-torsor ϕ_S over Bun_G.

4.1.3. For any $x \in X$ the set $|\tilde{\mathcal{F}}(\operatorname{Spec} O_x)|$ has a single element. We use it to trivialize the Π-torsor $|\tilde{\mathcal{F}}(\operatorname{Spec} K_x)|$ (note that $\Pi = H^1\big(\operatorname{Spec} K_x, \Pi(1)\big)$). Thus the restriction to $\operatorname{Spec} K_s$, $s \in S$, defines a Res_s-affine map $\mathrm{Res}_{s,\mathcal{F}} : \phi_{S,\mathcal{F}} \to \Pi$ where $\mathrm{Res}_s : H^1\big(X \setminus S, \Pi(1)\big) \to \Pi$ is the residue at s. For $c \in \Pi$ set $\Pi^S_c := \{\pi_S = (\pi_s) : \sum \pi_s = c\} \subset \Pi^S$. The map $\mathrm{Res}_{S,\mathcal{F}} := (\mathrm{Res}_{s,\mathcal{F}}) : \phi_{S,\mathcal{F}} \to \Pi^S$ has image $\Pi^S_{c(\mathcal{F})}$.

4.1.4. Recall that $\Pi^\vee$ is the group dual to Π, so we have a non-degenerate pairing $(\ ) : \Pi \times \Pi^\vee \to \mu_\infty$.

Let $\mathcal{E}$ be a $\Pi^\vee$-torsor on X. Set $\mathcal{E}_S := \prod_{s\in S} \mathcal{E}_s$ = the set of trivializations of $\mathcal{E}$ at S; this is a $(\Pi^\vee)^S$-torsor. For any $e \in \mathcal{E}_S$ we have the class $cl(\mathcal{E}, e) \in H^1_c(X \setminus S, \Pi^\vee)$. Denote by $\ell_{S,\mathcal{E},\mathcal{F}}$ a μ_∞-torsor equipped with a map

$$(\ ,\ )_\ell : \phi_{S,\mathcal{F}} \times \mathcal{E}_S \to \ell_{S,\mathcal{E},\mathcal{F}} \tag{150}$$

such that for $\varphi \in \phi_{S,\mathcal{F}}$, $e = (e_s) \in \mathcal{E}_S$, $h \in H^1\big(X \setminus S, \Pi(1)\big)$, $\chi = (\chi_s) \in (\Pi^\vee)^S$ one has

$$\begin{gathered} (\varphi + h, e)_\ell = (h, cl(\mathcal{E}, e))_{\mathcal{P}}(\varphi, e)_\ell \\ (\varphi, \chi e)_\ell = (\operatorname{Res}_S \varphi, \chi)(\varphi, e)_\ell. \end{gathered} \tag{151}$$

Here $(\ ,\ )_{\mathcal{P}} : H^1\big(X \setminus S, \Pi(1)\big) \times H^1_c\big(X \setminus S, \Pi^\vee\big) \to \mu_\infty$ is the Poincaré pairing and $(\operatorname{Res}_S \varphi, \chi) := \prod_{s\in S}(\operatorname{Res}_s \varphi, \chi_s) \in \mu_\infty$. Such $(\ell_{S,\mathcal{E},\mathcal{F}}, (\ )_\ell))$ exists and is unique. If $S' \supset S$ then we have obvious maps $\phi_{S,\mathcal{F}} \hookrightarrow \phi_{S',\mathcal{F}}$, $\mathcal{E}_{S'} \twoheadrightarrow \mathcal{E}_S$, and there is a unique identification of μ_∞-torsors $\ell_{S,\mathcal{E},\mathcal{F}} = \ell_{S',\mathcal{E},\mathcal{F}}$ that makes these maps mutually adjoint for $(\ ,\ )_\ell$. Thus our μ_∞-torsor is independent of S and we denote it simply $\ell_{\mathcal{E},\mathcal{F}}$.

When $\mathcal{F}$ varies $\ell_{\mathcal{E},\mathcal{F}}$ become fibers of a μ_∞-torsor $\ell_{\mathcal{E}}$ over Bun_G. The functor

$$\ell = \ell^{\tilde{G}} : \Pi^\vee \operatorname{tors}(X) \to \mu_\infty \operatorname{tors}(\operatorname{Bun}_G), \tag{152}$$

$\mathcal{E} \mapsto \ell_{\mathcal{E}}$, has an obvious structure of Picard functor. The corresponding homomorphism of the automorphism groups $\Pi^\vee \to \Gamma(\operatorname{Bun}_G, \mu_\infty)$ is $\chi \mapsto (c, \chi)$.

Remark. In fact ℓ is a functor $\Pi^\vee \operatorname{tors}(X) \to \mu_m \operatorname{tors}(\operatorname{Bun}_G)$ where m is the order of Π. This follows from the construction or from the fact that (152) is a Picard functor.

4.1.5. For an abelian group A denote by $A\,\mathrm{gerbes}(X)$ the category associated to the 2-category of A-gerbes on X (so $A\,\mathrm{gerbes}(X)$ is the groupoid whose objects are A-gerbes on X and whose morphisms are 1-morphisms up to 2-isomorphism). In 4.1.2–4.1.4 we have in fact constructed a bi-Picard functor

$$\Pi^\vee\,\mathrm{tors}(X)\times\Pi(1)\,\mathrm{gerbes}(X)\to\mu_\infty\,\mathrm{tors} \tag{153}$$

where $\mu_\infty\,\mathrm{tors}$ denotes the category of μ_∞-torsors over a point. In this subsection (which can be skipped by the reader) we give a "scientific interpretation" of this construction.

In §1.4.11 from [Del73] Deligne associates a Picard category to a complex $K^{\cdot}$ of abelian groups such that $K^i=0$ for $i\neq 0,-1$. We denote this Picard category by $P(K^{\cdot})$. Its objects are elements of K^0 and a morphism from $x\in K^0$ to $y\in K^0$ is an element $f\in K^{-1}$ such that $df=y-x$.

In 4.1.4 we implicitly used the interpretation of $\Pi^\vee\,\mathrm{tors}(X)$ as $P(K^{\cdot}_S)$ where $K^0_S=H^1_c(X\backslash S,\Pi^\vee)=$ the set of isomorphism classes of $\Pi^\vee$-torsors on X trivialized over S, $K^{-1}_S=H^0(S,\Pi^\vee)$. In 4.1.3 we implicitly used the interpretation of $\Pi(1)\,\mathrm{gerbes}(X)$ as $P(L^{\cdot}_S)$ where $L^0_S=H^2_S(X,\Pi(1))=\Pi^S$, $L^{-1}_S=H^1(X\backslash S,\Pi(1))$ (L^0_S parametrizes $\Pi(1)$-gerbes on X with a fixed object over $X\backslash S$). The construction of the bi-Picard functor (153) given in 4.1.4 uses only the canonical pairing $K^{\cdot}_S\times L^{\cdot}_S\to\mu_\infty[1]$.

For $S'\supset S$ we have canonical quasi-isomorphisms $K^{\cdot}_{S'}\to K^{\cdot}_S$ and $L^{\cdot}_S\to L^{\cdot}_{S'}$. The corresponding equivalences $P(K^{\cdot}_{S'})\to P(K^{\cdot}_S)$ and $P(L^{\cdot}_S)\to P(L^{\cdot}_{S'})$ are compatible with our identifications of $P(K^{\cdot}_S)$ and $P(K^{\cdot}_{S'})$ with $\Pi^\vee\,\mathrm{tors}(X)$ and also with the identifications of $P(L^{\cdot}_S)$ and $P(L^{\cdot}_{S'})$ with $\Pi(1)\,\mathrm{gerbes}(X)$. The morphism $L^{\cdot}_S\to L^{\cdot}_{S'}$ is adjoint to $K^{\cdot}_{S'}\to K^{\cdot}_S$ with respect to the pairings $K^{\cdot}_S\times L^{\cdot}_S\to\mu_\infty[1]$ and $K^{\cdot}_{S'}\times L^{\cdot}_{S'}\to\mu_\infty[1]$. Therefore (153) does not depend on S.

Remarks

(i) Instead of $K^{\cdot}_S$ and $L^{\cdot}_S$ it would be more natural to use their images in the derived category, i.e., $(\tau_{\leq 1} R\Gamma(X, \Pi^\vee))[1]$ and $(\tau_{\geq 1} R\Gamma(X, \Pi(1)))[2]$. However the usual derived category is not enough: according to §§1.4.13–1.4.14 from [Del73] the image of $K^{\cdot}$ in the derived category only gives $P(K^{\cdot})$ up to equivalence unique up to *non-unique* isomorphism. So one needs a refined version of the notion of derived category, which probably cannot be found in the literature.

(ii) From the non-degeneracy of the pairing $K^{\cdot}_S \times L^{\cdot}_S \to \mu_\infty[1]$ one can easily deduce that (153) induces an equivalence between $\Pi^\vee\,\mathrm{tors}(X)$ and the category of Picard functors $\Pi(1)\,\mathrm{gerbes}(X) \to \mu_\infty\,\mathrm{tors}$ (this is a particular case of the equivalence (1.4.18.1) from [Del73]).

4.1.6. The definition of $\ell_{\mathcal{E}}$ from 4.1.4 can be reformulated as follows. Let $S \subset X$ be finite and non-empty. For a fixed $e \in \mathcal{E}_S$ we have the class $c = cl(\mathcal{E}, e) \in H^1_c(X \backslash S, \Pi^\vee)$ and therefore a morphism $\lambda_e : H^1(X \backslash S, \Pi(1)) \to \mu_\infty$ defined by $\lambda_e(h) = (h, c)_{\mathcal{P}}$. Denote by $\ell_{\mathcal{E},e}$ the λ_e-pushforward of the $H^1(X \backslash S, \Pi(1))$-torsor ϕ_S from 4.1.2. The torsors $\ell_{\mathcal{E},e}$ for various $e \in \mathcal{E}_S$ are identified as follows.

Let $\tilde{e} = \chi e$, $\chi \in (\Pi^\vee)^S$. Then $\lambda_{\tilde{e}}(h)/\lambda_e(h) = (\mathrm{Res}_S(h), \chi)$ where Res_S is the boundary morphism $H^1(X \backslash S, \Pi(1)) \to H^2_S(X, \Pi(1)) = \Pi^S$. So $\ell_{\mathcal{E},\tilde{e}}/\ell_{\mathcal{E},e}$ is the pushforward of the Π^S-torsor $(\mathrm{Res}_S)_* \phi_S$ via $\chi : \Pi^S \to \mu_\infty$. The map $\mathrm{Res}_{S,\mathcal{F}} : \phi_{S,\mathcal{F}} \to \Pi^S$ from 4.1.3 induces a canonical trivialization of $(\mathrm{Res}_S)_* \phi_S$ and therefore a canonical isomorphism $\ell_{\mathcal{E},e} \xrightarrow{\sim} \ell_{\mathcal{E},\tilde{e}}$. So we can identify $\ell_{\mathcal{E},e}$ for various $e \in \mathcal{E}_S$ and obtain a μ_∞-torsor on Bun_G, which does not depend on $e \in \mathcal{E}_S$. Clearly it does not depend on S. This is $\ell_{\mathcal{E}}$.

4.1.7. Let $S \subset X$ be a non-empty finite set, $O_S := \prod\limits_{x \in S} O_x$, $K_S := \prod\limits_{x \in S} K_x$ where O_x is the completed local ring of x and K_x is its field of fractions. Denote by $\underline{S}$ the formal neighbourhood of S and by $\mathrm{Bun}_{G,\underline{S}}$ the moduli scheme of G-bundles on X trivialized over $\underline{S}$ (in 2.3.1 we introduced $\mathrm{Bun}_{G,\underline{x}}$, which corresponds to $S = \{x\}$). One defines an action of $G(K_S)$ on $\mathrm{Bun}_{G,\underline{S}}$

extending the action of $G(O_S)$ by interpreting a G-bundle on X as a G-bundle on $X \setminus S$ with a trivialization of its pullback to $\operatorname{Spec} K_S$ (see 2.3.4 and 2.3.7).

Let $\ell_{\mathcal{E}}$ be the μ_∞-torsor on Bun_G corresponding to a $\Pi^\vee$-torsor $\mathcal{E}$ on X (see 4.1.4, 4.1.6). Denote by $\ell_{\mathcal{E}}^S$ the inverse image of $\ell_{\mathcal{E}}$ on $\mathrm{Bun}_{G,\underline{S}}$. The action of $G(O_S)$ on $\mathrm{Bun}_{G,\underline{S}}$ canonically lifts to its action on $\ell_{\mathcal{E}}^S$. We claim that a trivialization of $\mathcal{E}$ over S defines an action of $G(K_S)$ on $\ell_{\mathcal{E}}^S$ extending the above action of $G(O_S)$ and compatible with the action of $G(K_S)$ on $\mathrm{Bun}_{G,\underline{S}}$. Indeed, once $e \in \mathcal{E}_S$ is chosen $\ell_{\mathcal{E}}^S$ can be identified with $\ell_{\mathcal{E},e}^S = (\lambda_e)_* \tilde{\phi}_S$ where $\tilde{\phi}_S$ is the pullback of ϕ_S to $\mathrm{Bun}_{G,\underline{S}}$ and λ_e was defined in 4.1.6. $G(K_S)$ acts on $\tilde{\phi}_S$ because $\phi_{S,\mathcal{F}}$ depends only on the restriction of $\mathcal{F}$ to $X \setminus S$. So $G(K_S)$ acts on $\ell_{\mathcal{E},e}^S$.

The isomorphism $\ell_{\mathcal{E},e}^S \xrightarrow{\sim} \ell_{\mathcal{E},\tilde{e}}^S$ induced by the isomorphism $\ell_{\mathcal{E},e} \xrightarrow{\sim} \ell_{\mathcal{E},\tilde{e}}$ from 4.1.6 is *not* $G(K_S)$-equivariant. Indeed, if $\tilde{e} = \chi e$, $\chi \in (\Pi^\vee)^S$, then according to 4.1.6 $\ell_{\mathcal{E},\tilde{e}}^S / \ell_{\mathcal{E},e}^S$ is the pushforward of the Π^S-torsor $(\mathrm{Res})_* \tilde{\phi}_S$ via $\chi : \Pi^S \to \mu_\infty$. The identification $(\mathrm{Res})_* \tilde{\phi}_S = \mathrm{Bun}_{G,\underline{S}} \times \Pi^S$ from 4.1.6 becomes $G(K_S)$-equivariant if $G(K_S)$ acts on Π^S via the boundary morphism $\varphi : G(K_S) \to H^1(\operatorname{Spec} K_S, \Pi(1)) = \Pi^S$ (we should check the sign!!!). Therefore the trivial μ_∞-torsor $\ell_{\mathcal{E},\tilde{e}}^S / \ell_{\mathcal{E},e}^S$ is equipped with a nontrivial action of $G(K_S)$: it acts by $\chi\varphi : G(K_S) \to \mu_\infty$.

So to each $e \in \mathcal{E}_S$ there corresponds an action of $G(K_S)$ on $\tilde{\phi}_S$, and if e is replaced by χe, $\chi \in (\Pi^\vee)^S = \mathrm{Hom}(\Pi^S, \mu_\infty)$, then the action is multiplied by $\chi\varphi : G(K_S) \to \mu_\infty$.

Remark. By the way, we have proved that the coboundary map $\varphi : G(K_S) \to H^1(\operatorname{Spec} K_S, \Pi(1)) = \Pi^S$ is locally constant[27] (indeed, $G(K_S)$ acts on $(\mathrm{Res})_* \widetilde{\phi_S}$ as a group ind-scheme, so φ is a morphism of ind-schemes, i.e., φ is locally constant. The proof can be reformulated as follows. Without loss of generality we may assume that S consists of a single point x. The group ind-scheme $G(K_x)$ acts on $\mathrm{Bun}_{G,\underline{x}}$ (see 2.3.3 –

[27] See also 4.5.4.

2.3.4), so it acts on $\pi_0(\mathrm{Bun}_{G,\underline{x}}) = \pi_0(\mathrm{Bun}_G)$. One has the "first Chern class" map $c : \pi_0(\mathrm{Bun}_G) \to \Pi$. It is easy to show that $c(gu) = \varphi(g)c(u)$ for $u \in \pi_0(\mathrm{Bun}_G)$, $g \in G(K_x)$ where $\varphi : G(K_x) \to H^1(K_x, \Pi(1)) = \Pi$ is the coboundary map. So φ is locally constant.

4.1.8. Denote by $\widetilde{G(K_S)}_{\mathcal{E}}$ the group generated by μ_∞ and elements $\langle g, e\rangle$, $g \in G(K_S)$, $e \in \mathcal{E}_S$, with the defining relations

$$\begin{aligned} \langle g_1 g_2, e\rangle &= \langle g_1, e\rangle\langle g_2, e\rangle \\ \langle g_1, \chi e\rangle &= \chi(\varphi(g))\cdot\langle g, e\rangle\,, \quad \chi \in (\Pi^\vee)^S = \mathrm{Hom}(\Pi^S, \mu_\infty) \\ \alpha\langle g, e\rangle &= \langle g, e\rangle\alpha\,, \quad \alpha \in \mu_\infty \end{aligned}$$

$\widetilde{G(K_S)}_{\mathcal{E}}$ is a central extension of $G(K_S)$ by μ_∞. The extension is trivial: a choice of $e \in \mathcal{E}_S$ defines a splitting

$$\sigma_e : G(K_S) \to \widetilde{G(K_S)}_{\mathcal{E}}\,, \quad g \mapsto \langle g, e\rangle\,. \tag{154}$$

It follows from 4.1.7 that $\widetilde{G(K_S)}_{\mathcal{E}}$ acts on $\ell^S_{\mathcal{E}}$ so that $\mu_\infty \subset \widetilde{G(K_S)}_{\mathcal{E}}$ acts in the obvious way and the action of $G(K_S)$ on $\ell^S_{\mathcal{E}}$ corresponding to $e \in \mathcal{E}_S$ (see 4.1.7) comes from the splitting (154).

4.1.9. Consider the point of $\mathrm{Bun}_{G,\underline{S}}$ corresponding to the trivial G-bundle on X with the obvious trivialization over $\underline{S}$. Acting by $G(K_S)$ on this point one obtains a morphism $f : G(K_S) \to \mathrm{Bun}_{G,\underline{S}}$. Suppose that G is semisimple. Then f induces an isomorphism.

$$G(K_S)/G(A_S) \xrightarrow{\sim} \mathrm{Bun}_{G,\underline{S}} \tag{155}$$

where $A_S := H^0(X \setminus S, \mathcal{O}_X)$ (see Theorem 1.3 from [La-So] and its proof in §3 of loc.cit). It is essential that $G(K_S)$ and $G(A_S)$ are considered as group ind-schemes and $G(K_S)/G(A_S)$ as an fppf quotient, so (155) is more than a bijection between the sets of $\mathbb{C}$-points. We also have an isomorphism

$$G(O_S) \setminus G(K_S)/G(A_S) \xrightarrow{\sim} \mathrm{Bun}_G\,. \tag{156}$$

It is easy to see that the μ_∞-torsors $\ell_{\mathcal{E}}$ and $\ell_{\mathcal{E}}^S$ defined in 4.1.4 and 4.1.7 can be described as

$$\ell_{\mathcal{E}}^S = \widetilde{G(K_S)}_{\mathcal{E}}/G(A_S) \tag{157}$$

$$\ell_{\mathcal{E}} = G(O_S) \setminus \widetilde{G(K_S)}_{\mathcal{E}}/G(A_S) \tag{158}$$

where $\widetilde{G(K_S)}_{\mathcal{E}}$ is the central extension from 4.1.8. Here the embeddings $i : G(O) \to \widetilde{G(K_S)}_{\mathcal{E}}$ and $j : G(A_S) \to \widetilde{G(K_S)}_{\mathcal{E}}$ are defined by

$$i(g) = \langle g, e\rangle\,, \quad e \in \mathcal{E}_S \tag{159}$$

$$j(g) = \langle g, e\rangle \cdot (\psi(g), cl(\mathcal{E}, e))_{\mathcal{P}}^{-1}\,, \quad e \in \mathcal{E}_S \tag{160}$$

(we should check the sign!!!) where ψ is the boundary morphism $G(A_S) \to H^1(X \setminus S, \Pi(1))$ and $cl(\mathcal{E}, e) \in H_c^1(X \setminus S, \Pi^\vee)$ is the class of $(\mathcal{E}, e)$ (the r.h.s. of (159) and (160) do not depend on e).

Remark. The morphisms $\varphi : G(K_S) \to \Pi^S$ and $\psi : G(A_S) \to H^1(X \setminus S, \Pi(1))$ induce a morphism

$$\mathrm{Bun}_G = G(O_S) \setminus G(K_S)/G(A_S) \to \Pi^S/H^1(X \setminus S, \Pi(1)) \tag{161}$$

where the r.h.s. of (161) is understood as a quotient *stack*. Clearly $\ell_{\mathcal{E}}$ is the pullback of a certain μ_∞-torsor on the stack $\Pi^S/H^1(X \setminus S, \Pi(1))$.

4.1.10. The reader can skip the remaining part of 4.1.

Let C be a groupoid. Denote by $\underline{C}$ the corresponding constant sheaf of groupoids on the category of $\mathbb{C}$-schemes equipped with the fppf topology. If the automorphism groups of objects of C are finite then $\underline{C}$ is an algebraic stack. By abuse of notation we will often write C instead of $\underline{C}$ (e.g., if C is a set then $\underline{C} = C \times \operatorname{Spec} \mathbb{C}$ is usually identified with C).

Examples. 1) If C has a single object and G is its automorphism group then $\underline{C}$ is the classifying stack of G.

2) If $C = P(K^\cdot)$ (see 4.1.5) then $\underline{C}$ is the quotient stack of K^0 with respect to the action of K^{-1}. So according to 4.1.5 the r.h.s. of (161) is the stack corresponding to the groupoid $\Pi(1)\,\text{gerbes}(X)$.

3) If $C = A\,\mathrm{gerbes}(X)$ then $\underline{C}$ is the sheaf of groupoids associated to the presheaf $S \mapsto A\,\mathrm{gerbes}(X \times S)$.

Consider the groupoid $\Pi(1)\,\mathrm{gerbes}(X)$ as an algebraic stack. In 4.1.2 we defined a canonical morphism

$$\tilde{c} : \mathrm{Bun}_G \to \Pi(1)\,\mathrm{gerbes}(X) \tag{162}$$

that associates to a G-bundle $\mathcal{F}$ the $\Pi(1)$-gerbe of $\tilde{G}$-liftings of $\mathcal{F}$ (by the way, the morphism (161) defined for semisimple G coincides with $\tilde{c}$). $\tilde{c}$ is a refinement of the Chern class map $c : \mathrm{Bun}_G \to H^2(X, \Pi(1)) = \Pi$; more precisely, c is the composition of $\tilde{c}$ and the canonical morphism $\Pi(1)\,\mathrm{gerbes}(X) \to H^2(X, \Pi(1)) =$ the set of isomorphism classes of $\Pi(1)\,\mathrm{gerbes}(X)$.

The μ_∞-torsors on Bun_G constructed in 4.1.4 come from μ_∞-torsors on $\Pi(1)\,\mathrm{gerbes}(X)$. The following proposition shows that if $\tilde{G}$ is the universal covering of G then *any* local system on Bun_G comes from a unique local system on $\Pi(1)\,\mathrm{gerbes}(X)$.

4.1.11. *Proposition.* Suppose that $\tilde{G}$ is the universal covering of G (so $\Pi = \pi_1(G)$). Then the morphism (162) induces an equivalence between the fundamental groupoid of Bun_G and $\Pi(1)\,\mathrm{gerbes}(X)$.

Let us sketch a transcendental proof (since it is transcendental we will not distinguish between Π and $\Pi(1)$). Denote by X^{top} the C^∞ manifold corresponding to X; for a G-bundle $\mathcal{F}$ on X denote by $\mathcal{F}^{\mathrm{top}}$ the corresponding G-bundle on X^{top}. Consider the groupoid $\mathrm{Bun}_G^{\mathrm{top}}$ whose objects are G-bundles on X^{top} and morphisms are isotopy classes of C^∞ isomorphisms between G-bundles. It is easy to show that the natural functor $\mathrm{Bun}_G^{\mathrm{top}} \to \Pi\,\mathrm{gerbes}(X^{\mathrm{top}}) = \Pi\,\mathrm{gerbes}(X)$ is an equivalence. So we must prove that for a G-bundle ξ on X^{top} the stack of G-bundles $\mathcal{F}$ on X equipped with an isotopy class of isomorphisms $\mathcal{F}^{\mathrm{top}} \xrightarrow{\sim} \xi$ is non-empty, connected, and simply connected. This is clear if a G-bundle on X is interpreted as a G-bundle on X^{top} equipped with a $\bar{\partial}$-connection.

Remark. In 4.1.2 we defined the $H^1(X\setminus S, \Pi(1))$-torsor $\phi_S \to \mathrm{Bun}_G$. If $S = \{x\}$ for some $x \in X$ then $H^1(X\setminus S, \Pi(1)) = H^1(X, \Pi(1))$, so $\phi_{\{x\}} \to \mathrm{Bun}_G$ is a $H^1(X, \Pi(1))$-torsor. Proposition 4.1.11 can be reformulated as follows: if $\tilde{G}$ is the universal covering of G then the Chern class map $\pi_0(\mathrm{Bun}_G) \to \Pi$ is bijective and the restriction of $\phi_{\{x\}} \to \mathrm{Bun}_G$ to each connected component of Bun_G is a universal covering. This is really a reformulation because a choice of x defines an equivalence.

$$\Pi(1)\,\mathrm{gerbes}(X) \xrightarrow{\sim} \Pi \times H^1(X, \Pi(1))\,\mathrm{tors} \tag{163}$$

(to a $\Pi(1)$-gerbe on X one associates its class in $H^2(X, \Pi(1)) = \Pi$ and the $H^1(X, \Pi(1))$-torsor of isomorphism classes of its objects over $X \setminus \{x\}$).

4.2. **Pfaffians I.** In this subsection we assume that for $(\mathbb{Z}/2\mathbb{Z})$-graded vector spaces A and B the identification of $A \otimes B$ with $B \otimes A$ is defined by $a \otimes b \mapsto (-1)^{p(a)p(b)} b \otimes a$ where $p(a)$ is the parity of a. Following [Kn-Mu] for a vector space V of dimension $n < \infty$ we consider $\det V$ as a $(\mathbb{Z}/2\mathbb{Z})$-graded space of degree n mod 2.

4.2.1. Let X be a smooth complete curve over $\mathbb{C}$. An *ω-orthogonal bundle* on X is a vector bundle $\mathcal{Q}$ equipped with a non-degenerate symmetric pairing $\mathcal{Q} \otimes \mathcal{Q} \to \omega_X$. Denote by ω-Ort the stack of ω-orthogonal bundles on X. There is a well known line bundle $\det R\Gamma$ on ω-Ort (its fiber over $\mathcal{Q}$ is $\det R\Gamma(X, \mathcal{Q})$). Laszlo and Sorger [La-So] construct a $(\mathbb{Z}/2\mathbb{Z})$-graded line bundle on ω-Ort (which they call the *Pfaffian*) and show that the tensor square of the Pfaffian is $\det R\Gamma$. For our purposes it is more convenient to use another definition of Pfaffian. Certainly it should be equivalent to the one from [La-So], but we did not check this.

We will construct a line bundle Pf on ω-Ort which we call the Pfaffian; its fiber over an ω-orthogonal bundle $\mathcal{Q}$ is denoted by $\mathrm{Pf}(\mathcal{Q})$. The action of $-1 \in \mathrm{Aut}\,\mathcal{Q}$ on $\mathrm{Pf}(\mathcal{Q})$ defines a $(\mathbb{Z}/2\mathbb{Z})$-grading on Pf. Since Pf is a line bundle, "grading" just means that there is a locally constant $p : (\omega\text{-Ort}) \to \mathbb{Z}/2\mathbb{Z}$ such that $\mathrm{Pf}(\mathcal{Q})$ has degree $p(\mathcal{Q})$. Actually $p(\mathcal{Q}) = \dim H^0(\mathcal{Q}) \mod 2$

(the fact that $\dim H^0(\mathcal{Q}) \mod 2$ is locally constant was proved by M. Atiyah and D. Mumford [At, Mu]).

For an ω-orthogonal bundle $\mathcal{Q}$ denote by $\mathcal{Q}^-$ the same bundle $\mathcal{Q}$ equipped with the *opposite* pairing $\mathcal{Q}\otimes\mathcal{Q}\to\omega_X$. Set $\mathrm{Pf}^-(\mathcal{Q}) := \mathrm{Pf}(\mathcal{Q}^-)$. We will define a canonical isomorphism $\mathrm{Pf}\otimes\mathrm{Pf}^- \xrightarrow{\sim} \det R\Gamma$. Define isomorphisms $f_{\pm i} : \mathrm{Pf}(\mathcal{Q}) \xrightarrow{\sim} \mathrm{Pf}(\mathcal{Q}^-)$ by $f_{\pm i} := (\varphi_{\pm i})_*$ where $i=\sqrt{-1}$ and $\varphi_i : \mathcal{Q}\xrightarrow{\sim}\mathcal{Q}^-$ is multiplication by i. Identifying Pf and Pf^- by means of $f_{\pm i}$ we obtain isomorphisms $c_{\pm i} : \mathrm{Pf}^{\otimes 2} \xrightarrow{\sim} \det R\Gamma$ such that $(c_i)^{-1}c_{-i} : \mathrm{Pf}(\mathcal{Q})^{\otimes 2} \xrightarrow{\sim} \mathrm{Pf}(\mathcal{Q})^{\otimes 2}$ is multiplication by $(-1)^{p(\mathcal{Q})}$.

Remarks

(i) If $\mathcal{Q}$ is an ω-orthogonal bundle then by Serre's duality $H^1(X,\mathcal{Q}) = (H^0(X,\mathcal{Q}))^*$, so $\det R\Gamma(X,\mathcal{Q}) = \det H^0(X,\mathcal{Q})^{\otimes 2}$. The naive definition would be $\mathrm{Pf}^?(\mathcal{Q}) := \det H^0(X,\mathcal{Q})$, but this does not make sense for families of $\mathcal{Q}$'s because $\dim H^0(X,\mathcal{Q})$ can jump.

(ii) Let $\mathcal{Q}$ be the orthogonal direct sum of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Then $\det R\Gamma(X,\mathcal{Q}) = \det R\Gamma(X,\mathcal{Q}_1)\otimes\det R\Gamma(X,\mathcal{Q}_2)$. From the definitions of Pf and $\mathrm{Pf}\otimes\mathrm{Pf}^- \xrightarrow{\sim} \det R\Gamma$ it will be clear that there is a canonical ismorphism $\mathrm{Pf}(\mathcal{Q}) \xrightarrow{\sim} \mathrm{Pf}(\mathcal{Q}_1)\otimes\mathrm{Pf}(\mathcal{Q}_2)$ and the diagram

$$\begin{array}{ccc}
\mathrm{Pf}(\mathcal{Q})\otimes\mathrm{Pf}(\mathcal{Q}^-) & \xrightarrow{\sim} & \mathrm{Pf}(\mathcal{Q}_1)\otimes\mathrm{Pf}(\mathcal{Q}_1^-)\otimes\mathrm{Pf}(\mathcal{Q}_2)\otimes\mathrm{Pf}(\mathcal{Q}_2^-) \\
\downarrow \wr & & \downarrow \wr \\
\det R\Gamma(X,\mathcal{Q}) & \xrightarrow{\sim} & \det R\Gamma(X,\mathcal{Q}_1)\otimes\det R\Gamma(X,\mathcal{Q}_2)
\end{array}$$

is commutative. Therefore the isomorphisms $c_{\pm i} : \mathrm{Pf}(\mathcal{Q})^{\otimes 2} \xrightarrow{\sim} \det R\Gamma(X,\mathcal{Q})$ are compatible with decompositions $\mathcal{Q}=\mathcal{Q}_1\oplus\mathcal{Q}_2$.

(iii) One can define $c_\pm : \mathrm{Pf}(\mathcal{Q})^{\otimes 2} \xrightarrow{\sim} \det R\Gamma(X,\mathcal{Q})$ by $c_\pm = i^{\pm p(\mathcal{Q})^2}c_i$ where $p(\mathcal{Q})^2$ is considered as an element of $\mathbb{Z}/4\mathbb{Z}$. Then $c_\pm$ does not change if i is replaced by $-i$. However $c_\pm$ do not seem to be naturalobjects, e.g., they are not compatible with decompositions $\mathcal{Q}=\mathcal{Q}_1\oplus\mathcal{Q}_2$ (the "error" is $(-1)^{p(\mathcal{Q}_1)p(\mathcal{Q}_2)}$).

(iv) The construction of $\mathrm{Pf}(\mathcal{Q})$ works if $\mathbb{C}$ is replaced by any field k such that $\operatorname{char} k \neq 2$. The case $\operatorname{char} k = 2$ is discussed in 4.2.16.

4.2.2. A *Lagrangian triple* consists of an even-dimensional vector space V equipped with a non-degenerate bilinear symmetric form $(\ ,\ )$ and Lagrangian (= maximal isotropic) subspaces $L_+, L_- \subset V$. If X and $\mathcal{Q}$ are as in 4.2.1 and $\mathcal{Q}' \subset \mathcal{Q}$ is a subsheaf such that $H^0(X, \mathcal{Q}') = 0$ and $S := \operatorname{Supp}(\mathcal{Q}/\mathcal{Q}')$ is finite then one associates to $(\mathcal{Q}, \mathcal{Q}')$ a Lagrangian triple $(V; L_+, L_-)$ as follows (cf. [Mu]):

(1) $V := H^0(X, \mathcal{Q}''/\mathcal{Q}')$ where $\mathcal{Q}'' := \underline{\mathrm{Hom}}(\mathcal{Q}', \omega_X) \supset \mathcal{Q}$;
(2) $L_+ := H^0(X, \mathcal{Q}/\mathcal{Q}') \subset V$;
(3) $L_- := H^0(X, \mathcal{Q}'') \subset V$;
(4) the bilinear form on V is induced by the natural pairing $\mathcal{Q}''/\mathcal{Q}' \otimes \mathcal{Q}''/\mathcal{Q}' \to (j_*\omega_{X\setminus S})/\omega_X$ and the "sum of residues" map $H^0(X, (j_*\omega_{X\setminus S})/\omega_X) \to \mathbb{C}$ where j is the embedding $X\setminus S \to X$. In this situation one can identify $R\Gamma(X, \mathcal{Q})$ with the complex

$$0 \to L_- \to V/L_+ \to 0 \tag{164}$$

concentrated in degrees 0 and 1. In particular $H^0(X, \mathcal{Q}) = L_+ \cap L_-$, $H^1(X, \mathcal{Q}) = V/(L_+ + L_-)$ and Serre's pairing between $H^0(X, \mathcal{Q}) = L_+ \cap L_-$ and $H^1(X, \mathcal{Q}) = V/(L_+ + L_-)$ is induced by the bilinear form on V.

4.2.3. For a Lagrangian triple $(V; L_+, L_-)$ set

$$\det(V; L_+, L_-) := \det L_+ \otimes \det L_- \otimes (\det V)^*\,. \tag{165}$$

$\det(V; L_+, L_-)$ is nothing but the determinant of the complex (164). Formula (165) defines a line bundle det on the stack of Lagrangian triples. In 4.2.4 and 4.2.8 we will construct a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle Pf on this stack and a canonical isomorphism $\mathrm{Pf} \otimes \mathrm{Pf}^- \xrightarrow{\sim} \det$ where $\mathrm{Pf}^-(V; L_+, L_-) := \mathrm{Pf}(V^-; L_+, L_-)$ and V^- denotes V equipped with the form $-(\ ,\ )$. The naive "definition" would be $\mathrm{Pf}^?(V;L_+,L_-) := \det(L_+\cap L_-)$ or $\mathrm{Pf}^?(V;L_+,L_-)^* := \det((L_+\cap L_-)^*) = \det(V/(L_+ + L_-))$ (cf. Remark (i) from 4.2.1).

4.2.4. For a Lagrangian triple $(V;L_+,L_-)$ define $\mathrm{Pf}(V;L_+,L_-)$ as follows. Denote by $\mathrm{Cl}(V)$ the Clifford algebra equipped with the canonical $(\mathbb{Z}/2\mathbb{Z})$-grading ($V\subset \mathrm{Cl}(V)$ is odd). Let M be an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathrm{Cl}(V)$-module (actually M is irreducible even without taking the grading into account). M is defined uniquely up to tensoring by a 1-dimensional $(\mathbb{Z}/2\mathbb{Z})$-graded vector space. Set $M_{L_-} = M/L_-M$, $M^{L_+} := \{m\in M | L_+m = 0\}$. Then M^{L_+} and M_{L_-} are 1-dimensional $(\mathbb{Z}/2\mathbb{Z})$-graded spaces. We set

$$\mathrm{Pf}(V;L_+,L_-) := M^{L_+}\otimes (M_{L_-})^* \,. \tag{166}$$

In particular we can take $M = \mathrm{Cl}(V)/\mathrm{Cl}(V)L_+$. Then $M^{L_+} = \mathbb{C}$, so

$$\mathrm{Pf}(V;L_+,L_-)^* = \mathrm{Cl}(V)/(L_-\cdot \mathrm{Cl}(V) + \mathrm{Cl}(V)\cdot L_+)\,. \tag{167}$$

Clearly (166) or (167) defines Pf as a $(\mathbb{Z}/2\mathbb{Z})$-graded line bundle on the stack of Lagrangian triples.[28] The grading corresponds to the action of $-1\in \mathrm{Aut}(V;L_+,L_-)$ on $\mathrm{Pf}(V;L_+,L_-)$.

If V is the orthogonal direct sum of V_1 and V_2 then $\mathrm{Cl}(V)$ is the tensor product of the superalgebras $\mathrm{Cl}(V_1)$ and $\mathrm{Cl}(V_2)$. Therefore if $(V^1;L^1_+,L^1_-)$ and $(V^2;L^2_+,L^2_-)$ are Lagrangian triples one has a canonical isomorphism

$$\mathrm{Pf}(V^1\oplus V^2; L^1_+\oplus L^2_+, L^1_-\oplus L^2_-) = \mathrm{Pf}(V^1;L^1_+,L^1_-)\otimes \mathrm{Pf}(V^2;L^2_+,L^2_-)\,. \tag{168}$$

where $\oplus$ denotes the orthogonal direct sum.

$\mathrm{Pf}(V;L_+,L_-)$ is even if and only if $\dim(L_+\cap L_-)$ is even. This follows from (168) and statement (i) of the following lemma.

[28] In other words, passing from individual Lagrangian triples to families is obvious. This principle holds for all our discussion of Pfaffians (only in the infinite-dimensional setting of 4.2.14 we explicitly consider families because this really needs some care).

4.2.5. *Lemma.*

(i) Any Lagrangian triple $(V; L_+, L_-)$ can be represented as an orthogonal direct sum of Lagrangian triples $(V^1; L^1_+, L^1_-)$ and $(V^2; L^2_+, L^2_-)$ such that $L^1_+ \cap L^1_- = 0$, $L^2_+ = L^2_-$.

(ii) Moreover, if a subspace $\Lambda \subset L_+$ is fixed such that $L_+ = \Lambda \oplus (L_+ \cap L_-)$ then one can choose the above decomposition $(V; L_+, L_-) = (V^1; L^1_+, L^1_-) \oplus (V^2; L^2_+, L^2_-)$ so that $L^1_+ = \Lambda$.

Proof

(i) Choose a subspace $P \subset V$ such that $V = (L_+ + L_-) \oplus P$. Then set $V^2 := (L_1 \cap L_2) \oplus P$, $V^1 := (V^2)^\perp$.

(ii) Choose a subspace $P \subset \Lambda^\perp$ such that $\Lambda^\perp = L_+ \oplus P$ (this implies that $V = (L_+ + L_-) \oplus P$ because $\Lambda^\perp / L_+ \to V/(L_+ + L_-)$ is an isomorphism). Then proceed as above. $\square$

4.2.6. In this subsection (which can be skipped by the reader) we construct a canonical isomorphism between $\mathrm{Pf}(V; L_+, L_-)$ and the naive $\mathrm{Pf}^?(V; L_+, L_-)$ from 4.2.3. Recall that $\mathrm{Pf}^?(V; L_+, L_-) := \det(L_+ \cap L_-)$, so $\mathrm{Pf}^?(V; L_+, L_-)^* = \det((L_+ \cap L_-)^*) = \det(V/(L_+ + L_-))$, it being understood that the pairing $\det W \otimes \det W^* \to \mathbb{C}$, $W := L_+ \cap L_-$, is defined by $(e_1 \wedge \ldots \wedge e_k) \otimes (e^k \wedge \ldots \wedge e^1) \mapsto 1$ where $e_1, \ldots, e_k$ is a base of W and $e^1, \ldots, e^k$ is the dual base of W^* (this pairing is reasonable from the "super" point of view; e.g., it is compatible with decompositions $W = W_1 \oplus W_2$).

To define the isomorphism $\mathrm{Pf}(V; L_+, L_-) \xrightarrow{\sim} \mathrm{Pf}^?(V; L_+, L_-)$ we use the canonical filtration on $\mathrm{Cl}(V)$ defined by

$$\mathrm{Cl}_0(V) = \mathbb{C}, \quad \mathrm{Cl}_{k+1}(V) = \mathrm{Cl}_k(V) + V \cdot \mathrm{Cl}_k(V)\,. \tag{169}$$

We have $\mathrm{Cl}_k(V)/\mathrm{Cl}_{k-1}(V) = \bigwedge^k V$. Set $r := \dim(L_+ \cap L_-)$. One has the canonical epimorphism $\varphi : \mathrm{Cl}_r(V) \to \bigwedge^r V \to \bigwedge^r(V/(L_+ + L_-)) = \det(V/(L_+ + L_-)) = \mathrm{Pf}^?(V; L_+, L_-)^*$. It is easy to deduce from 4.2.5(i) that the canonical mapping $\mathrm{Cl}_r(V) \to \mathrm{Cl}(V)/(L_- \cdot \mathrm{Cl}(V) + \mathrm{Cl}(V) \cdot L_+) = \mathrm{Pf}(V; L_+, L_-)^*$ factors through φ and the induced map

$f : \mathrm{Pf}^?(V; L_+, L_-)^* \to \mathrm{Pf}(V; L_+, L_-)^*$ is an isomorphism. f^* is the desired isomorphism $\mathrm{Pf}(V; L_+, L_-) \xrightarrow{\sim} \mathrm{Pf}^?(V; L_+, L_-)$.

Here is an equivalent definition. Let M be an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded $\mathrm{Cl}(V)$-module. The canonical embedding $\det(L_+ \cap L_-) \subset \bigwedge^*(L_+ \cap L_-) = \mathrm{Cl}(L_+ \cap L_-) \subset \mathrm{Cl}(V)$ induces a map $\det(L_+ \cap L_-) \otimes M_{L_+ \cap L_-} \to M^{L_+ \cap L_-}$, which is actually an isomorphism. It is easy to deduce from 4.2.5(i) that the composition $M^{L_+} \to M^{L_+ \cap L_-} \xrightarrow{\sim} \det(L_+ \cap L_-) \otimes M_{L_+ \cap L_-} \to \det(L_+ \cap L_-) \otimes M_{L_-}$ is an isomorphism. It induces an isomorphism $\mathrm{Pf}(V; L_+, L_-) := M^{L_+} \otimes (M_{L_-})^{\otimes -1} \to \det(L_+ \cap L_-) = \mathrm{Pf}^?(V; L_+, L_-)$, which is actually inverse to the one constructed above.

4.2.7. Before constructing the isomorphism $\mathrm{Pf} \otimes \mathrm{Pf}^- \xrightarrow{\sim} \det$ we will construct a canonical isomorphism

$$\mathrm{Pf}(V \oplus V^*; L_+ \oplus L_+^\perp, L_- \oplus L_-^\perp) \xrightarrow{\sim} \det(V; L_+, L_-) \tag{170}$$

where V is a finite dimensional vector space without any bilinear form on it, $L_\pm \subset V$ are arbitrary subspaces and $V \oplus V^*$ is equipped with the obvious bilinear form (the l.h.s. of (170) makes sense because $L_\pm \oplus L_\pm^\perp$ is Lagrangian, the r.h.s. of (170) is defined by (165)). Set

$$M = \bigwedge V \otimes (\det L_+)^*\,, \quad \bigwedge V := \bigoplus_i \bigwedge^i V\,. \tag{171}$$

M is the irreducible $\mathrm{Cl}(V \oplus V^*)$-module with $M^{L_+ \oplus L_+^\perp} = \mathbb{C}$, so according to (166) $\mathrm{Pf}(V \oplus V^*; L_+ \oplus L_+^\perp, L_- \oplus L_-^\perp) = (M_{L_- \oplus L_-^\perp})^*$. Clearly $M_{L_-} = \bigwedge(V/L_-) \otimes (\det L_+)^*$ and $M_{L_- \oplus L_-^\perp} = \det(V/L_-) \otimes (\det L_+)^* = \det(V; L_+, L_-)^*$ (see (165)). So we have constructed the isomorphism (170).

4.2.8. Now let $(V; L_+, L_-)$ be a Lagrangian triple. We will construct a canonical isomorphism

$$\mathrm{Pf}(V; L_+, L_-) \otimes \mathrm{Pf}(V^-; L_+, L_-) \xrightarrow{\sim} \det(V; L_+, L_-) \tag{172}$$

where V^- denotes V equipped with the bilinear form $-(\ ,\ )$. If W is a finite dimensional vector space equipped with a nondegenerate symmetric bilinear form then $(V\otimes W; L_+\otimes W, L_-\otimes W)$ is a Lagrangian triple. (170) can be rewritten as a canonical isomorphism.

$$\det(V;L_+,L_-)\xrightarrow{\sim}\mathrm{Pf}(V\otimes H;L_+\otimes H,L_-\otimes H) \tag{173}$$

where H denotes $\mathbb{C}^2$ equipped with the bilinear form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. On the other hand (168) yields an isomorphism

$$\mathrm{Pf}(V;L_+,L_-)\otimes\mathrm{Pf}(V^-;L_+,L_-)\xrightarrow{\sim}\mathrm{Pf}(V\otimes H';L_+\otimes H',L_-\otimes H') \tag{174}$$

where H' denotes $\mathbb{C}^2$ equipped with the bilinear form $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. So an isomorphism $\varphi:\ H'\xrightarrow{\sim}H$ induces an isomorphism

$$\varphi_*:\ \mathrm{Pf}(V;L_+,L_-)\otimes\mathrm{Pf}(V^-;L_+,L_-)\xrightarrow{\sim}\det(V;L_+,L_-)\,.$$

Lemma. If $\psi\in\mathrm{Aut}\,H'$ then

$$(\varphi\psi)_*=(\det\psi)^n\varphi_*\,,\quad n=\dim(L_+\cap L_-)\,. \tag{175}$$

Proof. $\mathrm{Aut}\,H'$ acts on the r.h.s. of (174) by some character $\chi:\ \mathrm{Aut}\,H'\to\mathbb{C}^*$. Any character of $\mathrm{Aut}\,H'$ is of the form $\psi\mapsto(\det\psi)^m$, $m\in\mathbb{Z}/2\mathbb{Z}$. $\chi\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)=(-1)^n$, $n:=\dim(L_+\cap L_-)$, because $-1\in\mathrm{Aut}(V;L_+,L_-)$ acts on $\mathrm{Pf}(V;L_+,L_-)$ as $(-1)^n$ (see 4.2.4). So $m=n \mod 2$. $\square$

We define (172) to be φ_* for any $\varphi:\ H'\xrightarrow{\sim}H$ such that $\det\varphi=1$.

Remarks

(i) (172) is compatible with decompositions of $(V;L_+,L_-)$ into orthogonal direct sums; i.e., if one has such a decomposition $(V;L_+,L_-)=(V^1;L^1_+,L^1_-)\oplus(V^2;L^2_+,L^2_-)$ then the isomorphisms (172) for $(V;L_+,L_-)$, $(V^1;L^1_+,L^1_-)$, and $(V^2;L^2_+,L^2_-)$ are compatible with (168) and the canonical isomorphism $\det(V;L_+,L_-)=\det(V^1;L^1_+,L^1_-)\otimes\det(V^2;L^2_+,L^2_-)$.

(ii) (170) is compatible with decompositions of $(V;L_+,L_-)$ into direct sums.

4.2.9. In this subsection (which can be skipped by the reader) we give an equivalent construction of (172). We will use the superalgebra anti-isomorphism $* : \mathrm{Cl}(V^-) \xrightarrow{\sim} \mathrm{Cl}(V)$ identical on V (for any $v_1, \dots, v_k \in V$ one has $(v_1 \dots v_k)^* = (-1)^{k(k-1)/2} v_k \dots v_1$). We also use the canonical map $\mathrm{sTr} : \mathrm{Cl}(V) = \mathrm{Cl}_n(V) \to \mathrm{Cl}_n(V)/\mathrm{Cl}_{n-1}(V) = \det V$ where $n = \dim V$ and $\mathrm{Cl}_k(V)$ is defined by (169). It has the "supertrace property"

$$\mathrm{sTr}(ab) = (-1)^{p(a)p(b)} \,\mathrm{sTr}(ba) \tag{176}$$

where $a, b \in \mathrm{Cl}(V)$ are homogeneous of degrees $p(a), p(b) \in \mathbb{Z}/2\mathbb{Z}$. Indeed, it is enough to prove (176) in the case $a \in V$, $p(ab) = n \mod 2$; then $b \in \mathrm{Cl}_{n-1}(V)$ and (176) is obvious. Or one can check that $\mathrm{sTr}(a)$ coincides up to a sign with the supertrace of the operator $a : M \to M$ where M is an irreducible $\mathrm{Cl}(V)$-module.

Now consider the map

$$\det L_- \otimes \mathrm{Pf}(V; L_+, L_-)^* \otimes \det L_+ \otimes \mathrm{Pf}(V^-; L_+, L_-)^* \to \det V \tag{177}$$

defined by $a_- \otimes x \otimes a_+ \otimes y \mapsto \mathrm{sTr}(a_- x a_+ y^*)$. Here $a_\pm \in \det L_\pm \subset \Lambda^*(L_\pm) = \mathrm{Cl}(L_\pm) \subset \mathrm{Cl}(V)$, $x \in \mathrm{Pf}(V; L_+, L_-)^* = \mathrm{Cl}(V)/(L_- \cdot \mathrm{Cl}(V) + \mathrm{Cl}(V) \cdot L_+)$, $y^* \in \mathrm{Cl}(V)/(L_+ \cdot \mathrm{Cl}(V) + \mathrm{Cl}(V) \cdot L_-)$, so (177) is well-defined. It is easy to see (e.g., from 4.2.5 (i)) that (177) is an isomorphism. It induces an isomorphism

$$\mathrm{Pf}(V; L_+, L_-) \otimes \mathrm{Pf}(V^-; L_+, L_-) \xrightarrow{\sim} \det L_+ \otimes \det L_- \otimes (\det V)^* = \det(V; L_+, L_-)$$

One can show that this isomorphism equals (172).

4.2.10. Let X and $\mathcal{Q}$ be as in 4.2.1 and $\mathcal{Q}' \subset \mathcal{Q}$ as in 4.2.2. To these data we have associated a Lagrangian triple $(V; L_+, L_-)$ such that $\det(V; L_+, L_-) = \det R\Gamma(X, \mathcal{Q})$ (see 4.2.2). Set $\mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}) := \mathrm{Pf}(V; L_+, L_-)$. According to 4.2.9 we have a canonical isomorphism $\mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}) \otimes \mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}^-) \xrightarrow{\sim} \det R\Gamma(X, \mathcal{Q})$. To define $\mathrm{Pf}(\mathcal{Q})$ it is enough to define a compatible system of isomorphisms $\mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}) \xrightarrow{\sim} \mathrm{Pf}_{\widetilde{\mathcal{Q}}'}(\mathcal{Q})$ for all pairs $(\mathcal{Q}', \widetilde{\mathcal{Q}}')$ such that $\mathcal{Q}' \subset \widetilde{\mathcal{Q}}'$. To define

$\mathrm{Pf}(\mathcal{Q}) \otimes \mathrm{Pf}(\mathcal{Q}^-) \xrightarrow{\sim} \det R\Gamma(X, \mathcal{Q})$ it suffices to prove the commutativity of

$$\begin{array}{ccc} \mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}) \otimes \mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q}^-) & \xrightarrow{\sim} & \det R\Gamma(X, \mathcal{Q}) \\ \downarrow\wr & & \\ \mathrm{Pf}_{\widetilde{\mathcal{Q}}'}(\mathcal{Q}) \otimes \mathrm{Pf}_{\widetilde{\mathcal{Q}}'}(\mathcal{Q}^-) & \xrightarrow{\sim} & \end{array}$$

The Lagrangian triple $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ corresponding to $\widetilde{\mathcal{Q}}'$ is related to the triple $(V; L_+, L_-)$ corresponding to $\mathcal{Q}'$ as follows: if $\Lambda = H^0(X, \widetilde{\mathcal{Q}}'/\mathcal{Q}') \subset H^0(X, \mathcal{Q}/\mathcal{Q}') = L_+$ then

$$\widetilde{V} = \Lambda^\perp/\Lambda, \quad \widetilde{L}_+ = L_+/\Lambda \subset \widetilde{V}, \quad \widetilde{L}_- = L_- \cap \Lambda^\perp \hookrightarrow \widetilde{V} \tag{178}$$

(notice that $\Lambda \cap L_- = H^0(X, \widetilde{\mathcal{Q}}') = 0$). So it remains to do some linear algebra (see 4.2.11). It is easy to check that our definition of $\mathrm{Pf}(\mathcal{Q})$ and $\mathrm{Pf}(\mathcal{Q}) \otimes \mathrm{Pf}(\mathcal{Q}^-) \xrightarrow{\sim} \det R\Gamma(X, \mathcal{Q})$ makes sense for families of $\mathcal{Q}$'s.

4.2.11. Let $(V; L_+, L_-)$ be a Lagrangian triple, $\Lambda \subset L_+$ a subspace such that $\Lambda \cap L_- = 0$. Then $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ defined by (178) is a Lagrangian triple. In this situation we will say that $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ is a *subquotient* of $(V; L_+, L_-)$. It is easy to show that a subquotient of a subquotient is again a subquotient. So we can consider the category T with Lagrangian triples as objects such that a morphism from $(V; L_+, L_-)$ to $(V'; L'_+, L'_-)$ is defined to be an isomorphism between $(V; L_+, L_-)$ and a subquotient of $(V'; L'_+, L'_-)$. Consider also the category C whose objects are finite complexes of finite dimensional vector spaces and morphisms are quasi-isomorphisms. Denote by $\mathbb{I}$ the category whose objects are $(\mathbb{Z}/2\mathbb{Z})$-graded 1-dimensional vector spaces and morphisms are isomorphisms preserving the grading. The complex (164) considered as an object of C depends functorially on $(V; L_+, L_-) \in T$: if $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ is the subquotient of $(V; L_+, L_-)$ corresponding to $\Lambda \subset L_+$ then we have the quasi-isomorphism

$$\begin{array}{ccc} L_- & \longrightarrow & V/L_+ \\ \hookleftarrow & & \hookleftarrow \\ \widetilde{L}_- & \longrightarrow & \widetilde{V}/\widetilde{L}_+ = \Lambda^\perp/L_+ \end{array}$$

Applying the functor $\det : C \to \mathbb{I}$ from [Kn-Mu] we see that $\det(V; L_+, L_-) \in \mathbb{I}$ depends functorially on $(V; L_+, L_-) \in T$. If $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ is the subquotient of $(V; L_+, L_-)$ corresponding to $\Lambda \subset L_+$ then the isomorphism between $\det(V; L_+, L_-) = (\det L_+) \otimes (\det L_-) \otimes (\det V)^*$ and $\det(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-) = (\det \widetilde{L}_+) \otimes (\det \widetilde{L}_-) \otimes (\det \widetilde{V})^*$ comes from the natural isomorphisms $\det L_+ = \det \Lambda \otimes \det \widetilde{L}_+$, $\det L_- = \det \widetilde{L}_- \otimes \det(V/\Lambda^\perp)$, $\det V = \det \Lambda \otimes \det \widetilde{V} \otimes \det(V/\Lambda^\perp)$.

As explained in 4.2.10 we have to define Pf as a functor $T \to \mathbb{I}$ and to show that the isomorphism $\mathrm{Pf}(V; L_+, L_-) \otimes \mathrm{Pf}^-(V; L_+, L_-) \xrightarrow{\sim} \det(V; L_+, L_-)$ from 4.2.8 is functorial.

If $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ is the subquotient of $(V; L_+, L_-)$ corresponding to $\Lambda \subset L_+$ then

$$\begin{aligned} \mathrm{Pf}(V; L_+, L_-)^* &= \mathrm{Cl}(V)/(L_- \cdot \mathrm{Cl}(V) + \mathrm{Cl}(V) \cdot L_+) \\ \mathrm{Pf}(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)^* &= \mathrm{Cl}(\Lambda^\perp)/((L_- \cap \Lambda^\perp) \cdot \mathrm{Cl}(\Lambda^\perp) + \mathrm{Cl}(\Lambda^\perp) \cdot L_+) \,. \end{aligned}$$

So the embedding $\mathrm{Cl}(\Lambda^\perp) \to \mathrm{Cl}(V)$ induces a mapping

$$\mathrm{Pf}(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)^* \to \mathrm{Pf}(V; L_+, L_-)^* \,. \tag{179}$$

This defines Pf^* as a functor $T \to \{(\mathbb{Z}/2\mathbb{Z})\text{-graded 1-dimensional spaces}\}$ (it is easy to see that composition corresponds to composition). It remains to show that

a) (179) is an isomorphism,

b) (179) is compatible with the pairings $\mathrm{Pf}(V; L_+, L_-)^* \otimes \mathrm{Pf}(V^-; L_+, L_-)^* \xrightarrow{\sim} \det(V; L_+, L_-)^*$ and $\mathrm{Pf}(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)^* \otimes \mathrm{Pf}(\widetilde{V}^-; L_+, L_-)^* \xrightarrow{\sim} \det(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)^*$ from 4.2.8.

b) can be checked directly and a) follows from b). One can also prove a) by reducing to the case where $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ is a *maximal* subquotient, (i.e., $\Lambda \oplus (L_+ \cap L_-) = L_+$) and then using 4.2.5 (ii).

4.2.12. Let E be a vector bundle on X. Then $E \oplus (E^* \otimes \omega_X)$ has the obvious structure of ω-orthogonal bundle. We will construct a canonical isomorphism

$$\mathrm{Pf}(E \oplus (E^* \otimes \omega_X)) \xrightarrow{\sim} \det R\Gamma(X, E)\,. \tag{180}$$

Choose a subsheaf $E' \subset E$ and a locally free sheaf $E'' \supset E$ so that $H^0(X, E') = 0$, $H^1(X, E'') = 0$, and E''/E' has finite support. Set $V := H^0(X, E''/E')$, $L_+ := H^0(X, E'/E) \subset V$, $L_- := H^0(X, E'') \subset V$. Then $R\Gamma(X, E)$ can be identified with the complex $0 \to L_- \to V/L_+ \to 0$ and $\det R\Gamma(X, E)$ with $\det(V; L_+, L_-)$. On the other hand the Pfaffian of $\mathcal{Q} := E \oplus (E^* \otimes \omega_X)$ can be computed using the subsheaf $\mathcal{Q}' := E' \oplus ((E'')^* \otimes \omega_X) \subset \mathcal{Q}$. Then $\mathrm{Pf}_{\mathcal{Q}'}(\mathcal{Q})$ equals the l.h.s. of (170). So (170) yields the isomorphism (180). One checks that (180) does not depend on E' and E''.

4.2.13. The notion of Lagrangian triple has a useful infinite dimensional generalization. First let us recall some basic definitions.

Definition. A *Tate space* is a complete topological vector space having a base of neighbourhoods of 0 consisting of commensurable vector subspaces (i.e., $\dim U_1/(U_1 \cap U_2) < \infty$ for any U_1, U_2 from this base).

Remark. Tate spaces are implicit in his remarkable work [T]. In fact, the approach to residues on curves developed in [T] can be most naturally interpreted in terms of the canonical central extension of the endomorphism algebra of a Tate space, which is also implicit in [T]. A construction of the Tate extension can be found in 7.13.18.

Let V be a Tate space. A vector subspace $P \subset V$ is *bounded* if for every open subspace $U \subset V$ there exists a finite set $\{v_1, \dots, v_n\} \subset V$ such that $P \subset U + \mathbb{C}v_1 + \dots \mathbb{C}v_n$. The *topological dual* of V is the space V^* of continuous linear functionals on V equipped with the (linear) topology such that orthogonal complements of bounded subspaces of V form a base of neighbourhoods of $0 \in V^*$. Clearly V^* is a Tate space and the canonical morphism $V \to (V^*)^*$ is an isomorphism.

Example (coordinate Tate space). Let I be a set. We say that $A, B \subset I$ are commensurable if $A \backslash (A \cap B)$ and $B \backslash (B \cap A)$ are finite. Commensurability is an equivalence relation. Suppose that an equivalence class $\mathcal{A}$ of subsets $A \subset I$ is fixed. Elements of $\mathcal{A}$ are called *semi-infinite* subsets. Denote by $\mathbb{C}((I, \mathcal{A}))$ the space of formal linear combinations $\sum\limits_i c_i e_i$ where $c_i \in \mathbb{C}$ vanish when $i \notin A$ for some semi-infinite A. This is a Tate vector space (the topology is defined by subspaces $\mathbb{C}[[A]] := \{\sum\limits_{i \in A} c_i e_i\}$ where A is semi-infinite). The space dual to $\mathbb{C}((I, \mathcal{A}))$ is $\mathbb{C}((I, \mathcal{A}'))$ where $\mathcal{A}'$ consists of complements to subsets from $\mathcal{A}$. Any Tate vector space is isomorphic to $\mathbb{C}((I, \mathcal{A}))$ for appropriate I and $\mathcal{A}$; such an isomorphism is given by the corresponding subset $\{e_i\} \subset V$ called *topological basis* of V.

A *c-lattice* in V is an open bounded subspace. A *d-lattice**) in V is a discrete subspace $\Gamma \subset V$ such that $\Gamma + P = V$ for some c-lattice $P \subset V$. If $W \subset V$ is a d-lattice (resp. c-lattice) then there is a c-lattice (resp. d-lattice) $W' \subset V$ such that $V = W \oplus W'$. If $W \subset V$ is a d-lattice (resp. c-lattice) then $W^\perp \subset V^*$ is also a d-lattice (resp. c-lattice) and $(W^\perp)^\perp = W$.

A (continuous) bilinear form on a Tate space V is said to be *nondegenerate* if it induces a topological isomorphism $V \to V^*$. Let V be a Tate space equipped with a nondegenerate symmetric bilinear form. A subspace $L \subset V$ is *Lagrangian* if $L^\perp = L$.

Definition. A Tate Lagrangian triple consists of a Tate space V equipped with a nondegenerate symmetric bilinear form, a Lagrangian c-lattice $L_+ \subset V$, and a Lagrangian d-lattice $L_- \subset V$.

Example. Let $\mathcal{Q}$ be an ω-orthogonal bundle on X. If $x \in X$ let $\mathcal{Q} \otimes O_x$ (resp. $\mathcal{Q} \otimes K_x$) denote the space of global sections of the pullback of $\mathcal{Q}$ to $\operatorname{Spec} O_x$ (resp. $\operatorname{Spec} K_x$). $\mathcal{Q} \otimes K_x$ is a Tate space equipped with the nondegenerate symmetric bilinear form $\operatorname{Res}(\ ,\ )$. For every non-empty finite

*) c and d are the first letters of "compact" and "discrete".

$S \subset X$ we have the Tate Lagrangian triple

$$(181) \qquad V := \underset{x\in S}{\oplus}(\mathcal{Q}\otimes K_x)\,, \quad L_+ := \underset{x\in S}{\oplus}(\mathcal{Q}\otimes O_x)\,, \quad L_- := \Gamma(X\backslash S, \mathcal{Q})\,.$$

Let $(V; L_+, L_-)$ be a Tate Lagrangian triple. Then for any c-lattice $\Lambda \subset L_+$ such that $\Lambda \cap L_- = 0$ one has the finite-dimensional Lagrangian triple $(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ defined by (178). As explained in 4.2.11 $\mathrm{Pf}(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ and $\det(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$ do not depend on Λ. Set $\mathrm{Pf}(V; L_+, L_-) := \mathrm{Pf}(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$, $\det(V; L_+, L_-) := \det(\widetilde{V}; \widetilde{L}_+, \widetilde{L}_-)$. Equivalently one can define $\det(V; L_+, L_-)$ to be the determinant of the complex (164) and $\mathrm{Pf}(V; L_+, L_-)$ can be defined by (166) or (167) (the $\mathrm{Cl}(V)$-module M from (166) should be assumed *discrete*, which means that $\{v \in V | vm = 0\}$ is open for every $m \in M$).

Example. If $(V; L_+, L_-)$ is defined by (181) then $\mathrm{Pf}(V; L_+, L_-) = \mathrm{Pf}(\mathcal{Q})$, $\det(V; L_+, L_-) = \det R\Gamma(X, \mathcal{Q})$.

The constructions from 4.2.7 and 4.2.8 make sense in the Tate situation with the following obvious changes: a) in 4.2.7 one should suppose that L_+ is a c-lattice and L_- is a d-lattice, b) (171) should be replaced by the following formula:

$$(182) \qquad M = \varinjlim_{U} \bigwedge(V/U) \otimes \det(L_+/U)^*$$

where U belongs to the set of c-lattices in L_+. The r.h.s. of (182) is the *fermionic Fock space*, i.e., the direct sum of *semi-infinite powers* of V (cf. Lecture 4 from [KR] and references therein).

Remark. The expression for $\mathrm{Pf}(\mathcal{Q})$ in terms of the triple (181) can be reformulated as follows. For $x \in X$ consider the abelian Lie superalgebras $\mathfrak{a}_{O_x} \subset \mathfrak{a}_{K_x}$ such that the odd component of $\mathfrak{a}_{O_x}$ (resp. $\mathfrak{a}_{K_x}$) is $\mathcal{Q}\otimes O_x$ (resp. $\mathcal{Q}\otimes K_x$) and the even components are 0. The bilinear symmetric form on $\mathcal{Q}\otimes K_x$ defines a central extension $0 \to \mathbb{C} \to \tilde{\mathfrak{a}}_{K_x} \to \mathfrak{a}_{K_x} \to 0$ with a canonical splitting over $\mathfrak{a}_{O_x}$. The Clifford algebra $\mathrm{Cl}(\mathcal{Q}\otimes K_x)$ is the twisted universal enveloping algebra $U'\mathfrak{a}_{K_x}$ and $M_x := \mathrm{Cl}(\mathcal{Q}\otimes K_x)/\,\mathrm{Cl}(\mathcal{Q}\otimes K_x)\,\cdot$

$(\mathcal{Q}\otimes O_x)$ is the vacuum module over $U'\mathfrak{a}_{K_x}$. According to (167) $\mathrm{Pf}(\mathcal{Q})^*$ *is the space of coinvariants of the action of* $\Gamma(X\backslash S,\mathcal{Q})$ *on* $\underset{x\in S}{\otimes} M_x$.

4.2.14. In this subsection we discuss *families* of Tate Lagrangian triples. Let R be a commutative ring. We define a *Tate R-module* to be a topological R-module isomorphic to $P\oplus Q^*$ where P and Q are (infinite) direct sums of finitely generated projective R-modules (a base of neighbourhoods of $0\in P\oplus Q^*$ is formed by $M^\perp\subset Q^*$ for all possible finitely generated submodules $M\subset Q$). This bad*) definition is enough for our purposes. In fact, we mostly work with Tate R-modules isomorphic to $V_0\widehat{\otimes}R$ where V_0 is a Tate space.

The discussion of Tate linear algebra from 4.2.13 remains valid for Tate R-modules if one defines the notions of c-lattice and d-lattice as follows.

Definition. A *c-lattice* in a Tate R-module V is an open bounded submodule $P\subset V$ such that V/P is projective. A *d-lattice* in V is a submodule $\Gamma\subset V$ such that for some c-lattice $P\subset V$ one has $\Gamma\cap P=0$ and $V/(\Gamma+P)$ is a projective module of finite type.*)

Now if $\frac{1}{2}\in R$ we can define the notion of Tate Lagrangian triple just as in 4.2.13 (of course, if $\frac{1}{2}\notin R$ one should work with quadratic forms instead of bilinear ones, which is easy). The Pfaffian of a Tate Lagrangian triple $(V;L_+,L_-)$ over R is defined as in 4.2.13 with the following minor change: to pass to the finite-dimensional Lagrangian triple $(\widetilde{V};\widetilde{L}_+,\widetilde{L}_-)$ defined by (178) one has to assume that $\Lambda\subset L_+$ is a c-lattice such that $\Lambda\cap L_-=0$ and $V/(\Lambda+L_-)$ is projective (these two properties are equivalent to the following one: $\Lambda^\perp+L_-=V$).

Example. Let $D\subset X\otimes R$ be a closed subscheme finite over $\operatorname{Spec}R$ that can be locally defined by one equation (i.e., D is an effective relative Cartier

*) A projective $R((t))$-module of finite rank is not necessarily a Tate module in the above sense. Our notion of Tate R-module is not local with respect to $\operatorname{Spec}R$. There are also other drawbacks.

*) Then this holds for all c-lattices $P'\subset P$.

divisor). Let $\mathcal{Q}$ be a vector bundle on $X \otimes R$. Suppose that the morphism $D \to \operatorname{Spec} R$ is surjective. Then

$$V := \varprojlim_m \varinjlim_n H^0(X \otimes R, \mathcal{Q}(nD)/\mathcal{Q}(-mD))$$

is a Tate R-module*) ,

$$L_+ := \varprojlim_m H^0(X \otimes R, \mathcal{Q}/\mathcal{Q}(-mD)) \subset V$$

is a c-lattice, and

$$L_- := H^0((X \otimes R) \setminus D, \mathcal{Q}) \subset V$$

is a d-lattice. If $\mathcal{Q}$ is an ω-orthogonal bundle then $(V; L_+, L_-)$ is a Lagrangian triple and $\operatorname{Pf}(\mathcal{Q}) = \operatorname{Pf}(V; L_+, L_-)$ (cf. 4.2.13).

4.2.15. Denote by $\mathcal{B}$ the groupoid of finite dimensional vector spaces over $\mathbb{C}$ equipped with a nondegenerate symmetric bilinear form. In this subsection (which can be skipped by the reader) we construct canonical isomorphisms

(183)
$$\operatorname{Pf}(V \otimes W; L_+ \otimes W, L_- \otimes W) \xrightarrow{\sim} \operatorname{Pf}(V; L_+, L_-)^{\otimes \dim W} \otimes |\det W|^{\otimes p(V;L_+,L_-)},$$

(184)
$$\operatorname{Pf}(\mathcal{Q} \otimes W) \xrightarrow{\sim} \operatorname{Pf}(\mathcal{Q})^{\otimes \dim W} \otimes |\det W|^{\otimes p(\mathcal{Q})}$$

where $W \in \mathcal{B}$, $(V; L_+, L_-)$ is a (Tate) Lagrangian triple, $\mathcal{Q}$ is an ω-orthogonal bundle on X, $|\det W|$ is the determinant of W considered as a space (not super-space!), and $p(V; L_+, L_-)$, $p(\mathcal{Q}) \in \mathbb{Z}/2\mathbb{Z}$ are the parities of $\operatorname{Pf}(V; L_+, L_-)$, $\operatorname{Pf}(\mathcal{Q})$. $|\det W|^{\otimes n}$ makes sense for $n \in \mathbb{Z}/2\mathbb{Z}$ because one has the canonical isomorphism $|\det W|^{\otimes 2} \xrightarrow{\sim} \mathbb{C}$, $(w_1 \wedge \ldots \wedge w_r)^{\otimes 2} \mapsto \det(w_i, w_j)$.

*) In fact, V is isomorphic to $V_0 \widehat{\otimes} R$ for some Tate space V_0 over $\mathbb{C}$. Indeed, we can assume that R is finitely generated over $\mathbb{C}$ and then apply 7.12.11. We need 7.12.11 in the case where R is finitely generated over $\mathbb{C}$ and the projective module from 7.12.11 is a direct sum of finitely generated modules; in this case 7.12.11 follows from Serre's theorem (Theorem 1 of [Se]; see also [Ba68], ch.4, §2) and Eilenberg's lemma [Ba63].

To define (183) and(184) notice that $\mathcal{B}$ is a tensor category with $\oplus$ as a tensor "product" and both sides of (183) and (184) are tensor functors from $\mathcal{B}$ to the category of 1-dimensional superspaces (to define the r.h.s. of (184) as a tensor functor rewrite it as $|\operatorname{Pf}(\mathcal{Q})|^{\otimes \dim W} \otimes (\det W)^{\otimes p(\mathcal{Q})}$ where $|\operatorname{Pf}(\mathcal{Q})|$ is obtained from $\operatorname{Pf}(\mathcal{Q})$ by changing the $(\mathbb{Z}/2\mathbb{Z})$-grading to make it even and $\det W$ is the determinant of W considered as a *super*space).

We claim that *there is a unique way to define* (183) *and* (184) *as isomorphisms of tensor functors so that for* $W = (\mathbb{C}, 1)$ (183) *and* (184) *equal* id. Here 1 denotes the bilinear form $(x, y) \mapsto xy$, $x, y \in \mathbb{C}$.

To prove this apply the following lemma to the tensor functor F obtained by dividing the l.h.s. of (183) or (184) by the r.h.s.

Lemma. Every tensor functor $F : \mathcal{B} \to \{\text{1-dimensional vector spaces}\}$ is isomorphic to the tensor functor F_1 defined by $F_1(W) = L^{\otimes \dim W}$, $L := F(\mathbb{C}, 1)$. There is a unique isomorphism $F \xrightarrow{\sim} F_1$ that induces the identity map $F(\mathbb{C}, 1) \to F_1(\mathbb{C}, 1)$.

Proof. For every $W \in \mathcal{B}$ the functor F induces a homomorphism $f_W : \operatorname{Aut} W \to \mathbb{C}^*$. Since $\operatorname{Aut} W$ is an orthogonal group $f_W(g) = (\det g)^{n(W)}$ for some $n(W) \in \mathbb{Z}/2\mathbb{Z}$. Clearly $n(W) = n$ does not depend on W. Set $W_1 := (\mathbb{C}, 1)$. F maps the commutativity isomorphism $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) : W_1 \oplus W_1 \to W_1 \oplus W_1$ to id. So $n = 0$, i.e., f_W is trivial for every W. The rest is clear because the semigroup $|\mathcal{B}|$ of isomorphism classes of objects of $\mathcal{B}$ is $\mathbb{Z}_+$. $\square$

Remarks

(i) (183) was implicitly used in 4.2.8.

(ii) We will use (183) in 4.2.16.

4.2.16. In this subsection (which can certainly be skipped by the reader) we explain what happens if $\mathbb{C}$ is replaced by a field k of characteristic 2. In this case one must distinguish between quadratic forms (see [Bourb59], §3, n°4) and symmetric bilinear forms. In the definition of Lagrangian triple V should be equipped with a nondegenerate *quadratic* form. So in the definition of

ω-orthogonal bundle $\mathcal{Q}$ should be equipped with a nondegenerate *quadratic* form $\mathcal{Q} \to \omega_X$ (since k has characteristic 2 nondegeneracy implies that the rank of $\mathcal{Q}$ is even). The construction of $\mathrm{Pf} \otimes \mathrm{Pf}^- \xrightarrow{\sim} \det$ from 4.2.8 has to be modified. If $(V; L_+, L_-)$ is a Lagrangian triple and W is equipped with a nondegenerate symmetric *bilinear* form then $(V \otimes W; L_+ \otimes W, L_- \otimes W)$ is a Lagrangian triple. The bilinear forms $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are not equivalent in characteristic 2, but one can use (183) for $W = H$ and $W = H'$ to construct $\mathrm{Pf} \otimes \mathrm{Pf}^- \xrightarrow{\sim} \det$. Finally we have to construct (183) and (184) in characteristic 2. Let us assume for simplicity that k is perfect. Then the characteristic property [*)] of the isomorphisms (183) and (184) is formulated just as in 4.2.15, but the proof of their existence and uniqueness should be modified. The semigroup $|\mathcal{B}|$ (see the end of the proof of the lemma from 4.2.15) is no longer $\mathbb{Z}_+$; it has generators a and b with the defining relation $a + b = 3a$ (a corresponds to the matrix (1) of order 1 and b corresponds to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). So the group corresponding to $\mathcal{B}$ is $\mathbb{Z}$, which is enough.

4.3. **Pfaffians II.**

4.3.1. Fix an n-dimensional vector space W over $\mathbb{C}$ and a nondegenerate symmetric bilinear form ( ) on it. To simplify notation we write O_n and SO_n instead of $O(W)$ and $SO(W)$.

Let $\mathcal{F}$ be an SO_n-torsor on X. The corresponding rank n vector bundle $W_{\mathcal{F}}$ carries the bilinear form $(\ )_{\mathcal{F}}$, and we have a canonical isomorphism $\det W_{\mathcal{F}} = \mathcal{O}_X \otimes \det W$. Let $\mathcal{L} \in \omega^{1/2}(X)$, i.e., $\mathcal{L}$ is a square root of ω_X. Then $W_{\mathcal{F}} \otimes \mathcal{L}$ is an ω-orthogonal bundle, so $\mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L})$ makes sense (see 4.2). Consider the "normalized" Pfaffian

$$\mathrm{Pf}_{\mathcal{L},\mathcal{F}} := \mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}) \otimes \mathrm{Pf}(W \otimes \mathcal{L})^{\otimes -1} \tag{185}$$

[*)] To formulate this property in the non-perfect case one should consider $\mathcal{B}$ as a stack rather than a groupoid.

and the "normalized" determinant

$$\nu(\mathcal{F}) := \det R\Gamma(X, W_{\mathcal{F}}) \otimes \det R\Gamma(X, \mathcal{O}_X \otimes W)^{\otimes -1}. \tag{186}$$

As explained in 4.2.1 there are canonical isomorphisms $c_{\pm i} : \mathrm{Pf}^{\otimes 2} \xrightarrow{\sim} \det R\Gamma$. Using, e.g., c_i one obtains an isomorphism*)

$$\mathrm{Pf}_{\mathcal{L},\mathcal{F}}{}^{\otimes 2} \xrightarrow{\sim} \nu_{\mathcal{L}}(\mathcal{F}) \tag{187}$$

where

$$\nu_{\mathcal{L}}(\mathcal{F}) := \det R\Gamma(X, W_{\mathcal{F}} \otimes \mathcal{L}) \otimes \det R\Gamma(X, W \otimes \mathcal{L})^{\otimes -1}. \tag{188}$$

Construction 7.2 from [Del87] yields a canonical isomorphism

$$\nu_{\mathcal{L}}(\mathcal{F}) = \nu(\mathcal{F}) \otimes \left\langle \det W_{\mathcal{F}} \otimes (\det W)^{\otimes -1}, \mathcal{L} \right\rangle$$

Since $\det W_{\mathcal{F}} = \mathcal{O}_X \otimes \det W$ one has $\nu_{\mathcal{L}}(\mathcal{F}) = \nu(\mathcal{F})$ and

$$\mathrm{Pf}_{\mathcal{L},\mathcal{F}}{}^{\otimes 2} = \nu(\mathcal{F}). \tag{189}$$

When $\mathcal{F}$ varies $\mathrm{Pf}_{\mathcal{L},\mathcal{F}}$ and $\nu(\mathcal{F})$ become fibers of line bundles on Bun_{SO_n} which we denote by $\mathrm{Pf}_{\mathcal{L}}$ and ν.

Denote by $\nu^{1/2}(\mathrm{Bun}_{SO_n})$ the category of square roots of ν. We have the functor

$$\mathrm{Pf} : \omega^{1/2}(X) \to \nu^{1/2}(\mathrm{Bun}_{SO_n}) \tag{190}$$

defined by $\mathcal{L} \mapsto \mathrm{Pf}_{\mathcal{L}}$.

$\omega^{1/2}(X)$ and $\nu^{1/2}(\mathrm{Bun}_{SO_n})$ are Torsors over the Picard categories $\mu_2\,\mathrm{tors}(X)$ and $\mu_2\,\mathrm{tors}(\mathrm{Bun}_{SO_n})$. We have the Picard functor $\ell^{\mathrm{Spin}} : \mu_2\,\mathrm{tors}(X) \to \mu_2\,\mathrm{tors}(\mathrm{Bun}_{SO_n})$; this is the functor $\ell = \ell^{\tilde{G}}$ from 4.1 in the particular case $G = SO_n$, $\tilde{G} = \mathrm{Spin}_n$, $\Pi = \mathbb{Z}/2\mathbb{Z}$. In 4.3.8–4.3.15 we will show that the functor $\mathrm{Pf} : \omega^{1/2}(X) \to \nu^{1/2}(\mathrm{Bun}_{SO_n})$ has a canonical

---

*) So the isomorphism (187)=(189) depends on the choice of a square root of -1. This dependence disappears if one multiplies (187) by $i^{\pm p(\mathcal{F})^2}$ where p is the canonical map $\mathrm{Bun}_{SO_n} \to \pi_0(\mathrm{Bun}_{SO_n}) = \pi_1(SO_n) = \mathbb{Z}/2\mathbb{Z}$ and $p(\mathcal{F})^2 \in \mathbb{Z}/4\mathbb{Z}$. We prefer not to do it for the reason explained in Remark (iii) from 4.2.1.

structure of ℓ^{Spin}-affine functor. Before doing it we show in 4.3.2–4.3.7 that for a finite $S \subset X$ the action of $SO_n(K_S)$ on $\mathrm{Bun}_{SO_n,\underline{S}}$ defined in 4.1.7 lifts to an action of a certain central extension of $SO_n(K_S)$ on the pullback of $\mathrm{Pf}_{\mathcal{L}}$ to $\mathrm{Bun}_{SO_n,\underline{S}}$. Once this action is introduced it is easy to characterize the ℓ^{Spin}-affine structure on the functor Pf essentially by the $SO_n(K_S)$-invariance property (see 4.3.8–4.3.10).

4.3.2. Let V be a Tate space equipped with a nondegenerate symmetric bilinear form of *even type*, i.e., there exists a Lagrangian c-lattice $L \subset V$ (see 4.2.13); if $\dim V < \infty$ this means that $\dim V$ is even. Denote by $O(V)$ the group of topological automorphisms of V preserving the form. Let us remind the well known construction of a canonical central extension

$$0 \to \mathbb{C}^* \to \tilde{O}(V) \to O(V) \to 0\,. \tag{191}$$

Let M be an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded discrete module over the Clifford algebra $\mathrm{Cl}(V)$ (discreteness means that $\{v \in V \mid vm = 0\}$ is open for every $m \in M$). Then M is unique up to tensoring by a 1-dimensional $(\mathbb{Z}/2\mathbb{Z})$-graded space. So there is a natural projective representation of $O(V)$ in M. (191) is the extension corresponding to this representation, i.e.,

$$\tilde{O}(V) := \{(g,\varphi) \mid g \in O(V),\ \varphi \in \mathrm{Aut}_{\mathbb{C}} M,\ \varphi(vm) = g(v)\cdot\varphi(m) \text{ for } m \in M\}\,.$$

Clearly $\tilde{O}(V)$ does not depend on the choice of M (in fact $\mathrm{Aut}_{\mathbb{C}} M$ is the group of invertible elements of the natural completion of $\mathrm{Cl}(V)$). If $(g,\varphi) \in \tilde{O}(V)$ then φ is either even or odd. Let $\chi(g) \in \mathbb{Z}/2\mathbb{Z}$ denote the parity of φ. Then $\chi : O(V) \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism.

The preimages of $-1 \in O(V)$ in $\tilde{O}(V)$ are not central. Indeed, if $\varphi : M \to M$, $\varphi(m) = m$ for even m and $\varphi(m) = -m$ for odd m then $[-1] := (-1,\varphi) \in \tilde{O}(V)$ and

$$[-1]\cdot\tilde{g} = (-1)^{\chi(g)}\cdot\tilde{g}\cdot[-1]\,, \quad g \in O(V) \tag{192}$$

where $\tilde{g}$ denotes a preimage of g in $\tilde{O}(V)$.

$O(V)$ and $\operatorname{Aut}_{\mathbb{C}} M$ have natural structures of group ind-schemes. More precisely, the functors that associate to a $\mathbb{C}$-algebra R the sets $O(V\widehat{\otimes}R)$ and $\operatorname{Aut}_{\mathbb{C}}(M \otimes R)$ are ind-schemes (if $\dim V = \infty$ then they can be represented as a union of an *uncountable* filtered family of closed subschemes.) So $\tilde{O}(V)$ is a group ind-scheme.

Denote by $\operatorname{Lagr}(V)$ the set of Lagrangian c-lattices in V. It has a natural structure of ind-scheme: $\operatorname{Lagr}(V) = \varinjlim \operatorname{Lagr}(\Lambda^{\perp}/\Lambda)$ where Λ belongs to the set of isotropic c-lattices in V (so an R-point of $\operatorname{Lagr}(V)$ is a Lagrangian c-lattice in $V\widehat{\otimes}R$ in the sense of 4.2.14). Denote by $\mathcal{P} = \mathcal{P}_M$ the line bundle on $\operatorname{Lagr}(V)$ whose fiber over $L \in \operatorname{Lagr}(V)$ equals $M^L := \{m \in M | \, Lm = 0\}$. The action of $O(V)$ on $\operatorname{Lagr}(V)$ canonically lifts to an action of $\tilde{O}(V)$ on $\mathcal{P}$.

$\operatorname{Lagr}(V)$ has two connected components distinguished by the parity of the 1-dimensional $(\mathbb{Z}/2\mathbb{Z})$-graded space M^L, $L \in \operatorname{Lagr}(V)$. The proof of this statement is easily reduced to the case where $\dim V$ is finite (and even). The same argument shows that $L_1, L_2 \in \operatorname{Lagr}(V)$ belong to the same component if and only if $\dim(L_1/(L_1 \cap L_2))$ is even. Clearly the connected components of $\operatorname{Lagr}(V)$ are invariant with respect to $g \in O(V)$ if and only if $\chi(g) = 0$. Therefore $\chi : \, O(V) \to \mathbb{Z}/2\mathbb{Z}$ is a morphism of group ind-schemes.

Let us prove that (191) comes from an exact sequence of group ind-schemes

$$0 \to \mathbb{G}_m \to \tilde{O}(V) \to O(V) \to 0 \,. \tag{193}$$

We only have to show that the morphism $\tilde{O}(V) \to O(V)$ is a $\mathbb{G}_m$-torsor. To this end fix $L \in \operatorname{Lagr}(V)$ and set $M = \operatorname{Cl}(V)/\operatorname{Cl}(V)L$, so that the fiber of $\mathcal{P} = \mathcal{P}_M$ over L equals $\mathbb{C}$. Define $f : \, O(V) \to \operatorname{Lagr}(V)$ by $f(g) = gL$. Set $\mathcal{P}' := \mathcal{P} \setminus \{\text{zero section}\}$; this is a $\mathbb{G}_m$-torsor over $\operatorname{Lagr}(V)$. It is easy to show that the natural morphism $\tilde{O}(V) \to f^*\mathcal{P}'$ is an isomorphism, so $\tilde{O}(V)$ is a $\mathbb{G}_m$-torsor over $O(V)$.

Remark. Let $L \in \mathrm{Lagr}(V)$. Then (193) splits canonically over the stabilizer of L in $O(V)$: if $g \in O(V)$, $gL = L$, then there is a unique preimage of g in $\tilde{O}(V)$ that acts identically on M^L.

4.3.3. Set $O := \mathbb{C}[[t]]$, $K := \mathbb{C}((t))$. Denote by ω_O the (completed) module of differentials of O. Fix a square root of ω_O, i.e., a 1-dimensional free O-module $\omega_O^{1/2}$ equipped with an isomorphism $\omega_O^{1/2} \otimes \omega_O^{1/2} \xrightarrow{\sim} \omega_O$. Let W have the same meaning as in 4.3.1. We will construct a central extension of $O_n(K) := O(W \otimes K)$ considered as a group ind-scheme over $\mathbb{C}$.

Set $\omega_K^{1/2} := \omega_O^{1/2} \otimes_O K$, $\omega_K := \omega_O \otimes_O K$. Consider the Tate space $V := \omega_K^{1/2} \otimes W$. The bilinear form on W induces a K-bilinear form $V \times V \to \omega_K$. Composing it with $\mathrm{Res} : \omega_K \to \mathbb{C}$ one gets a nondegenerate symmetric bilinear form $V \times V \to \mathbb{C}$ of even type. Restricting the extension (193) to $O_n(K) \hookrightarrow O(V)$ one gets a central extension

$$0 \to \mathbb{G}_m \to \widetilde{O_n(K)} \to O_n(K) \to 0\,. \tag{194}$$

It splits canonically over $O_n(O) \subset O_n(K)$ (use the remark at the end of 4.3.2 for $L = \omega_O^{1/2} \otimes W \subset V$). The group $\mathrm{Aut}\,\omega_O^{1/2} = \mu_2$ acts on the extension (194) preserving the splitting over $O_n(O)$.

4.3.4. *Lemma.* The automorphism of $\widetilde{O_n(K)}$ induced by $-1 \in \mathrm{Aut}\,\omega_O^{1/2}$ maps $\tilde{g} \in \widetilde{O_n(K)}$ to $(-1)^{\theta(g)}\tilde{g}$ where g is the image of $\tilde{g}$ in $O_n(K)$ and $\theta : O_n(K) \to K^*/(K^*)^2 = \mathbb{Z}/2\mathbb{Z}$ is the spinor norm.

Proof. According to (192) we only have to show that $\chi(g) = \theta(g)$ for $g \in O_n(K) \subset O(V)$. According to the definition of θ (see [D71], ch. II, §7) it suffices to prove that if g is the reflection with respect to the orthogonal complement of a non-isotropic $x \in K^n$ then $\chi(g)$ equals the image of $(x, x) \in K^*$ in $K^*/(K^*)^2 = \mathbb{Z}/2\mathbb{Z}$. We can assume that $x \in O^n$, $x \notin tO^n$. $L := \omega_O^{1/2} \otimes W$ is a Lagrangian c-lattice in V, so $\chi(g)$ is the parity of $\dim L/(L \cap gL) = \dim O/(x, x)O$. □

Remarks

(i) Instead of using reflections one can compute the restriction of χ to a split Cartan subgroup of $SO_n(K)$ and notice that $\chi(g) = 0$ for $g \in O_n(\mathbb{C})$.

(ii) The restriction of θ to $SO_n(K)$ is the boundary morphism

$$SO_n(K) \to H^1(K, \mu_2) = \mathbb{Z}/2\mathbb{Z} \tag{195}$$

for the exact sequence $0 \to \mu_2 \to \mathrm{Spin}_n \to SO_n \to 0$.

(iii) If $g \in O_n(K) = O(W \otimes K)$ then $\dim(W \otimes O)/((W \otimes O) \cap g(W \otimes O))$ is even if and only if $\theta(g) = 0$. This follows from the proof of Lemma 4.3.4.

4.3.5. Consider the restriction of the extension (194) to $SO_n(K)$:

$$0 \to \mathbb{G}_m \to \widetilde{SO_n(K)} \to SO_n(K) \to 0\,. \tag{196}$$

It splits canonically over $SO_n(O)$. The extension (196) depends on the choice of $\omega_O^{1/2}$, so one should rather write $\widetilde{SO_n(K)}_{\mathcal{C}}$ where $\mathcal{C}$ is a square root of ω_O. Let $\mathcal{C}'$ be another square root of ω_O, then $\mathcal{C}' = \mathcal{C} \otimes \mathcal{A}$ where $\mathcal{A}$ is a μ_2-torsor over $\operatorname{Spec} O$ (or over $\operatorname{Spec} \mathbb{C}$, which is the same). Consider the (trivial) extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{G}_m$ such that $\mathcal{A}$ is the μ_2-torsor of its splittings. Its pullback by (195) is a (trivial) extension

$$0 \to \mathbb{G}_m \to \widetilde{SO_n(K)}_{\mathcal{A}} \to SO_n(K) \to 0 \tag{197}$$

equipped with a splitting over $SO_n(O)$ (in 4.1.8 we have already introduced this extension in a more general situation).

Lemma 4.3.4 yields a canonical isomorphism between $\widetilde{SO_n(K)}_{\mathcal{C}'}$ and the sum of the extensions $\widetilde{SO_n(K)}_{\mathcal{C}}$ and $\widetilde{SO_n(K)}_{\mathcal{A}}$. It is compatible with the splittings over $SO_n(O)$.

4.3.6. Let S, O_S, and K_S have the same meaning as in 4.1.7. Fix $\mathcal{L} \in \omega^{1/2}(X)$ and denote by $\omega_{K_S}^{1/2}$ the space of sections of the pullback of $\mathcal{L}$ to $\operatorname{Spec} K_S$. Then proceed as in 4.3.3: set $V := \omega_{K_S}^{1/2} \otimes W$, define the scalar product on V using the "sum of residues" map $\omega_{K_S} \to \mathbb{C}$, embed $SO_n(K_S)$ into $O(V)$ and finally get a central extension

$$0 \to \mathbb{G}_m \to \widetilde{SO_n(K_S)}_{\mathcal{L}} \to SO_n(K_S) \to 0 \tag{198}$$

with a canonical splitting over $SO_n(O_S)$.

Remark. (198) is the "super-sum" of the extensions (196) for $K = K_x$, $x \in S$. Let us explain that if G_i, $i \in I$, are groups equipped with morphisms $\theta_i : G_i \to \mathbb{Z}/2\mathbb{Z}$ and $\tilde{G}_i$ are central extensions of G_i by $\mathbb{G}_m$ then the super-sum of these extensions is the extension of $\bigoplus\limits_i G_i$ by $\mathbb{G}_m$ obtained from the usual sum by adding the pullback of the standard extension

$$0 \to \mathbb{G}_m \to A \to \bigoplus_{i \in I} (\mathbb{Z}/2\mathbb{Z}) \to 0$$

where A is generated by $\mathbb{G}_m$ and elements e_i, $i \in I$, with the defining relations $e_i^2 = 1$, $ce_i = e_i c$ for $c \in \mathbb{G}_m$, $e_i e_j = (-1) \cdot e_j e_i$ for $i \neq j$. In our situation $\theta_x : SO_n(K_x) \to \mathbb{Z}/2\mathbb{Z}$ is the spinor norm.

If $\mathcal{L}, \mathcal{L}' \in \omega^{1/2}(X)$ then $\mathcal{L}' = \mathcal{L} \otimes \mathcal{E}$ where $\mathcal{E}$ is a μ_2-torsor. It follows from 4.3.5 that there is a canonical isomorphism between $\widetilde{SO_n(K_S)}_{\mathcal{L}'}$ and the sum of the extensions $\widetilde{SO_n(K_S)}_{\mathcal{L}}$ and $\widetilde{SO_n(K_S)}_{\mathcal{E}}$ (see 4.1.8 for the definition of $\widetilde{SO_n(K_S)}_{\mathcal{E}}$).

4.3.7. In 4.3.1 we defined the line bundles $\mathrm{Pf}_{\mathcal{L}}$ on Bun_{SO_n}, $\mathcal{L} \in \omega^{1/2}(X)$. Denote by $\mathrm{Pf}_{\mathcal{L}}^S$ the pullback of $\mathrm{Pf}_{\mathcal{L}}$ to the scheme $\mathrm{Bun}_{SO_n \underline{S}}$ defined in 4.1.7. We have the obvious action of $SO_n(O_S) \times \mathbb{G}_m$ on $\mathrm{Pf}_{\mathcal{L}}^S$ ($\lambda \in \mathbb{G}_m$ acts as multiplication by λ). We are going to extend it to an action of $\widetilde{SO_n(K_S)}_{\mathcal{L}}$ on $\mathrm{Pf}_{\mathcal{L}}^S$ compatible with the action of $SO_n(K_S)$ on $\mathrm{Bun}_{SO_n, \underline{S}}$.

Let $u \in \mathrm{Bun}_{SO_n, \underline{S}}$, $\tilde{g} \in \widetilde{SO_n(K_S)}_{\mathcal{L}}$. Denote by $\mathcal{F}$ and $\mathcal{F}'$ the $SO(W)$-bundles corresponding to u and gu where $g \in SO_n(K_S)$ is the image of $\tilde{g}$. We must define an isomorphism $\mathrm{Pf}_{\mathcal{L},\mathcal{F}} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L},\mathcal{F}'}$, i.e., an isomorphism $\mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}) \xrightarrow{\sim} \mathrm{Pf}(W_{\mathcal{F}'} \otimes \mathcal{L})$. According to 4.2.13 it suffices to construct an isomorphism $\mathrm{Pf}(V; L_+, L_-) \xrightarrow{\sim} \mathrm{Pf}(V; L_+, L'_-)$ where V is the Tate space from 4.3.6, $L_+ = \omega_{O_S}^{1/2} \otimes W \subset V$, and $L_-, L'_- \subset V$ are discrete Lagrangian subspaces such that $L'_- = gL_-$. According to (166) this is equivalent to constructing an isomorphism $f : (M_{L_-})^{\otimes -1} \xrightarrow{\sim} (M_{gL_-})^{\otimes -1}$. We define f to be induced by the action of $\tilde{g} \in \tilde{O}(V)$ on M.

Attention: $\lambda \in \mathbb{G}_m \subset \widetilde{SO_n(K_S)}_{\mathcal{L}}$ acts on $\mathrm{Pf}_{\mathcal{L}}^S$ as multiplication by λ^{-1}.

4.3.8. As explained in 4.3.1 our goal is to define a canonical ℓ^{Spin}-affine structure on the functor (190). This means that for $\mathcal{L} \in \omega^{1/2}(X)$ and a μ_2-torsor $\mathcal{E}$ on X we must define an isomorphism

$$\mathrm{Pf}_{\mathcal{L}} \otimes \ell_{\mathcal{E}}^{\mathrm{Spin}} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L}'}\,, \quad \mathcal{L}' := \mathcal{L} \otimes \mathcal{E}\,. \tag{199}$$

We must also check certain compatibility properties for the isomorphisms (199).

To simplify notation we will write $\ell_{\mathcal{E}}$ instead of $\ell_{\mathcal{E}}^{\mathrm{Spin}}$. Let $S \subset X$ be finite. In 4.1.7–4.1.8 we constructed an action of the central extension $\widetilde{SO_n(K_S)}_{\mathcal{E}}$ on $\ell_{\mathcal{E}}^S :=$ the pullback of $\ell_{\mathcal{E}}$ to $\mathrm{Bun}_{SO_n,\underline{S}}$. So it follows from 4.3.6–4.3.7 that $\widetilde{SO_n(K_S)}_{\mathcal{L}'}$ acts both on $\mathrm{Pf}_{\mathcal{L}}^S \otimes \ell_{\mathcal{E}}^S$ and $\mathrm{Pf}_{\mathcal{L}'}^S$. Recall that the fibers of both sides of (199) over the trivial SO_n-bundle equal $\mathbb{C}$.

4.3.9. *Theorem.* There is a unique isomorphism (199) such that for every S the corresponding isomorphism $\mathrm{Pf}_{\mathcal{L}}^S \otimes \ell_{\mathcal{E}}^S \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L}'}^S$ is $\widetilde{SO_n(K_S)}_{\mathcal{L}'}$-equivariant and the isomorphism between the fibers over the trivial SO_n-bundle induced by (199) is identical.

The proof will be given in 4.3.11–4.3.13. See §5.2 from [BLaSo] for a short proof of a weaker statement.

4.3.10. *Proposition.* The isomorphisms (199) define an ℓ^{Spin}-affine structure on the functor $\mathrm{Pf} : \omega^{1/2}(X) \to \nu^{1/2}(\mathrm{Bun}_{SO_n})$.

The proof will be given in 4.3.15.

4.3.11. Let us start to prove Theorem 4.3.9. The uniqueness of (199) is clear if $n > 2$: in this case SO_n is semisimple, so one has the isomorphism (155) for $G = SO_n$, $S \neq \emptyset$. If $n = 2$ the action of $SO_n(K_S)$ on $\mathrm{Bun}_{SO_n,\underline{S}}$ is not transitive, but SO_n over the adeles acts transitively on $\varprojlim\limits_{S} \mathrm{Bun}_{SO_n,\underline{S}}(\mathbb{C})$, which is enough for uniqueness.

While proving the existence of (199) we will assume that $n>2$. The case $n=2$ can be treated using the embedding $SO_2 \hookrightarrow SO_3$ and the corresponding morphism $\mathrm{Bun}_{SO_2} \to \mathrm{Bun}_{SO_3}$ or using the remark at the end of 4.3.14.

Consider the $SO_n(K_S)$-equivariant line bundle $C_S := \mathrm{Pf}^S_{\mathcal{L}} \otimes \ell^S_{\mathcal{E}} \otimes (\mathrm{Pf}^S_{\mathcal{L}'})^*$ on $\mathrm{Bun}_{SO_n,\underline{S}}$. The stabilizer of the point of $\mathrm{Bun}_{SO_n,\underline{S}}$ corresponding to the trivial SO_n-bundle with the obvious trivialization over $\underline{S}$ equals $SO_n(A_S)$, $A_S := H^0(X \setminus S, \mathcal{O}_X)$. So the action of $SO_n(K_S)$ on C_S induces a morphism $f_S : SO_n(A_S) \to \mathbb{G}_m$. It suffices to prove that f_S is trivial for all S (then for $S \neq \emptyset$ one can use (155) to obtain a $SO_n(K_S)$-equivariant trivialization of C_S and of course these trivializations are compatible with each other).

Denote by Σ the scheme of finite subschemes of X (so Σ is the disjoint union of the symmetric powers of X). A_S, O_S, and K_S make sense for a non-necessarily reduced*) $S \in \Sigma$ (e.g., O_S is the ring of functions on the completion of X along S) and the rings A_S, O_S, K_S are naturally organized into families (i.e., there is an obvious way to define three ring ind-schemes over Σ whose fibers over $S \in \Sigma$ are equal to A_S, O_S, K_S respectively).

It is easy to show that *the morphisms f_S form a family* (i.e., they come from a morphism of group ind-schemes over Σ). Clearly if $S \subset S'$ then the restriction of $f_{S'}$ to $SO_n(A_S)$ equals f_S. In 4.3.12–4.3.13 we will deduce from these two facts that $f_S = 1$.

4.3.12. Let Y be a separated scheme of finite type over $\mathbb{C}$ and R a $\mathbb{C}$-algebra. Set $Y_{\mathrm{rat}}(R) = \varinjlim\limits_{U} \mathrm{Mor}(U, Y)$ where the limit is over all open $U \subset X \otimes R$ such that the fiber of U over any point of $\operatorname{Spec} R$ is non-empty. In other words, elements of $Y_{\mathrm{rat}}(R)$ are families of rational maps $X \to Y$ parameterized by

*) This is important when S varies. For a fixed S the rings A_S, O_S and K_S depend only on S_{red}.

$\operatorname{Spec} R$. The functor Y_{rat} is called the *space of rational maps* $X \to Y$. It is easy to show that Y_{rat} is a sheaf for the fppf topology, i.e., a "space" in the sense of [LMB93].

We have the spaces $Y(A_S)$, $S \in \Sigma$, which form a family (i.e., there is a natural space over Σ whose fiber over each S equals $Y(A_S)$). So a regular function on Y_{rat} defines a family of regular functions f_S on $Y(A_S)$, $S \in \Sigma$, such that for $S \subset S'$ the pullback of $f_{S'}$ to $Y(A_S)$ equals f_S. It is easy to see that a function on Y_{rat} is *the same* as a family of functions f_S with this property.

4.3.13. *Proposition.* Let G be a connected algebraic group.

(i) Every regular function on G_{rat} is constant. In particular every group morphism $G_{\mathrm{rat}} \to \mathbb{G}_m$ is trivial.

(ii) Moreover, for every $\mathbb{C}$-algebra R every regular function on $G_{\mathrm{rat}} \otimes R$ is constant (i.e., an element of R).

Proof. Represent G as $\bigcup\limits_i U_i$ where U_i are open sets isomorphic to $(\mathbb{A}^1 \setminus \{0\})^r \times \mathbb{A}^s$ (e.g., let $U \subset G$ be the big cell with respect to some Borel subgroup, then G is covered by a finite number of sets of the form gU, $g \in G$). One has the open covering $G_{\mathrm{rat}} = \bigcup\limits_i (U_i)_{\mathrm{rat}}$ and $(U_i)_{\mathrm{rat}} \cap (U_j)_{\mathrm{rat}} \neq \emptyset$. So it is enough to prove the proposition for $G = (\mathbb{G}_m)^r \times (\mathbb{G}_a)^s$. Moreover, it suffices to prove (ii) for $\mathbb{G}_a$ and $\mathbb{G}_m$.

Consider, e.g., the $\mathbb{G}_m$ case. Choose an ample line bundle $\mathcal{A}$ on X and set $V_n := H^0(X, \mathcal{A}^{\otimes n})$, $V_n' := V_n \setminus \{0\}$. Define $\pi_n : V_n' \times V_n' \to (\mathbb{G}_m)_{\mathrm{rat}}$ by $(f, g) \mapsto f/g$. A regular function φ on $(\mathbb{G}_m)_{\mathrm{rat}} \otimes R$ defines a regular function $\pi_n^* \varphi$ on $(V_n' \times V_n') \otimes R$, which is invariant with respect to the obvious action of $\mathbb{G}_m$ on $V_n' \times V_n'$. For n big enough $\dim V_n > 1$ and therefore $\pi_n^* \varphi$ extends to a $\mathbb{G}_m$-invariant regular function on $(V_n \times V_n) \otimes R$, which is necessarily a constant. So φ is constant. $\square$

4.3.14. This subsection is not used in the sequel (except the definition of GRAS_G needed in 5.3.10).

Let G be a connected algebraic group. The following approach to Bun_G seems to be natural.

Denote by GRAS_G the space of G-torsors on X equipped with a rational section. The precise definition of this space is quite similar to the definition of Y_{rat} from 4.3.12. We would call GRAS_G the *big Grassmannian* corresponding to G and X because for a fixed finite $S \subset X$ the space of G-bundles on X trivialized over $X \setminus S$ can be identified with the ind-scheme $G(K_S)/G(O_S) = \prod\limits_{x\in X} G(K_x)/G(O_x)$ (see 5.3.10), and $G(K_x)/G(O_x)$ is called the *affine Grassmannian* or *loop Grassmannian* (see 4.5 or [MV]).

The morphism $\pi : \mathrm{GRAS}_G \to \mathrm{Bun}_G$ is a G_{rat}-torsor for the smooth topology (the existence of a section $S \to \mathrm{GRAS}_G$ for some smooth surjective morphism $S \to \mathrm{Bun}_G$ is obvious if the reductive part of G equals GL_n, SL_n, or Sp_n; for a general G one can use [DSim]).

Consider the functor

$$\pi^* : \mathrm{Vect}(\mathrm{Bun}_G) \to \mathrm{Vect}(\mathrm{GRAS}_G) \tag{200}$$

where Vect denotes the category of vector bundles. It follows from 4.3.13 that (200) is fully faithful. One can show that for any scheme T every vector bundle on $G_{\mathrm{rat}} \times T$ comes from T. This implies that (200) is an equivalence.

Remark. Our construction of (199) can be interpreted as follows: we constructed an isomorphism between the pullbacks of the l.h.s. and r.h.s. of (199) to GRAS_{SO_n}, then we used the fact that (200) is fully faithful. It was not really necessary to use the isomorphism (155). So the construction of (199) also works in the case of SO_2.

4.3.15. Let us prove Proposition 4.3.10. The isomorphisms (199) are compatible with each other (use the uniqueness statement from 4.3.9). It remains to show that the tensor square of (199) equals the composition

$$\mathrm{Pf}_{\mathcal{L}}^{\otimes 2} \xrightarrow{\sim} \nu_{\mathcal{L}} \xrightarrow{\sim} \nu \xrightarrow{\sim} \nu_{\mathcal{L}'} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L}'}^{\otimes 2} \tag{201}$$

where $\nu_{\mathcal{L}}$ is defined by (188).

Fix an SO_n-torsor $\mathcal{F}$ on X and its trivialization over $X \setminus S$ for some non-empty finite $S \subset X$. Using the trivialization we will compute the isomorphisms $\mathrm{Pf}_{\mathcal{L},\mathcal{F}}^{\otimes 2} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L}',\mathcal{F}}^{\otimes 2}$ induced by (199) and (201).

Recall that $\mathrm{Pf}_{\mathcal{L},\mathcal{F}} := \mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}) \otimes \mathrm{Pf}(W \otimes \mathcal{L})^{\otimes -1}$. According to 4.2.13

$$\mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}) = \mathrm{Pf}(V; L_+, L_-), \quad \mathrm{Pf}(W \otimes \mathcal{L}) = \mathrm{Pf}(V; L_+^0, L_-)$$

where $V = \mathcal{L}_{K_S} \otimes W$, $L_- = \Gamma(X \setminus S, \mathcal{L} \otimes W)$, $L_+^0 = \mathcal{L}_{O_S} \otimes W$, and L_+ is the space of sections of the pullback of $W_{\mathcal{F}} \otimes \mathcal{L}$ to $\mathrm{Spec}\, O_S$ (we use the notation of 4.3.6). Using (166) one gets

$$\mathrm{Pf}_{\mathcal{L},\mathcal{F}} = M^{L_+} \otimes (M^{L_+^0})^* \tag{202}$$

where M is an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded discrete module over $\mathrm{Cl}(V)$. $\mathrm{Pf}_{\mathcal{L}',\mathcal{F}}$ has a similar description in terms of V', L'_+, $(L_+^0)'$, L'_- where $V' = \mathcal{L}'_{K_S} \otimes W$, etc. Fix a trivialization of the μ_2-torsor $\mathcal{E}$ from 4.3.8 over S. It yields a trivialization of $\mathcal{E}$ over $\mathrm{Spec}\, O_S$ and therefore an identification

$$(V', L'_+, (L_+^0)') = (V, L_+, L_+^0). \tag{203}$$

Since L_- is not involved in (202) we obtain an isomorphism $\mathrm{Pf}_{\mathcal{L},\mathcal{F}} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L}',\mathcal{F}}$. It is easy to show that it coincides with the one induced by (199) (notice that the trivialization of $\mathcal{F}$ over $X \setminus S$ and the trivialization of $\mathcal{E}$ over S induce a trivialization of $\ell_{\mathcal{E}}^{\mathrm{Spin}}$ over $\mathcal{F}$ because the l.h.s. of (150) has a distinguished element).

Now we have to show that the isomorphism $\mathrm{Pf}_{\mathcal{L},\mathcal{F}}^{\otimes 2} \xrightarrow{\sim} \mathrm{Pf}_{\mathcal{L}',\mathcal{F}}^{\otimes 2}$ induced by (201) is the identity provided $\mathrm{Pf}_{\mathcal{L},\mathcal{F}}$ and $\mathrm{Pf}_{\mathcal{L}',\mathcal{F}}$ are identified with the r.h.s. of (202).

The trivialization of $\mathcal{F}$ over $X \setminus S$ yields an isomorphism $\nu_{\mathcal{L}}(\mathcal{F}) \xrightarrow{\sim} d(L_+^0, L_+)$ where $d(L_+^0, L_+)$ is the relative determinant, i.e., $d(L_+^0, L_+) = \det(L_+/U) \otimes \det(L_+^0/U)^{\otimes -1}$ for any c-lattice $U \subset L \cap L_+^0$. We have a similar identification $\nu_{\mathcal{L}'}(\mathcal{F}) = d((L_+^0)', L'_+)$. The isomorphism $\nu_{\mathcal{L}}(\mathcal{F}) \xrightarrow{\sim} \nu_{\mathcal{L}'}(\mathcal{F})$ from (201) is defined in [Del87] as follows. One chooses *any* isomorphism f between the pullbacks of $\mathcal{L}$ and $\mathcal{L}'$ to $\mathrm{Spec}\, O_S$. f yields an isomorphism $f_* : (V, L_+, L_+^0) \xrightarrow{\sim} (V', L'_+, (L_+^0)')$ and therefore an isomorphism $d(L_+^0, L_+) \xrightarrow{\sim} d(L'_+, (L_+^0)')$, which actually does not depend on the choice of f. It is convenient to define f using the above trivialization of the μ_2-torsor $\mathcal{E} = \mathcal{L}' \otimes \mathcal{L}^{\otimes -1}$ over $\operatorname{Spec} O_S$. Then f_* coincides with (203).

Thus we have identified $\nu_{\mathcal{L}}(\mathcal{F})$ and $\nu_{\mathcal{L}'}(\mathcal{F})$ with $d(L_+^0, L_+)$ so that the isomorphism $\nu_{\mathcal{L}}(\mathcal{F}) \xrightarrow{\sim} \nu_{\mathcal{L}'}(\mathcal{F})$ from (201) becomes the identity map. We have identified both $\mathrm{Pf}_{\mathcal{L},\mathcal{F}}$ and $\mathrm{Pf}_{\mathcal{L}',\mathcal{F}}$ with the r.h.s. of (202). It remains to show that the isomorphism (187) and its analog for $\mathcal{L}'$ induce the same isomorphism

$$(M^{L_+} \otimes (M^{L_+^0})^*)^{\otimes 2} \xrightarrow{\sim} d(L_+^0, L_+) \tag{204}$$

According to 4.2.8 and 4.2.13 the isomorphism (204) induced by (187) can be described as follows. We have the canonical isomorphism

$$N^{L_+ \otimes H} \otimes (N^{L_+^0 \otimes H})^* \xrightarrow{\sim} d(L_+^0, L) \tag{205}$$

where N is an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded discrete module over the Clifford algebra $\mathrm{Cl}(V \oplus V^*) = \mathrm{Cl}(V \oplus V) = \mathrm{Cl}(V \otimes H)$ and H denotes $\mathbb{C}^2$ equipped with the bilinear form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (to construct (205) take for N the r.h.s. of (182)). On the other hand, $P := M \otimes M$ is an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded discrete module over $\mathrm{Cl}(V) \otimes \mathrm{Cl}(V) = \mathrm{Cl}(V \otimes H'')$ where H'' denotes $\mathbb{C}^2$ with the bilinear form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Rewrite the l.h.s. of (204) as $P^{L_+ \otimes H''} \otimes (P^{L_+^0 \otimes H''})^*$. So an orthogonal isomorphism $\psi : H'' \xrightarrow{\sim} H$ induces an isomorphism (204). To get the isomorphism (204) induced by (187) we must normalize ψ by $\det \psi = i$ (or $-i$?? we should check!).

Since L_- is not involved in the above description the analog of (187) for $\mathcal{L}'$ induces the same isomorphism (204), QED.

4.3.16. This subsection and 4.3.17 will be used in 4.4.14 (end of the proof of the horizontality theorem 2.7.3) and in the proof of Theorem 5.4.5 (which is the main result of this work). However the reader can skip them for the moment.

As usual, we set $O := \mathbb{C}[[t]]$, $K := \mathbb{C}((t))$. Fix $\mathcal{L} \in \omega^{1/2}(X)$, i.e., $\mathcal{L}$ is a square root of ω_X. Fix also a square root of ω_O and denote it by $\omega_O^{1/2}$. Then one defines a 2-sheeted covering $X_2^\wedge$ of the scheme $X^\wedge$ from 2.6.5. Recall that an R-point of $X^\wedge$ is an R-morphism $\alpha : \operatorname{Spec}(R\widehat{\otimes}O) \to X \otimes R$ whose differential does not vanish over any point of $\operatorname{Spec} R$. Denote by $\mathcal{L}_R$ the pullback of $\mathcal{L}$ to $X \otimes R$. By definition, the fiber of $X_2^\wedge(R)$ over $\gamma \in X^\wedge(R)$ is the set of isomorphisms $H^0(\operatorname{Spec} R\widehat{\otimes}O, \alpha^*\mathcal{L}_R) \xrightarrow{\sim} R\widehat{\otimes}\omega_O^{1/2})$ in the groupoid of square roots of $R\widehat{\otimes}\omega_O$.

The group ind-scheme $\operatorname{Aut}_2 O := \operatorname{Aut}(O, \omega_O^{1/2})$ introduced in 3.5.2 acts on $X_2^\wedge$ by transport of structure.

Let M be the scheme from 2.8.1 in the particular case $G = SO(W) = SO_n$. Denote by $M_2^\wedge$ the fiber product of M and $X_2^\wedge$ over X (so $M_2^\wedge$ is a 2-sheeted covering of the scheme $M^\wedge$ from 2.8.3). Then the semidirect product $\operatorname{Aut}_2 O \ltimes SO_n(K)$ acts on $M_2^\wedge$. Indeed, $M_2^\wedge$ is the fiber product of $M^\wedge$ and $X_2^\wedge$ over $X^\wedge$, and $\operatorname{Aut}_2 O \ltimes SO_n(K)$ acts on the diagram

$$\begin{array}{ccc} & & M^\wedge \\ & & \downarrow \\ X_2^\wedge & \longrightarrow & X^\wedge \end{array}$$

(the action of $\operatorname{Aut} O \ltimes SO_n(K)$ on $M^\wedge$ was defined in 2.8.4; $\operatorname{Aut}_2 O \ltimes SO_n(K)$ acts on $X_2^\wedge$ and $X^\wedge$ via its quotients $\operatorname{Aut}_2 O$ and $\operatorname{Aut} O$).

Denote by $\mathrm{Pf}_{\mathcal{L}}^\wedge$ the pullback to $M_2^\wedge$ of the line bundle $\mathrm{Pf}_{\mathcal{L}}$ on Bun_{SO_n} defined in 4.3.1. We will lift the action of $\operatorname{Aut}_2 O \ltimes SO_n(K)$ on $M_2^\wedge$ to an action of $\operatorname{Aut}_2 O \ltimes \widetilde{SO_n(K)}$ on $\mathrm{Pf}_{\mathcal{L}}^\wedge$, where $\widetilde{SO_n(K)}$ is the central extension (196) corresponding to $\omega_O^{1/2}$. The action of $\operatorname{Aut}_2 O$ on $\mathrm{Pf}_{\mathcal{L}}^\wedge$ is clear because $\operatorname{Aut}_2 O$ acts on $M_2^\wedge$ considered as a scheme over Bun_{SO_n}. On the other hand, $\widetilde{SO_n(K)}$ acts on $\mathrm{Pf}_{\mathcal{L},\widehat{x}}$:=the restriction of $\mathrm{Pf}_{\mathcal{L}}^\wedge$ to the fiber of $M_2^\wedge$ over $\widehat{x} \in X_2^\wedge$. Indeed, this fiber equals $\mathrm{Bun}_{SO_n,\underline{x}}$ where x is the image of $\widehat{x}$ in X, and by 4.3.7 the central extension $\widetilde{SO_n(K_x)}_{\mathcal{L}}$ acts on the pullback of $\mathrm{Pf}_{\mathcal{L}}$ to $\mathrm{Bun}_{SO_n,\underline{x}}$. This extension depends only on $\mathcal{L}_x$:=the pullback of $\mathcal{L}$ to $\operatorname{Spec} O_x$. Since $\widehat{x}$ defines an isomorphism between $(O, \omega_O^{1/2})$ and

$(O_x, H^0(\operatorname{Spec} O_x, \mathcal{L}_x))$ we get an isomorphism $\widetilde{SO_n(K_x)}_{\mathcal{L}} \xrightarrow{\sim} \widetilde{SO_n(K)}$ and therefore the desired action of $\widetilde{SO_n(K)}$.

4.3.17. *Proposition.*

(i) The action of $\widetilde{SO_n(K)}$ on $\mathrm{Pf}^{\wedge}_{\mathcal{L},\widehat{x}}$, $\widehat{x} \in X_2^{\wedge}$, comes from an (obviously unique) action of $\widetilde{SO_n(K)}$ on $\mathrm{Pf}^{\wedge}_{\mathcal{L}}$.

(ii) The actions of $\operatorname{Aut}_2 O$ and $\widetilde{SO_n(K)}$ on $\mathrm{Pf}^{\wedge}_{\mathcal{L}}$ define an action of $\operatorname{Aut}_2 O \ltimes \widetilde{SO_n(K)}$.

Remark. Statement (ii) can be interpreted in the spirit of 2.8.2: the action of $\operatorname{Aut}_2 O$ yields a connection along X on $\pi^* \mathrm{Pf}_{\mathcal{L}}$ where π is the morphism $M \to \mathrm{Bun}_G$, and the compatibility of the action of $\operatorname{Aut}_2 O$ with that of $\widetilde{SO_n(K)}$ means that the action on $\pi^* \mathrm{Pf}_{\mathcal{L}}$ of a certain central extension $\widetilde{J}^{\mathrm{mer}}(SO_n)_{\mathcal{L}}$ is horizontal.

Proof. To define the action of $\operatorname{Aut}_2 O \ltimes \widetilde{SO_n(K)}$ on $\mathrm{Pf}^{\wedge}_{\mathcal{L}}$ with the desired properties we proceed as in 4.3.7. Let R be a $\mathbb{C}$-algebra. Consider an R-point u of $M_2^{\wedge}$ and an R-point $\widetilde{g}$ of $\operatorname{Aut}_2 O \ltimes \widetilde{SO_n(K)}$. Recall that SO_n is an abbreviation for $SO(W)$. Denote by $\mathcal{F}$ and $\mathcal{F}'$ the $SO(W)$-torsors on $X \otimes R$ corresponding to u and gu where g is the image of $\widetilde{g}$ in $\operatorname{Aut}_2 O \ltimes SO_n(K)$. We have to define an isomorphism

$$\mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}_R) \xrightarrow{\sim} \mathrm{Pf}(W_{\mathcal{F}'} \otimes \mathcal{L}_R) \tag{206}$$

where $\mathcal{L}_R$ is the pullback of $\mathcal{L}$ to $X \otimes R$.

Set $V := \omega_O^{1/2} \otimes_O K \otimes W$. This is a Tate space over $\mathbb{C}$ equipped with a nondegenerate symmetric bilinear form (see 4.3.3). By 4.2.14

$$\mathrm{Pf}(W_{\mathcal{F}} \otimes \mathcal{L}_R) = \mathrm{Pf}(V \widehat{\otimes} R; L_+ \widehat{\otimes} R, L_-^u) \tag{207}$$

where $L_+ := \omega_O^{1/2} \otimes W \subset V$ (so L_+ is a Lagrangian c-lattice in V) and the Lagrangian d-lattice $L_-^u \subset V \widehat{\otimes} R$ is defined as follows. The point $u \in M_2^{\wedge}(R)$ is a quadruple $(\alpha, \mathcal{F}, \gamma, f)$ where α, $\mathcal{F}$, γ have the same meaning as in 2.8.4 (in our special case $G = SO(W)$) and f is an isomorphism between

$H^0(\operatorname{Spec} R\widehat{\otimes}K, \alpha^*\mathcal{L}_R)$ and $R\widehat{\otimes}\omega_O^{1/2}$ in the groupoid of square roots of $R\widehat{\otimes}\omega_O$. Let Γ_α have the same meaning as in 2.8.4. Then

$$L_-^u := H^0((X\otimes R)\setminus\Gamma_\alpha, W_{\mathcal{F}}\otimes\mathcal{L}_R)\subset H^0(\operatorname{Spec} R\widehat{\otimes}K, \alpha^*W_{\mathcal{F}}\otimes\alpha^*\mathcal{L}_R)\overset{\varphi}{\underset{\sim}{\longrightarrow}} V\widehat{\otimes}R$$

(the isomorphism φ is induced by γ and f).

Taking (207) into account we see that constructing (206) is equivalent to defining an isomorphism

$$\operatorname{Pf}(V\widehat{\otimes}R; L_+\otimes R, L_-^u)\xrightarrow{\sim}\operatorname{Pf}(V\widehat{\otimes}R; L_+\otimes R, L_-^{gu})\,. \tag{208}$$

The group ind-scheme $\operatorname{Aut}_2 O\ltimes SO(W\otimes K)$ acts on V in the obvious way, and it is easy to see that $L_-^{gu}=gL_-^u$. By (166) the l.h.s. of (208) is inverse to $(M\otimes R)_{L_-}$ where$M$ is the Clifford module $\operatorname{Cl}(V)/\operatorname{Cl}(V)L_+$ and $L_-:=L_-^u$. So it remains to construct an isomorphism $(M\otimes R)_{L_-}\xrightarrow{\sim}(M\otimes R)_{gL_-}$. We define it to be induced by the action*) of $\widetilde{g}$ on $M\otimes R$. □

4.4. **Half-forms on** Bun_G.

4.4.1. Let G be semisimple. Fix a G-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. Set $n:=\dim\mathfrak{g}$ and write SO_n instead of $SO(\mathfrak{g})$. The adjoint representation $G\to SO(\mathfrak{g})$ induces a morphism $f:\operatorname{Bun}_G\to\operatorname{Bun}_{SO_n}$. For $\mathcal{L}\in\omega^{1/2}(X)$ set $\lambda'_{\mathcal{L}}:=f^*\operatorname{Pf}_{\mathcal{L}}$ where $\operatorname{Pf}_{\mathcal{L}}$ is the line bundle from 4.3.1; so the fiber of $\lambda'_{\mathcal{L}}$ over $\mathcal{F}\in\operatorname{Bun}_G$ equals $\operatorname{Pf}(\mathfrak{g}_{\mathcal{F}}\otimes\mathcal{L})\otimes\operatorname{Pf}(\mathfrak{g}\otimes\mathcal{L})^{\otimes-1}$. The isomorphism (189) induces an isomorphism

$$(\lambda'_{\mathcal{L}})^{\otimes 2}=\omega^\sharp_{\operatorname{Bun}_G} \tag{209}$$

Here $\omega^\sharp_{\operatorname{Bun}_G}$ is the normalized canonical bundle (146); according to 2.1.1 the fiber of $\omega^\sharp_{\operatorname{Bun}_G}$ over $\mathcal{F}\in\operatorname{Bun}_G$ equals $\det R\Gamma(X,\mathfrak{g}_{\mathcal{F}})\otimes(\det R\Gamma(X,\mathfrak{g}\otimes\mathcal{O}_X))^{\otimes-1}$.

---

*)Recall that g is an R-point of $\operatorname{Aut}_2 O\ltimes\widetilde{SO_n(K)}=\operatorname{Aut}_2 O\ltimes\widetilde{SO(W\otimes K)}$. By the definition of $\widetilde{SO_n(K)}$ it acts on M. The group ind-scheme $\operatorname{Aut}_2 O$ acts on (V, L_+) and therefore on M.

4.4.2. Consider the functor

$$\lambda' : \omega^{1/2}(X) \to (\omega^\sharp)^{1/2}(\mathrm{Bun}_G), \tag{210}$$

$\mathcal{L} \mapsto \lambda'_{\mathcal{L}}$. By 4.3.10 λ' is affine with respect to the Picard functor $\tilde{\ell}' : \mu_2\,\mathrm{tors}(X) \to \mu_2\,\mathrm{tors}(\mathrm{Bun}_G)$ that sends a μ_2-torsor $\mathcal{E}$ on X to $\tilde{\ell}'_{\mathcal{E}} :=$ the pullback to Bun_G of the torsor $\ell_{\mathcal{E}}^{\mathrm{Spin}}$ on Bun_{SO_n}.

4.4.3. *Proposition.* $\tilde{\ell}' = \ell'$ where ℓ' is the composition of the functor $\mu_2\,\mathrm{tors}(X) \to Z\,\mathrm{tors}(X)$ induced by (56) and the functor $\ell : Z\,\mathrm{tors}(X) \to \mu_\infty\,\mathrm{tors}(\mathrm{Bun}_G)$ constructed in 4.1.1–4.1.4. Here $Z = \pi_1(G)^\vee$ =the center of ${}^LG$ (see the Remark from 4.1.1).

Assuming the proposition we define a canonical ℓ-affine functor

$$\lambda : Z\,\mathrm{tors}_\theta(X) \to \mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G) \tag{211}$$

by $\mathcal{E}\cdot\mathcal{L} \mapsto \lambda_{\mathcal{E}\cdot\mathcal{L}} := \ell_{\mathcal{E}}\cdot\lambda'_{\mathcal{L}}$, $\mathcal{E} \in Z\,\mathrm{tors}(X)$, $\mathcal{L} \in \omega^{1/2}(X)$. (Attention: normalization problem!!!???)

To prove Proposition 4.4.3 notice that $\tilde{\ell}'$ is the functor (152) corresponding to the extension of G by μ_2 induced by the spinor extension of $SO(\mathfrak{g})$. Therefore $\tilde{\ell}'$ is the composition of $\ell : Z\,\mathrm{tors}(X) \to \mu_\infty\,\mathrm{tors}(\mathrm{Bun}_G)$ and the functor $\mu_2\,\mathrm{tors}(X) \to Z\,\mathrm{tors}(X)$ induced by the morphism $\mu_2 \to Z = \pi_1(G)^\vee$ dual to $\pi_1(G) \to \pi_1(SO(\mathfrak{g})) = \mathbb{Z}/2\mathbb{Z}$. So it suffices to prove the following.

4.4.4. *Lemma.* The morphism $\pi_1(G) \to \pi_1(SO(\mathfrak{g})) = \mathbb{Z}/2\mathbb{Z}$ is dual to the morphism (56) for the group ${}^LG$.

Proof. We have the canonical isomorphism $f : P/P_G \xrightarrow{\sim} \mathrm{Hom}(\pi_1(G)(1), \mu_\infty)$ where P_G is the group of weights of G and P is the group of weights of its universal covering $\widetilde{G}$; a weight $\lambda \in P$ is a character of the Cartan subgroup $\widetilde{H} \subset \widetilde{G}$ and $f(\lambda)$ is its restriction to $\pi_1(G)(1) \subset \widetilde{H}$. Let M be a spinor representation of $so(\mathfrak{g})$. Then $\widetilde{G}$ acts on M and $\pi_1(G)(1) \subset \tilde{G}$ acts according to some character $\chi \in \mathrm{Hom}(\pi_1(G)(1), \mu_\infty)$. According to the definition of (56) (see also the definition of $\lambda^\#$ in 3.4.1) the lemma just says that $\chi = f(\rho)$ where $\rho \in P$ is the sum of fundamental weights.

Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra. Choose a $\mathfrak{b}$-invariant flag $0 \subset V_1 \subset \ldots \subset V_n = \mathfrak{g}$ such that $\dim V_k = k$, $V_k^\perp = V_{n-k}$, and $\mathfrak{b}$ is one of the V_k. Let $\mathfrak{b}'$ be the stabilizer of this flag in $so(\mathfrak{g})$. This is a Borel subalgebra of $so(\mathfrak{g})$ containing $\mathfrak{b}$. Let $m \in M$ be a highest vector with respect to $\mathfrak{b}'$. Then $\mathbb{C}m$ is $\mathfrak{b}$-invariant and the corresponding character of $\mathfrak{b}$ equals one half of the sum of the positive roots, i.e., ρ. So $\chi = f(\rho)$. $\square$

Remark. According to Kostant (cf. the proof of Lemma 5.9 from [Ko61]) the $\mathfrak{g}$-module M is isomorphic to the sum of $2^{[r/2]}$ copies of the irreducible $\mathfrak{g}$-module with highest wight ρ (where r is the rank of $\mathfrak{g}$).

4.4.5. Our construction of (211) slightly depends on the choice of a scalar product on $\mathfrak{g}$ (see 4.4.1). Since there are several "canonical" scalar products on $\mathfrak{g}$ the reader may prefer the following version of (211).

To simplify notation let us assume that G is simple. Then the space of invariant symmetric bilinear forms on $\mathfrak{g}$ is 1-dimensional. Denote it by β. Choose a square root of β, i.e., a 1-dimensional vector space $\beta^{1/2}$ equipped with an isomorphism $\beta^{1/2} \otimes \beta^{1/2} \xrightarrow{\sim} \beta$. So $\mathfrak{g} \otimes \beta^{1/2}$ carries a *canonical* bilinear form. Consider the representation $G \to SO(\mathfrak{g} \otimes \beta^{1/2})$ and then proceed as in 4.4.1–4.4.3 (e.g., now the fiber of $\lambda'_{\mathcal{L}}$ over $\mathcal{F} \in \mathrm{Bun}_G$ equals $\mathrm{Pf}(\mathfrak{g}_{\mathcal{F}} \otimes \mathcal{L} \otimes \beta^{1/2}) \otimes \mathrm{Pf}(\mathfrak{g} \otimes \mathcal{L} \otimes \beta^{1/2})^{\otimes -1})$. The functor (211) thus obtained slightly depends on the choice of $\beta^{1/2}$. More precisely, $-1 \in \mathrm{Aut}\, \beta^{1/2}$ acts on $\lambda'_{\mathcal{L}}$ and therefore on $\lambda_{\mathcal{M}}$, $\mathcal{M} \in Z\,\mathrm{tors}_\theta(X)$, as multiplication by $(-1)^p$ where $p : \mathrm{Bun}_G \to \mathbb{Z}/2\mathbb{Z}$ is the composition

$$\mathrm{Bun}_G \to \pi_0(\mathrm{Bun}_G) = \pi_1(G) \to \pi_1(SO(\mathfrak{g})) = \mathbb{Z}/2\mathbb{Z}.$$

Do we want to consider $\lambda_{\mathcal{M}}$ as a SUPER-sheaf??!

4.4.6. We have associated to $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$ a line bundle $\lambda_{\mathcal{L}}$ on Bun_G (see 4.4.1–4.4.3). For $x \in X$ denote by $\lambda_{\mathcal{L},\underline{x}}$ the pullback of $\lambda_{\mathcal{L}}$ to $\mathrm{Bun}_{G,\underline{x}}$. In 4.4.7–4.4.10 we will define a central extension $\widetilde{G(K_x)}_{\mathcal{L}}$ of $G(K_x)$ that acts on $\lambda_{\mathcal{L},\underline{x}}$. In 4.4.11–4.4.13 we consider the Lie algebra of $\widetilde{G(K_x)}_{\mathcal{L}}$.

4.4.7. Let O, K and ω_O have the same meaning as in 4.3.3. Fix a square root $\mathcal{L}$ of ω_O. Then we construct a central extension of group ind-schemes

$$0 \to \mathbb{G}_m \to \widetilde{G(K)}_{\mathcal{L}} \to G(K) \to 0 \tag{212}$$

as follows. $\mathcal{L}$ defines the central extension (196). Fix a non-degenerate invariant symmetric bilinear form*) on $\mathfrak{g}$ and write SO_n instead of $SO(\mathfrak{g})$, $n := \dim \mathfrak{g}$. We define (212) to be the central extension of $G(K)$ *opposite* to the one induced from (196) via the adjoint representation $G \to SO(\mathfrak{g}) = SO_n$. The extension (212) splits over $G(O)$.

Remark. In the case $G = SO_r$ our notation is ambiguous: $\widetilde{G(K)} \neq \widetilde{SO_r(K)}$. Hopefully this ambiguity is harmless.

4.4.8. Let $\mathcal{L} \in \omega^{1/2}(X)$, $x \in X$. According to 4.4.7 the restriction of $\mathcal{L}$ to $\operatorname{Spec} O_x$ defines a central extension of $G(K_x)$, which will be denoted by $\widetilde{G(K_x)}_{\mathcal{L}}$. Denote by $\lambda'_{\mathcal{L},\underline{x}}$ the pullback to $\operatorname{Bun}_{G,\underline{x}}$ of the line bundle $\lambda'_{\mathcal{L}}$ from 4.4.1. It follows from 4.3.7 that the action of $G(K_x)$ on $\operatorname{Bun}_{G,\underline{x}}$ lifts to a canonical action of $\widetilde{G(K_x)}_{\mathcal{L}}$ on $\lambda'_{\mathcal{L}}$. The subgroup $\mathbb{G}_m \subset \widetilde{G(K_x)}_{\mathcal{L}}$ acts on $\lambda'_{\mathcal{L}}$ in the natural way (see the definition of $\widetilde{G(K_x)}_{\mathcal{L}}$ in 4.4.7 and the last sentence of 4.3.7). The action of $G(O_x) \subset \widetilde{G(K_x)}_{\mathcal{L}}$ on $\lambda'_{\mathcal{L},\underline{x}}$ is the obvious one.

4.4.9. In 4.4.7 we defined a functor

$$\omega^{1/2}(O) \to \{\text{central extensions of } G(K) \text{ by } \mathbb{G}_m\} \tag{213}$$

where $\omega^{1/2}(O)$ is the groupoid of square roots of ω_O. The l.h.s. of (213) is a μ_2-category in the sense of 3.4.4. The r.h.s. of (213) is a Z-category, $Z := \pi_1(G)^\vee = \operatorname{Hom}(\pi_1(G), \mathbb{G}_m)$. Indeed, the coboundary morphism*)

$$G(K) \to H^1(K, \pi_1^{\mathrm{et}}(G)) = \pi_1(G) = Z^\vee \tag{214}$$

---

*)Instead of fixing the form on $\mathfrak{g}$ the reader can proceed as in 4.4.5.

*)A priori (214) is a morphism of abstract groups, but according to the Remark from 4.1.7 it is, in fact, a morphism of group ind-schemes. See also 4.5.4.

induces a morphism*)

$$Z \to \operatorname{Hom}(G(K), \mathbb{G}_m)\,, \tag{215}$$

i.e., a Z-structure on the r.h.s. of (213). Using the morphism $\mu_2 \to Z$ defined by (56) we consider the r.h.s. of (213) as a μ_2-category. Then (213) is a μ_2-functor (use 4.3.4, Remark (ii) from 4.3.4, and 4.4.4). So by 3.4.4 the functor (213) yields a Z-functor

$$Z\,\mathrm{tors}_\theta(O) \to \{\text{central extensions of } G(K) \text{ by } \mathbb{G}_m\}\,. \tag{216}$$

The central extension of $G(K)$ corresponding to $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$ by (213) will be denoted by $\widetilde{G(K)}_{\mathcal{L}}$. The extension

$$0 \to \mathbb{G}_m \to \widetilde{G(K)}_{\mathcal{L}} \to G(K) \to 0 \tag{217}$$

splits over $G(O)$.

Remarks

(i) According to 3.4.7 (i) the Z-structure on the r.h.s. of (213) yields a Picard functor

$$Z\,\mathrm{tors}(O) = Z\,\mathrm{tors} \to \{\text{central extensions of } G(K) \text{ by } \mathbb{G}_m\}\,. \tag{218}$$

Explicitly, (218) is the composition of the canonical equivalence

$$\begin{gathered}\{\text{trivial extensions of } Z^\vee \text{ by } \mathbb{G}_m\} = Z\,\mathrm{tors}\\ \text{an extension} \mapsto \text{the } Z\text{-torsor of its splittings}\end{gathered} \tag{219}$$

and the functor from the l.h.s. of (219) to the r.h.s. of (218) induced by (214). In other words, (218) is the functor $\mathcal{E} \mapsto \widetilde{G(K)}_{\mathcal{E}}$ from 4.1.8.

(ii) By 3.4.7 (iv) the functor (216) is affine with respect to the Picard functor (218).

---

*) In fact, an isomorphism (see 4.5.4)

4.4.10. Let $\mathcal{L} \in Z\operatorname{tors}_\theta(X)$. According to 4.4.9 the image of $\mathcal{L}$ in $Z\operatorname{tors}_\theta(O_x)$ defines a central extension of $G(K_x)$, which will be denoted by $\widetilde{G(K_x)}_{\mathcal{L}}$. Denote by $\lambda_{\mathcal{L},\underline{x}}$ the pullback of $\lambda_{\mathcal{L}}$ to $\mathrm{Bun}_{G,\underline{x}}$. The action of $G(K_x)$ on $\mathrm{Bun}_{G,\underline{x}}$ lifts to a canonical action of $\widetilde{G(K_x)}_{\mathcal{L}}$ on $\lambda_{\mathcal{L},\underline{x}}$ (use 4.3.7–4.3.9, 4.1.8, and the Remarks from 4.4.9). $G(O_x) \times \mathbb{G}_m \subset \widetilde{G(K_x)}_{\mathcal{L}}$ acts on $\lambda_{\mathcal{L},\underline{x}}$ in the obvious way.

4.4.11. *Proposition.* The Lie algebra extension corresponding to (217) is the extension

$$0 \to \mathbb{C} \to \widetilde{\mathfrak{g} \otimes K} \to \mathfrak{g} \otimes K \to 0$$

from 2.5.1.

Proof. The Lie algebra extension corresponding to (217) does not depend on $\mathcal{L} \in Z\operatorname{tors}_\theta(O)$, so instead of (217) one can consider (212) and finally (194). So it is enough to use the Kac–Peterson–Frenkel theorem which says that the Lie algebra extension

$$0 \to \mathbb{C} \to \widetilde{o_n(K)} \to o_n(K) \to 0 \tag{220}$$

corresponding to (194) is defined by the cocycle $(u, v) \mapsto \frac{1}{2}\operatorname{Res}\operatorname{Tr}(du, v)$, $u, v \in o_n(K)$. In fact, to use [KP] or Proposition I.3.11 from [Fr81] one has to use the following characterization of $\widetilde{o_n(K)}$ (which does not involve the group $\widetilde{O_n(K)}$): let V have the same meaning as in 4.3.3 and let M be an irreducible discrete module over $\mathrm{Cl}(V)$, then one has a representation of $\widetilde{o_n(K)}$ in M compatible with the action of $\widetilde{o_n(K)}$ on $\mathrm{Cl}(V)$ and such that $1 \in \mathbb{C} \subset \widetilde{o_n(K)}$ acts on M identically. $\square$

4.4.12. Let $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{L},\underline{x}}$ have the same meaning as in 4.4.10. According to 4.4.10 and 4.4.11 the action of $\mathfrak{g} \otimes K_x$ on $\mathrm{Bun}_{G,\underline{x}}$ lifts to a canonical action of $\widetilde{\mathfrak{g} \otimes K_x}$ on $\lambda_{\mathcal{L},\underline{x}}$ whose restriction to $\mathbb{C} \times (\mathfrak{g} \otimes O_x) \subset \widetilde{\mathfrak{g} \otimes K_x}$ is the obvious one; in particular $\mathbf{1} \in \mathbb{C} \subset \widetilde{\mathfrak{g} \otimes K_x}$ acts as multiplication by 1.

$\lambda_{\mathcal{L}}$ is equipped with an isomorphism $\lambda_{\mathcal{L}}^{\otimes 2n} \xrightarrow{\sim} (\omega_{\mathrm{Bun}_G}^{\sharp})^{\otimes n}$ for some $n \neq 0$, so the sheaf of differential operators acting on $\lambda_{\mathcal{L}}$ is D'. Therefore according to 1.2.5 the action of $\widetilde{\mathfrak{g}\otimes K_x}$ on $\lambda_{\mathcal{L},\underline{x}}$ induces a canonical morphism

$$h_x : \mathfrak{z}_x \to \Gamma(\mathrm{Bun}_G, D')\,.$$

Clearly h_x does not depend on $\mathcal{L} \in Z\mathrm{tors}_\theta(X)$.

4.4.13. In this subsection we prove that the h_x from 4.4.12 coincides with the h_x from 2.5.4. The reader can skip this proof and simply forget the old definition of h_x (it was introduced only to avoid the discussion of square roots of ω_{Bun_G} in Section 2).

To prove that the two definitions of h_x are equivalent it suffices to show that if $\mathcal{L}$ is a square root of ω_X then the isomorphism $\lambda_{\mathcal{L}}^{\otimes 2} \xrightarrow{\sim} \omega^\sharp_{\mathrm{Bun}_G}$ induces a $\widetilde{\mathfrak{g}\otimes K_x}$-equivariant isomorphism between their pullbacks to $\mathrm{Bun}_{G,\underline{x}}$. This can be proved directly, but in fact it cannot be otherwise. Indeed, the obstruction to $\widetilde{\mathfrak{g}\otimes K_x}$-equivariance is a 1-cocycle $\widetilde{\mathfrak{g}\otimes K_x} \to H^0(\mathrm{Bun}_{G,\underline{x}}, \mathcal{O})$. Since $\mathrm{Hom}(\widetilde{\mathfrak{g}\otimes K_x}, \mathbb{C}) = 0$ it is enough to show that every regular function f on $\mathrm{Bun}_{G,\underline{x}}$ is locally constant. According to 2.3.1 $\mathrm{Bun}_{G,\underline{x}}$ is the inverse limit of $\mathrm{Bun}_{G,nx}$, $n \in \mathbb{N}$. Clearly f comes from a regular function on $\mathrm{Bun}_{G,nx}$ for some n. So it suffices to prove the following lemma.

Lemma. Every regular function on $\mathrm{Bun}_{G,nx}$ is locally constant.

Proof. Choose $y \in X\backslash\{x\}$ and consider the scheme M parametrizing G-bundles on X trivialized over nx and the formal neighbourhood of y (here the divisor nx is considered as a subscheme). $G(K_y)$ acts on M and a regular function f on $\mathrm{Bun}_{G,nx}$ is a $G(O_y)$-invariant element of $H^0(M, \mathcal{O}_M)$. Clearly $H^0(M, \mathcal{O}_M)$ is an integrable discrete $\mathfrak{g}\otimes K_y$-module. It is well known and very easy to prove that a $(\mathfrak{g}\otimes O_y)$-invariant element of such a module is $(\mathfrak{g}\otimes K_y)$-invariant. So f is $(\mathfrak{g}\otimes K_y)$-invariant. Since the action of $\mathfrak{g}\otimes K_y$ on M is (formally) transitive f is locally constant. $\square$

Remark. The above lemma is well known. A standard way to prove it would be to represent $\mathrm{Bun}_{G,nx}$ as $\Gamma\backslash G(K_y)/G(O_y)$ for some $\Gamma \subset G(K_y)$ (see

[La-So] for the case $n = 0$) and then to use the fact that a regular function on $G(K_y)/G(O_y)$ is locally constant.

4.4.14. Now we will finish the proof of the horizontality theorem 2.7.3 (see 2.8.3 – 2.8.5 for the beginning of the proof).

Let M be the scheme over X whose fiber over $x \in X$ is $\mathrm{Bun}_{G,\underline{x}}$. Fix $\mathcal{L} \in \omega^{1/2}(X)$ and $\mathcal{L}^{\mathrm{loc}} \in \omega^{1/2}(O)$ (i.e., $\mathcal{L}$ is a square root of ω_X, $\mathcal{L}^{\mathrm{loc}}$ is a square root of ω_O). Then one has the scheme $X_2^\wedge$ defined in 4.3.16. Denote by $M_2^\wedge$ the fiber product of M and $X_2^\wedge$ over X. The semidirect product $\mathrm{Aut}_2 O \ltimes G(K)$ acts on $M_2^\wedge$ (cf. 4.3.16).

One has its central extension $\mathrm{Aut}_2 O \ltimes \widetilde{G(K)}$ where $\widetilde{G(K)}$ is the central extension (212) corresponding to $\mathcal{L}^{\mathrm{loc}}$ and $\mathrm{Aut}_2 O = \mathrm{Aut}(O, \mathcal{L}^{\mathrm{loc}})$ acts on $\widetilde{G(K)} = \widetilde{G(K)}_{\mathcal{L}^{\mathrm{loc}}}$ by transport of structure. Denote by $\lambda_{\mathcal{L}}^\wedge$ the pullback to $M_2^\wedge$ of the Pfaffian line bundle $\lambda'_{\mathcal{L}}$ from 4.4.1. Since $\mathrm{Aut}_2 O$ acts on $M_2^\wedge$ as on a scheme over Bun_G one gets the action of $\mathrm{Aut}_2 O$ on $\lambda_{\mathcal{L}}^\wedge$. On the other hand, $\widetilde{G(K)}$ acts on $\lambda_{\mathcal{L},\widehat{x}}^\wedge$:=the restriction of $\lambda_{\mathcal{L}}^\wedge$ to the fiber of $M_2^\wedge$ over $\widehat{x} \in X_2^\wedge$. Indeed, this fiber equals $\mathrm{Bun}_{G,\underline{x}}$ where x is the image of $\widehat{x}$ in X, and by 4.4.8 the central extension $\widetilde{G(K_x)}_{\mathcal{L}}$ acts on $\lambda'_{\mathcal{L},\underline{x}} = \lambda_{\mathcal{L},\widehat{x}}^\wedge$. This extension depends only on $\mathcal{L}_x$:=the pullback of $\mathcal{L}$ to $\mathrm{Spec}\, O_x$. Since $\widehat{x}$ defines an isomorphism $(O_x, \mathcal{L}_x) \xrightarrow{\sim} (O, \mathcal{L}^{\mathrm{loc}})$ we get an isomorphism $\widetilde{G(K_x)}_{\mathcal{L}} \xrightarrow{\sim} \widetilde{G(K)}$ and therefore an action of $\widetilde{G(K)}$ on $\lambda_{\mathcal{L},\widehat{x}}^\wedge$. As explained in 2.8.5, to finish the proof of 2.7.3 it suffices to show that

i) the action of $\widetilde{G(K)}$ on $\lambda_{\mathcal{L},\widehat{x}}^\wedge$ corresponding to various $\widehat{x} \in X_2^\wedge$ come from an (obviously unique) action of $\widetilde{G(K)}$ on $\lambda_{\mathcal{L}}^\wedge$,

ii) this action is compatible with that of $\mathrm{Aut}_2 O$ (i.e., we have, in fact, an action of $\mathrm{Aut}_2 O \ltimes \widetilde{G(K)}$ on $\lambda_{\mathcal{L}}^\wedge$).

This follows immediately from 4.3.17.

4.4.15. In this subsection and the following one we formulate and prove a generalization of statements i) and ii) from 4.4.14, which will be used in the proof of the main result of this work (Theorem 5.4.5). The generalization is obvious ($\omega^{1/2}(X)$ is replaced by $Z\,\mathrm{tors}_\theta(X)$, etc.), and the reader can certainly skip these subsections for the moment.

Fix $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$ and $\mathcal{L}^{\mathrm{loc}} \in Z\,\mathrm{tors}_\theta(O)$. Denote by $X^\wedge_Z$ the etale Z-covering of $X^\wedge$ such that the preimage in $X^\wedge_Z(R)$ of a point of $X^\wedge(R)$ corresponding to a morphism $\alpha : \mathrm{Spec}(R\widehat{\otimes}O) \to X$ is the set of isomorphisms $\mathcal{L}^{\mathrm{loc}}_R \xrightarrow{\sim} \alpha^*\mathcal{L}$ in the groupoid*) $Z\,\mathrm{tors}_\theta(R\widehat{\otimes}O)$, where $\mathcal{L}^{\mathrm{loc}}_R$ is the pullback of $\mathcal{L}^{\mathrm{loc}}$ to $\mathrm{Spec}\,R\widehat{\otimes}O$. The group ind-scheme $\mathrm{Aut}_Z\,O = \mathrm{Aut}(O, \mathcal{L}^{\mathrm{loc}})$ from 4.6.6 acts on $X^\wedge_Z$ by transport of structure. Denote by $M^\wedge_Z$ the fiber product of M and $X^\wedge_Z$ over X. Let $\lambda^\wedge_{\mathcal{L}}$ denote the pullback to $M^\wedge_Z$ of the line bundle $\lambda_{\mathcal{L}}$ defined in 4.4.3. The semidirect product $\mathrm{Aut}_Z\,O \ltimes G(K)$ acts on $M^\wedge_Z$. One has its central extension $\mathrm{Aut}_Z\,O \ltimes \widetilde{G(K)}$, where $\widetilde{G(K)}$ is the central extension (217) corresponding to $\mathcal{L}^{\mathrm{loc}}$ and $\mathrm{Aut}_Z\,O = \mathrm{Aut}(O, \mathcal{L}^{\mathrm{loc}})$ acts on $\widetilde{G(K)} = \widetilde{G(K)}_{\mathcal{L}^{\mathrm{loc}}}$ by transport of structure. Let us lift the action of $\mathrm{Aut}_Z\,O \ltimes G(K)$ on $M^\wedge_Z$ to an action of $\mathrm{Aut}_Z\,O \ltimes \widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L}}$.

Just as in 4.4.14 one defines the action of $\mathrm{Aut}_Z\,O$ on $\lambda^\wedge_{\mathcal{L}}$ and the action of $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L},\widehat{x}}$:=the restriction of $\lambda^\wedge_{\mathcal{L}}$ to the fiber of $M^\wedge_Z$ over $\widehat{x} \in \widehat{X}_Z$.

4.4.16. *Proposition.*

(i) The actions of $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L},\widehat{x}}$ corresponding to various $\widehat{x} \in X^\wedge_Z$ come from an (obviously unique) action of $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L}}$.

(ii) The actions of $\mathrm{Aut}_Z\,O$ and $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L}}$ define an action of $\mathrm{Aut}_Z\,O \ltimes \widetilde{G(K)}$.

Proof. Represent $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$ as $\mathcal{L} = \mathcal{E} \cdot \mathcal{L}_0$, $\mathcal{E} \in Z\,\mathrm{tors}(X)$, $\mathcal{L}_0 \in \omega^{1/2}(X)$. By definition, $\lambda_{\mathcal{L}} = l_{\mathcal{E}} \otimes \lambda'_{\mathcal{L}_0}$ (see 4.1.4 or 4.1.6 for the definition of the μ_∞-torsor $l_{\mathcal{E}}$ on Bun_G).

Consider $\mathcal{L}^{\mathrm{loc}}$ as an object of $\omega^{1/2}(O)$ (this is possible because both $Z\,\mathrm{tors}_\theta(O)$ and $\omega^{1/2}(O)$ have one and only one isomorphism class of objects). Using $\mathcal{L}_0$ and $\mathcal{L}^{\mathrm{loc}}$ construct $X^\wedge_2$, $M^\wedge_2$, and $\lambda^\wedge_{\mathcal{L}_0}$ (see 4.4.14).

---

*) Here it is convenient to use the definition $Z\,\mathrm{tors}_\theta$ from 3.4.5

Consider $\mathcal{E}$ as a Z-covering $\mathcal{E} \to X$. Set $X^\wedge_{\mathcal{E}} := \mathcal{E} \times_X X^\wedge$, $M^\wedge_{\mathcal{E}} := \mathcal{E} \times_X M^\wedge$, where $X^\wedge$ and $M^\wedge$ have the same meaning as in 2.6.5 and 2.8.3. Denote by $l^\wedge_{\mathcal{E}}$ the pullback of $l_{\mathcal{E}}$ to $M^\wedge_{\mathcal{E}}$.

Set $M^\wedge_{\mathcal{E},2} := \mathcal{E} \times_X M^\wedge_2$. One has the etale coverings $M^\wedge_{\mathcal{E},2} \to M^\wedge_2$, $M^\wedge_{\mathcal{E},2} \to M^\wedge_{\mathcal{E}}$, and $p : M^\wedge_{\mathcal{E},2} \to M^\wedge_Z$. Clearly $p^*\lambda^\wedge_{\mathcal{L}}$ is the tensor product of the pullbacks of $l^\wedge_{\mathcal{E}}$ and $\lambda^\wedge_{\mathcal{L}_0}$ to $M^\wedge_{\mathcal{E},2}$. Now consider $l^\wedge_{\mathcal{E}}$ and $\lambda^\wedge_{\mathcal{L}_0}$ separately.

The semidirect product $\operatorname{Aut} O \ltimes G(K)$ acts on $M^\wedge_{\mathcal{E}}$, and the action of $\operatorname{Aut} O$ on $M^\wedge_{\mathcal{E}}$ lifts canonically to its action on $l^\wedge_{\mathcal{E}}$ (cf. 4.4.14 or 2.8.5). $G(K)$ acts on the restriction of $l^\wedge_{\mathcal{E}}$ to the fiber over each point of $X^\wedge_{\mathcal{E}}$ (see 4.1.7). It is easy to see that these actions come from an action of $\operatorname{Aut} O \ltimes G(K)$ on $l^\wedge_{\mathcal{E}}$. On the other hand, by 4.4.14 we have a canonical action of $\operatorname{Aut}_2 O \ltimes \widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L}_0}$.

So we get an action of $\operatorname{Aut}_2 O \ltimes \widetilde{G(K)}$ on $p^*\lambda^\wedge_{\mathcal{L}}$, which is compatible with the action of $\operatorname{Aut}_2 O$ on $\lambda^\wedge_{\mathcal{L}}$ and with the action of $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L},\widehat{x}}$, $\widehat{x} \in X^\wedge_Z$. Since p is etale and surjective the action of $\operatorname{Aut}_2 O \ltimes \widetilde{G(K)}$ on $p^*\lambda^\wedge_{\mathcal{L}}$ descends to an action of $\operatorname{Aut}_2 O \ltimes \widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L}}$. Since $\operatorname{Aut}_Z O$ is generated by $\operatorname{Aut}_2 O$ and Z it remains to show that the action of $Z \subset \operatorname{Aut}_Z O$ on $\lambda^\wedge_{\mathcal{L}}$ is compatible with that of $\widetilde{G(K)}$. This is clear because the actions of Z and $\widetilde{G(K)}$ on $\lambda^\wedge_{\mathcal{L},\widehat{x}}$ are compatible for every $\widehat{x} \in X^\wedge_Z$. $\square$

4.5. **The affine Grassmannian.** The *affine Grassmannian* $\mathcal{GR}$ is the fpqc quotient $G(K)/G(O)$ where $O = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$. In this section we recall some basic properties of $\mathcal{GR}$. In 4.6 we construct and investigate the *local Pfaffian bundle*; this is a line bundle on $\mathcal{GR}$.

The affine Grassmannian will play an essential role in the proof of our main theorem 5.2.6. However the reader can skip this section for the moment.

In 4.5.1 – ? G denotes an arbitrary connected affine algebraic group. Connectedness is a harmless assumption because $G(K)/G(O) = G^0(K)/G^0(O)$ where G^0 is the connected component of G.

4.5.1. *Theorem.*

(i) The fpqc quotient $G(K)/G(O)$ is an ind-scheme of ind-finite type.

(ii) $G(K)/G(O)$ is formally smooth.*)

(iii) The projection $p: G(K) \to G(K)/G(O)$ admits a section locally for the Zariski topology.

(iv) $G(K)/G(O)$ is ind-proper if and only if G is reductive.

(v) $G(K)$, or equivalently $G(K)/G(O)$, is reduced if and only if $\mathrm{Hom}(G, \mathbb{G}_m) = 0$.

Remark. The theorem is well known. The essential part of the proof given below consists of references to works by Faltings, Beauville, Laszlo, and Sorger.

Proof. (i) and (iv) hold for $G = GL_n$. Indeed, there is an ind-proper ind-scheme $Gr(K^n)$ parametrizing c-lattices in K^n (see 7.11.2(iii) for details). $GL_n(K)/GL_n(O)$ is identified with the closed sub-ind-scheme of $Gr(K^n)$ parametrizing O-invariant c-lattices. To prove (i) and (iv) for any G we need the following lemma.

Lemma. Let $G_1 \subset G_2$ be affine algebraic groups such that the quotient $U := G_1 \setminus G_2$ is quasiaffine, i.e., U is an open subscheme of an affine scheme Z. Suppose that the fpqc quotient $G_2(K)/G_2(O)$ is an ind-scheme of ind-finite type. Then this also holds for $G_1(K)/G_1(O)$ and the morphism

$$G_1(K)/G_1(O) \to G_2(K)/G_2(O) \tag{221}$$

is a locally closed embedding. If U is affine then (221) is a closed embedding.

The reader can easily prove the lemma using the global interpretation of $G(K)/G(O)$ from 4.5.2. We prefer to give a local proof.

Proof. Consider the morphism $f: G_1(K) \to Z(K)$. Clearly $Z(O)$ is a closed subscheme of $Z(K)$, and $U(O)$ is an open subscheme of $Z(O)$. So $Y := f^{-1}(U(O))$ is a locally closed sub-ind-scheme of $G_2(K)$; it is closed if

*) The definition of formal smoothness can be found in 7.11.1.

U is affine. Clearly $Y \cdot G_2(O) = Y$, so Y is the preimage of a locally closed sub-ind-scheme $Y' \subset G_2(K)/G_2(O)$; if U is affine then Y' is closed. Since $G_1(K) \subset Y$ we have a natural morphism

$$G_1(K) \to Y' . \tag{222}$$

We claim that (222) is a $G_1(O)$-torsor ($G_1(O)$ acts on $G_1(K)$ by right translations) and therefore $G_1(K)/G_1(O) = Y'$. To see that (222) is a $G_1(O)$-torsor notice that the morphism $Y \to Y'$ is a $G_2(O)$-torsor, the morphism $\varphi : Y \to U(O) = G_1(O) \setminus G_2(O)$ is $G_2(O)$-equivariant, and $G_1(K) = \varphi^{-1}(\overline{e})$ where $\overline{e} \in G_1(O) \setminus G_2(O)$ is the image of $e \in G_2(O)$. $\square$

Let us prove (i) and (iv) for any G. Choose an embedding $G \hookrightarrow GL_n$. If G is reductive then GL_n/G is affine, so the lemma shows that $G(K)/G(O)$ is an ind-proper ind-scheme. For any G we will construct an embedding $i : G \hookrightarrow G' := GL_n \times \mathbb{G}_m$ such that $G'/i(G)$ is quasiaffine; this will imply (i). To construct i take a GL_n-module V such that $G \subset GL_n$ is the stabilizer of some 1-dimensional subspace $l \subset V$. The action of G in l is defined by some $\chi : G \to \mathbb{G}_m$. Define $i : G \hookrightarrow G' := GL_n \times \mathbb{G}_m$ by $i(g) = (g, \chi(g)^{-1})$. To show that $G'/i(G)$ is quasiaffine consider V as a G'-module ($\lambda \in \mathbb{G}_m$ acts as multiplication by λ) and notice that the stabilizer of a nonzero $v \in l$ in G' equals $i(G)$. So $G'/i(G) \simeq G'v$ and $G'v$ is quasiaffine.

Let us finish the proof of (iv). If $G(K)/G(O)$ is ind-proper and G' is a normal subgroup of G then according to the lemma $G'(K)/G'(O)$ is also ind-proper. Clearly $\mathbb{G}_a(K)/\mathbb{G}_a(O)$ is not ind-proper. Therefore $G(K)/G(O)$ is ind-proper only if G is reductive.

To prove (iii) it suffices to show that $p : G(K) \to G(K)/G(O)$ admits a section over a neighbourhood of any $\mathbb{C}$-point $x \in G(K)/G(O)$ (here we use that $\mathbb{C}$-points are dense in $G(K)/G(O)$ by virtue of (i)). Since p is $G(K)$-equivariant we are reduced to the case where x is the image of $e \in G(K)$. So one has to construct a sub-ind-scheme $\Gamma \subset G(K)$ containing e such that the morphism

$$\Gamma \times G(O) \to G(K)\,, \quad (\gamma, g) \mapsto \gamma g \tag{223}$$

is an open immersion. According to Faltings [Fal94, p.350–351] the morphism (223) is an open immersion if the set of R-point of Γ is defined by

$$\Gamma(R) = \mathrm{Ker}(G(R[t^{-1}]) \xrightarrow{f} G(R)) \subset G(R((t))) = G(R\widehat{\otimes}K)$$

where f is evaluation at $t = \infty$. The proof of this statement is due to Beauville and Laszlo (Proposition 1.11 from [BLa94]). It is based on the global interpretation of $G(K)/G(O)$ in terms of $X = \mathbb{P}^1$ (see 4.5.2) and on the following property of G-bundles on $\mathbb{P}^1$: for a G-bundle $\mathcal{F}$ on $S \times \mathbb{P}^1$ the points $s \in S$ such that the restriction of $\mathcal{F}$ to $s \times \mathbb{P}^1$ is trivial form an *open* subset of S (indeed, $H^1(\mathbb{P}^1, \mathcal{O} \otimes \mathfrak{g}) = 0$, $\mathfrak{g} := \mathrm{Lie}\, G$).

Let us deduce[*)] (ii) from (iii). Since $G(K)$ is formally smooth it follows from (iii) that each point of $G(K)/G(O)$ has a formally smooth neighbourhood. Since $G(K)/G(O)$ is of ind-finite type this implies (ii).

It remains to consider (v). $G(O)$ is reduced. So $G(K)$ is reduced if and only if $G(K)/G(O)$ is reduced. Laszlo and Sorger prove that if $\mathrm{Hom}(G, \mathbb{G}_m) = 0$ then $G(K)/G(O)$ is reduced (see the proof of Proposition 4.6 from [La-So]); their proof is based on a theorem of Shafarevich. If $\mathrm{Hom}(G, \mathbb{G}_m) \neq 0$ there exist morphisms $f : \mathbb{G}_m \to G$ and $\chi : G \to \mathbb{G}_m$ such that $\chi f = \varphi_n$, $n \neq 0$, where $\varphi_n(\lambda) := \lambda^n$. The image of the morphism $\mathbb{G}_m(K) \to \mathbb{G}_m(K)$ induced by φ_n is not contained in $\mathbb{G}_m(K)_{\mathrm{red}}$, so $G(K)$ is not reduced. $\square$

4.5.2. Let X be a connected smooth projective curve over $\mathbb{C}$, $x \in X(\mathbb{C})$, O_x the completed local ring of x, and K_x its field of fractions. Then according to Beauville – Laszlo (see 2.3.4) the fpqc quotient $G(K_x)/G(O_x)$ can be

[*)] In fact, one can prove (ii) without using (iii).

interpreted as the moduli space of pairs $(\mathcal{F},\gamma)$ consisting of a principal G-bundle $\mathcal{F}$ on X and its section (=trivialization) $\gamma : X\setminus\{x\}\to\mathcal{F}$: to $(\mathcal{F},\gamma)$ one assigns the image of γ/γ_x in $G(K_x)/G(O_x)$ where γ_x is a section of $\mathcal{F}$ over $\operatorname{Spec} O_x$ and γ/γ_x denotes the element $g\in G(K_x)$ such that $\gamma=g\gamma_x$ (we have identified $G(K_x)/G(O_x)$ with the moduli space of pairs $(\mathcal{F},\gamma)$ at the level of $\mathbb{C}$-points; the readers can easily do it for R-points where R is any $\mathbb{C}$-algebra).

4.5.3. Let us recall the algebraic definition of the topological fundamental group of G. Denote by $\pi_1^{\mathrm{et}}(G)$ the fundamental group of G in Grothendieck's sense. A character $f : G\to\mathbb{G}_m$ induces a morphism $\pi_1^{\mathrm{et}}(G)\to\pi_1^{\mathrm{et}}(\mathbb{G}_m)=\widehat{\mathbb{Z}}(1)$ and therefore a morphism $f_* : (\pi_1^{\mathrm{et}}(G))(-1)\to\widehat{\mathbb{Z}}$. Denote by $\pi_1(G)$ the set of $\alpha\in(\pi_1^{\mathrm{et}}(G))(-1)$ such that $f_*(\alpha)\in\mathbb{Z}$ for all $f\in\operatorname{Hom}(G,\mathbb{G}_m)$. We consider $\pi_1(G)$ as a discrete group. In fact, $\pi_1(G)$ does not change if G is replaced by its maximal reductive quotient. For reductive G one identifies $\pi_1(G)$ with the quotient of the group of coweights of G modulo the coroot lattice.

For any finite covering $p : \widetilde{G}\to G$ one has the coboundary map $G(K)\to H^1(K,A)=A(-1)$, $A:=\operatorname{Ker} p$. These maps yield a homomorphism $G(K)\to(\pi_1^{\mathrm{et}}(G))(-1)$. Its image is contained in $\pi_1(G)$. So we have constructed a canonical homomorphism

$$\varphi : G(K)\to\pi_1(G) \tag{224}$$

where $G(K)$ is understood in the naive sense (i.e., as the group of K-points of G or as the group of $\mathbb{C}$-points of the ind-scheme $G(K)$). The restriction of (224) to $G(O)$ is trivial, so (224) induces a map

$$G(K)/G(O)\to\pi_1(G) \tag{225}$$

where $G(K)/G(O)$ is also understood in the naive sense.

Now consider $G(K)$ and $G(K)/G(O)$ as ind-schemes. The set of $\mathbb{C}$-points of $G(K)/G(O)$ is dense in $G(K)/G(O)$, and the same is true for $G(K)$.

4.5.4. *Proposition.*

(i) The maps (224) and (225) are locally constant.

(ii) The corresponding maps

$$\pi_0(G(K)) \to \pi_1(G) \tag{226}$$

$$\pi_0(G(K)/G(O)) \to \pi_1(G) \tag{227}$$

are bijective.

Proof. We already proved (i) using a global argument (see the Remark at the end of 4.1.7). The same argument can be reformulated using the interpretation of $G(K_x)/G(O_x)$ from 4.5.2: the map (225) equals minus the composition of the natural map $G(K_x)/G(O_x) \to \mathrm{Bun}_G$ and the "first Chern class" map $c: \pi_0(\mathrm{Bun}_G) \to \pi_1(G)$. For a local proof of (i) see 4.5.5.

Now let us prove (ii). The map $\pi_0(G(K)) \to \pi_0(G(K)/G(O))$ is bijective (because G is connected). So it suffices to consider (226). Since G can be represented as a semi-direct product of a reductive group and a unipotent group we can assume that G is reductive. Fix a Cartan subgroup $H \subset G$. We have $\pi_0(H(K)) = \pi_1(H)$ and the composition $\pi_0(H(K)) \to \pi_0(G(K)) \to \pi_1(G)$ is the natural map $\pi_1(H) \to \pi_1(G)$, which is surjective. So (226) is also surjective. The map $\pi_0(H(K)) \to \pi_0(G(K))$ is surjective (use the Bruhat decomposition for the abstract group $G(K)$). Therefore to prove the injectivity of (226) it suffices to show that the kernel of the natural morphism $f: \pi_0(H(K)) \to \pi_1(G)$ is contained in the kernel of $\pi_0(H(K)) \to \pi_0(G(K))$. Since $\operatorname{Ker} f$ is the coroot lattice it is enough to prove that for any coroot $\gamma : \mathbb{G}_m \to H$ the image of $\mathbb{G}_m(K)$ in $G(K)$ belongs to the connected component of $e \in G(K)$. A coroot $\mathbb{G}_m \to H$ extends to a morphism $SL(2) \to G$, so it suffices to notice that $SL(2, K)$ is connected (because any matrix from $SL(2, K)$ can be represented as a product of unipotent matrices). $\square$

In the next subsection we give a local proof of 4.5.4(i).

4.5.5. *Lemma.* Let $M = \operatorname{Spec} R$ be a connected affine variety, A a finite abelian group, $\alpha \in H^1_{\text{et}}(\operatorname{Spec} R((t)), A)$. For $x \in M(\mathbb{C})$ denote by $\alpha(x)$ the restriction of α to the fiber of $\operatorname{Spec} R((t)) \to \operatorname{Spec} R$ over x, so $\alpha(x) \in H^1_{\text{et}}(\operatorname{Spec} \mathbb{C}((t)), A) = A(-1)$. Then $\alpha(x) \in A(-1)$ does not depend on x.

Proof. It suffices to show that for any smooth connected M' and any morphism $M' \to M$ the pullback of α to $M'(\mathbb{C})$ is constant*). So we can assume that M is smooth. Set $V := \operatorname{Spec} R[[t]]$, $V' := \operatorname{Spec} R((t))$. We can assume that $A = \mu_n$. Then α corresponds to a μ_n-torsor on V', i.e., a line bundle $\mathcal{A}$ on V' equipped with an isomorphism $\psi : \mathcal{A}^{\otimes n} \xrightarrow{\sim} \mathcal{O}_{V'}$. Since V is regular $\mathcal{A}$ extends to a line bundle $\widetilde{\mathcal{A}}$ on V. Then ψ induces an isomorphism $\widetilde{\mathcal{A}}^{\otimes n} \xrightarrow{\sim} t^k \mathcal{O}_V$ for some $k \in \mathbb{Z}$. Clearly $\alpha(x) \in \mathbb{Z}/n\mathbb{Z}$ is the image of k. $\square$

Here is a local proof of 4.5.4(i). Since $G(K)/G(O)$ is of ind-finite type it suffices to prove that for every connected affine variety $M = \operatorname{Spec} R$ and any morphism $f : M \to G(K)$ the composition $M(\mathbb{C}) \to G(K) \to \pi_1(G)$ is constant. For any finite abelian group A an exact sequence $0 \to A \to \widetilde{G} \to G \to 0$ defines a map $\pi_1(G) \to A(-1)$ and it is enough to show that the composition $M(\mathbb{C}) \to G(K) \to \pi_1(G) \to A(-1)$ is constant. To prove this apply the lemma to $\alpha = \varphi^*\beta$ where $\varphi : \operatorname{Spec} R((t)) \to G$ corresponds to $f : \operatorname{Spec} R \to G(K)$ and $\beta \in H^1_{\text{et}}(G, A)$ is the class of $\widetilde{G}$ considered as an A-torsor on G.

Remark. In fact, one can prove that for every affine scheme $M = \operatorname{Spec} R$ over $\mathbb{C}$ the "Künneth morphism"

$$(228) \quad H^1_{\text{et}}(M, A) \oplus H^0(M, \mathbb{Z}) \otimes H^1_{\text{et}}(\operatorname{Spec} \mathbb{C}((t)), A) \to H^1_{\text{et}}(M((t)), A),$$
$$M((t)) := \operatorname{Spec} R((t)),$$

is an isomorphism (clearly this implies the lemma). A similar statement holds for any ring R such that the order of A is invertible in R.

---

*)In fact, it is enough to consider only those M' that are smooth curves.

4.5.6. *Proposition.* Let $A \subset G$ be a finite central subgroup, $G' := G/A$.

(i) The morphism $G(K)/G(O) \to G'(K)/G'(O)$ induces an isomorphism between $G(K)/G(O)$ and the union of some connected components of $G'(K)/G'(O)$.

(ii) The morphism $G(K) \to G'(K)$ is an etale covering.

Remark. By 4.5.4 the components mentioned in (i) are labeled by elements of $\mathrm{Im}(\pi_1(G) \to \pi_1(G'))$. The same is true for the connected components of the image of $G(K)$ in $G'(K)$.

Proof. Clearly (i) and (ii) are equivalent.

Let us prove (i) under the assumption of semisimplicity of G (which is equivalent to semisimplicity of G'). In this case the morphism $f : G(K)/G(O) \to G'(K)/G'(O)$ is ind-proper by 4.5.1(iv). By 4.5.4(i) the fibers of f over geometric points*) of components $C \subset G'(K)/G'(O)$ such that $f^{-1}(C) \neq \emptyset$ contain exactly one point, and it is easy to see that these fibers are reduced. By 4.5.1(v) $G'(K)/G'(O)$ is reduced. So in the semisimple case (i) is clear.

Now let us reduce the proof of (ii) to the semisimple case. We can assume that A is cyclic. It suffices to construct a morphism ρ from G to a semisimple group G_1 such that $\rho|_A$ is injective and $\rho(A) \subset G_1$ is central (then the morphism $G(K) \to G'(K)$ is obtained by base change from $G_1(K) \to G_1'(K)$, $G_1' := G_1/\rho(A)$). To construct G_1 and ρ one can proceed as follows. Fix an isomorphism $\chi : A \xrightarrow{\sim} \mu_n$. Let V be a finite-dimensional G-module such that Z acts on V via χ. Denote by W_{pq} the direct sum of p copies of V and q copies of $\mathrm{Sym}^{n-1} V^*$. If $p \cdot \dim V = q(n-1) \cdot \dim \mathrm{Sym}^{n-1} V$ then one can set $G_1 := SL(W_{pq})$ (indeed, the image of $GL(V)$ in $GL(W_{pq})$ is contained in $SL(W_{pq})$). $\square$

Remarks

*) The statement for $\mathbb{C}$-points follows immediately from 4.5.4(i). Since 4.5.4 remains valid if $\mathbb{C}$ is replaced by an algebraically closed field $E \supset \mathbb{C}$ the statement is true for E-points as well.

(i) Proposition 4.5.6 is an immediate consequence of the bijectivity of (228).

(ii) It is easy to prove Proposition 4.5.6 using the global interpretation of $G(K)/G(O)$ from 4.5.2.

4.5.7. Suppose that G is reductive. Denote by G_{ad} the quotient of G by its center. Set $T := G/[G,G]$, $G' := G_{\mathrm{ad}} \times T$. Then $G' = G/A$ for some finite central subgroup $A \subset G$. So by 4.5.6 $G(K)/G(O)$ can be identified with the union of certain connected components of $G'(K)/G'(O) = G_{\mathrm{ad}}(K)/G_{ad}(O) \times T(K)/T(O)$.

The structure of $T(K)/T(O)$ is rather simple. For instance, the reduced part of $\mathbb{G}_m(K)/\mathbb{G}_m(O)$ is the discrete space $\mathbb{Z}$ and the connected component of $1 \in \mathbb{G}_m(K)/\mathbb{G}_m(O)$ is the formal group with Lie algebra K/O.

4.5.8. From now on we assume that G is reductive and set $\mathcal{GR} := G(K)/G(O)$.

Recall that $G(O)$-orbits in $\mathcal{GR}$ are labeled by dominant coweights of G or, which is the same, by $P_+({}^LG) :=$ the set of dominant weights of ${}^LG$. More precisely, $\chi \in P_+({}^LG)$ defines a conjugacy class of morphisms $\nu : \mathbb{G}_m \to G$ and, by definition, Orb_χ is the $G(O)$-orbit of the image of $\nu(\pi)$ in $\mathcal{GR}$ where π is a prime element of O (this image does not depend on the choice of π). Clearly Orb_χ does not depend on the choice of ν inside the conjugacy class, so Orb_χ is well-defined. According to [IM] the map $\chi \mapsto \mathrm{Orb}_\chi$ is a bijection between $P_+({}^LG)$ and the set of $G(O)$-orbits in $\mathcal{GR}$. It is easy to show that

$$\dim \mathrm{Orb}_\chi = (\chi, 2\rho) \tag{229}$$

where 2ρ is the sum of positive roots of G.

Remark. Clearly Orb_χ is $\mathrm{Aut}^0 O$-invariant.

4.5.9. We have the bijection (227) between $\pi_0(\mathcal{GR})$ and $\pi_1(G)$. Let Z be the center of the Langlands dual group ${}^LG$. We identify $\pi_1(G)$ with

$Z^\vee := \mathrm{Hom}(Z, \mathbb{G}_m)$ using the duality between the Cartan tori of G and ${}^L G$. So the connected components of $\mathcal{GR}$ are labeled by elements of $Z^\vee$.

Remark. The connected component of $\mathcal{GR}$ containing Orb_χ corresponds to $\chi_Z \in Z^\vee$ where χ_Z is the restriction of $\chi \in P^+({}^L G)$ to Z.

4.5.10. There is a canonical morphism $\alpha : \mu_2 \to Z$. If G is semisimple we have already defined it by (56). If G is reductive this gives us a morphism $\mu_2 \to Z'$ where Z' is the center of the commutant of ${}^L G$; then we define α to be the composition $\mu_2 \to Z' \hookrightarrow Z$.

According to 4.4.4 the dual morphism $\alpha^\vee : \pi_1(G) \to \mathbb{Z}/2\mathbb{Z}$ is the morphism of fundamental groups that comes from the adjoint representation $G \to SO(\mathfrak{g}_{ss})$, $\mathfrak{g}_{ss} := [\mathfrak{g}, \mathfrak{g}]$.

The composition of (227) and $\alpha^\vee$ defines a locally constant *parity function*

$$p : \mathcal{GR} \to \mathbb{Z}/2\mathbb{Z}\,. \tag{230}$$

We say that a connected component of $\mathcal{GR}$ is *even* (resp. *odd*) if (230) maps it to 0 (resp. 1).

4.5.11. *Proposition.* All the $G(O)$-orbits of an even (resp. odd) component of $\mathcal{GR}$ have even (resp. odd) dimension.

Proof. Let $x = gG(O) \in \mathcal{GR}$. Using the relation between $\alpha^\vee$ and the adjoint representation (see 4.5.10) as well as Remarks (ii) and (iii) from 4.3.4 we see that x belongs to an even component of $\mathcal{GR}$ if and only if

$$\dim \mathfrak{g}_{ss} \otimes O / \big((\mathfrak{g}_{ss} \otimes O) \cap \mathrm{Ad}_g(\mathfrak{g}_{ss} \otimes O) \big) \tag{231}$$

is even. But (231) is the dimension of the $G(O)$-orbit of x. $\square$

Here is another proof. Using (229) and the Remark from 4.5.9 we see that the proposition is equivalent to the formula $\chi_Z(\alpha(-1)) = (-1)^{\langle \chi, 2\rho \rangle}$, which is obvious because according to (56) $\alpha : \mu_2 \to Z$ is the restriction of the morphism $\lambda^\# : \mathbb{G}_m \to H \subset G$ corresponding to 2ρ.

4.5.12. The following properties of $G(O)$-orbits in $\mathcal{GR}$ will not be used in this work but still we think they are worth mentioning.

The closure of Orb_χ is the union of $\mathrm{Orb}_{\chi'}$, $\chi' \leq \chi$. Indeed, if $\rho : G \to GL(V)$ is a representation with lowest weight λ then for $g \in \mathrm{Orb}_\chi$ one has $\rho(g) \in t^{(\chi,\lambda)} \operatorname{End}(V \otimes O)$, $\rho(g) \notin t^{(\chi,\lambda)+1} \operatorname{End}(V \otimes O)$. So if $\mathrm{Orb}_{\chi'} \subset \overline{\mathrm{Orb}_\chi}$ then $(\chi - \chi', \lambda) \leq 0$ for every antidominant weight λ of G and therefore $\chi - \chi'$ is a linear combination of simple coroots of G with non-negative coefficients; by 4.5.4(i) these coefficients are integer, so $\chi' \leq \chi$. On the other hand, a $GL(2)$ computation shows that the set of weights χ' of ${}^L G$ such that $\mathrm{Orb}_{\chi'} \subset \overline{\mathrm{Orb}_\chi}$ is saturated in the sense of [Bour75], Ch. VIII, §7, no. 2. So Proposition 5 from loc.cit shows that $\mathrm{Orb}_{\chi'} \subset \overline{\mathrm{Orb}_\chi}$ for every dominant χ' such that $\chi' \leq \chi$.

The above description of $\overline{\mathrm{Orb}_\chi}$ implies that Orb_χ is closed if and only if χ is minimal. If G is simple then χ is minimal if and only if $\chi = 0$ or χ is a microweight of ${}^L G$ (see [Bour68], Ch. VI, §2, Exercise 5). So on each connected component of $\mathcal{GR}$ there is exactly one closed $G(O)$-orbit (use 4.5.4 and the first part of the exercise from loc.cit). If Orb_χ is closed it is projective, so in this case $G(O)$ acts on Orb_χ via $G = G(O/tO)$ and Orb_χ is the quotient of G by a parabolic subgroup. In terms of 9.1.3 $\mathrm{Orb}_\chi = \mathrm{orb}_\chi = G/P_\chi^-$.

If G is simple then there is exactly one χ such that $\overline{\mathrm{Orb}_\chi} \setminus \mathrm{Orb}_\chi$ consists of a single point*); this χ is the coroot of $\mathfrak{g} := \operatorname{Lie} G$ corresponding to the maximal root $\alpha_{\max}$ of $\mathfrak{g}$ (see [Bour75], Ch. VIII, §7, Exercise 22). In this case $\overline{\mathrm{Orb}_\chi}$ can be described as follows. Set $V := \mathfrak{g} \otimes (\mathfrak{m}^{-1}/O)$ where $\mathfrak{m}$ is the maximal ideal of O. Denote by $\overline{V}$ the projective space containing V as an affine subspace. So $\overline{V}$ is the space of lines in $V \oplus \mathbb{C}$; in particular $V^* = \mathfrak{g}^* \otimes (\mathfrak{m}/\mathfrak{m}^2)$ acts on $\overline{V}$ preserving $0 \in V$. Denote by C the set of elements of V that are G-conjugate to $\mathfrak{g}_{\alpha_{\max}} \otimes (\mathfrak{m}^{-1}/O)$. This is a closed subvariety of V. Its projective closure $\overline{C} \subset \overline{V}$ is V^*-invariant because C is a

*) Of course, this point is the image of $e \in G(K)$.

cone. It is easy to show that the morphism $\exp : C \to G(K)/G(O)$ extends to an isomorphism $f : \overline{C} \xrightarrow{\sim} \overline{\mathrm{Orb}_\chi}$. Clearly f is $\mathrm{Aut}^0 O$-equivariant and G-equivariant. The action of $\mathrm{Ker}(G(O) \to G(O/\mathfrak{m}))$ on $\overline{C}$ induced by its action on $\overline{\mathrm{Orb}_\chi}$ comes from the action of V^* on $\overline{C}$ and the isomorphism

$$\mathrm{Ker}(G(O/\mathfrak{m}^2) \to G(O/\mathfrak{m})) \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} V^*$$

where the last arrow is induced by the invariant scalar product on $\mathfrak{g}$ such that $(\alpha_{\max}, \alpha_{\max}) = 2$.

4.6. **Local Pfaffian bundles.** Consider the affine Grassmannian $\mathcal{GR} := G(K)/G(O)$ where $O = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$. Set $Z := \mathrm{Hom}(\pi_1(G), \mathbb{G}_m)$ (by the Remark from 4.1.1 Z is the center of ${}^L G$). In this subsection we will construct and investigate a functor $\mathcal{L} \mapsto \lambda_{\mathcal{L}} = \lambda_{\mathcal{L}}^{\mathrm{loc}}$ from the groupoid $Z\,\mathrm{tors}_\theta(O)$ (see 3.4.3) to the category of line bundles on $\mathcal{GR}$. We call $\lambda_{\mathcal{L}}$ the *local Pfaffian bundle* corresponding to $\mathcal{L}$.

We recommend the reader to skip this subsection for the moment.

4.6.1. In 4.4.9 we defined a functor $\mathcal{L} \mapsto \widetilde{G(K)}_{\mathcal{L}}$ from $Z\,\mathrm{tors}_\theta(O)$ to the category of central extensions of $G(K)$ by $\mathbb{G}_m$. For $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$ we have the splitting $G(O) \to \widetilde{G(K)}_{\mathcal{L}}$ and therefore the principal $\mathbb{G}_m$-bundle

$$\widetilde{G(K)}_{\mathcal{L}}/G(O) \to G(K)/G(O) = \mathcal{GR}\ . \tag{232}$$

4.6.2. *Definition.* $\lambda_{\mathcal{L}}$ is inverse to the line bundle on $\mathcal{GR}$ corresponding to the $\mathbb{G}_m$-bundle (232).

Clearly $\lambda_{\mathcal{L}}$ depends functorially on $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$.

4.6.3. *Remark.* $\widetilde{G(K)}_{\mathcal{L}}$ depends on the choice of a non-degenerate invariant bilinear form on $\mathfrak{g}$ (see 4.4.7). So this is also true for $\lambda_{\mathcal{L}}$.

4.6.4. Let $\overline{e} \in \mathcal{GR}$ denote the image of the unit $e \in G$. Our $\lambda_{\mathcal{L}}$ is the unique $\widetilde{G(K)}_{\mathcal{L}}$-equivariant line bundle on $\mathcal{GR}$ trivialized over $\overline{e}$ such that any $c \in \mathbb{G}_m \subset \widetilde{G(K)}_{\mathcal{L}}$ acts on $\lambda_{\mathcal{L}}$ as multiplication by c^{-1}. Uniqueness follows from the equality $\mathrm{Hom}(G(O), \mathbb{G}_m) = 0$.

4.6.5. By 4.4.11 the action of $\widetilde{G(K)}_{\mathcal{L}}$ on $\lambda_{\mathcal{L}}$ induces an action of $\widetilde{\mathfrak{g}\otimes K}$ on $\lambda_{\mathcal{L}}$ such that $\mathbf{1}\in\mathbb{C}\subset\widetilde{\mathfrak{g}\otimes K}$ acts as multiplication by -1. It is compatible with the action of $\mathfrak{g}\otimes K$ on $\mathcal{GR}$ by left infinitesimal translations.

4.6.6. The push-forward of (63) by the morphism (56) is an exact sequence

$$0\to Z\to \mathrm{Aut}_Z O\to \mathrm{Aut} O\to 0\,. \tag{233}$$

For any $\mathcal{L}\in Z\,\mathrm{tors}(O)$ the exact sequence

$$0\to Z\to \mathrm{Aut}(O,\mathcal{L})\to \mathrm{Aut}\,O\to 0 \tag{234}$$

can be canonically identified with (233). Here $\mathrm{Aut}(O,\mathcal{L})$ is the group ind-scheme of pairs (σ,φ), $\sigma\in\mathrm{Aut}\,O$, $\varphi:\mathcal{L}\xrightarrow{\sim}\sigma_*\mathcal{L}$ (the reader may prefer to consider $\mathcal{L}$ as an object of the category $\widetilde{Z}\,\mathrm{tors}_\omega(O)$ from 3.4.5). The isomorphism between (233) and (234) is induced by the obvious morphism $\mathrm{Aut}_2\,O:=\mathrm{Aut}(O,\omega_O^{1/2})\to\mathrm{Aut}(O,\mathcal{L})$.

$\mathrm{Aut}_Z\,O=\mathrm{Aut}(O,\mathcal{L})$ acts on the exact sequence (217) by transport of structure; the action of $\mathrm{Aut}_Z\,O$ on $\mathbb{G}_m$ is trivial and its action on $G(K)$ comes from the usual action of $\mathrm{Aut}\,O$ on $G(K)$. The subgroup $G(O)\subset\widetilde{G(K)}_{\mathcal{L}}$ is $\mathrm{Aut}_Z\,O$-invariant.

4.6.7. It follows from 4.6.6 that the action of $\mathrm{Aut}\,O$ on $\mathcal{GR}$ lifts canonically to an action of $\mathrm{Aut}_Z\,O$ on the principal bundle (232) and the line bundle $\lambda_{\mathcal{L}}$. The action of $\mathrm{Aut}_Z\,O$ on $\lambda_{\mathcal{L}}$ induces an action of $\mathrm{Der}\,O=\mathrm{Lie}\,\mathrm{Aut}_Z\,O$ on $\lambda_{\mathcal{L}}$.

4.6.8. The action of $Z=\mathrm{Aut}\,\mathcal{L}$ on the extension (217) comes from (215). So Z acts on $\lambda_{\mathcal{L}}$ via the morphism

$$Z\to H^0(\mathcal{GR},\mathcal{O}^*_{\mathcal{GR}}) \tag{235}$$

inverse to the composition of (215) and the natural embedding $\mathrm{Hom}(G(K),\mathbb{G}_m)\hookrightarrow H^0(\mathcal{GR},\mathcal{O}^*_{\mathcal{GR}})$. Recall that $\pi_0(\mathcal{GR})=Z^\vee$ (see 4.5.9), so $z\in Z$ defines $f_z:\pi_0(\mathcal{GR})\to\mathbb{C}^*$ and (235) is the map $z\mapsto f_z^{-1}$.

4.6.9. *Remark.* (Do we need it ???). Consider the category of line bundles on $\mathcal{GR}$ as a Z-category in the sense of 3.4.4, the Z-structure being defined by (235). By 3.4.7 (i) we have a canonical Picard functor

$$Z\,\mathrm{tors}(O) = Z\,\mathrm{tors} \to \{\text{line bundles on } \mathcal{GR}\}\,. \tag{236}$$

Explicitly, (236) assigns to $\mathcal{E} \in Z\,\mathrm{tors}$ the $\mathcal{E}$-twist of $\mathcal{O}_{\mathcal{GR}}$ equipped with the Z-action (235). By 3.4.7 (iv) the functor $\mathcal{L} \mapsto \lambda_{\mathcal{L}}$, $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$, is affine with respect to the Picard functor (236).

4.6.10. The morphism $\alpha : \mu_2 \to Z$ defined by (56) induces an action of μ_2 on $\lambda_{\mathcal{L}}$, $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$. It defines a $(\mathbb{Z}/2\mathbb{Z})$-grading on $\lambda_{\mathcal{L}}$. In 4.5.10 we introduced the notions of even and odd component of $\mathcal{GR}$. According to 4.6.8 the restriction of the $(\mathbb{Z}/2\mathbb{Z})$-graded bundle $\lambda_{\mathcal{L}}$ to an even (resp. odd) component of $\mathcal{GR}$ is even (resp. odd).

4.6.11. The functor

$$Z\,\mathrm{tors}_\theta(O) \to \{\text{line bundles on } \mathcal{GR}\}, \quad \mathcal{L} \mapsto \lambda_{\mathcal{L}} \tag{237}$$

is a Z-functor in the sense of 3.4.4 provided the Z-structure on the r.h.s. of (237) is defined by (235). Since $Z\,\mathrm{tors}_\theta(O)$ is equivalent to $\omega^{1/2}(O) \otimes_{\mu_2} Z$ (see 3.4.4) the functor (237) is reconstructed from the corresponding functor

$$\omega^{1/2}(O) \to \{\text{line bundles on}\, \mathcal{GR}\} \tag{238}$$

where $\omega^{1/2}(O)$ is the groupoid of square roots of $\omega(O)$. Since the extension (212) essentially comes from the "Clifford extension" (193) it is easy to give a Cliffordian description of (238). Here is the answer.

Let $\mathcal{L} \in \omega^{1/2}(O)$. We have fixed a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, so the Tate space $V = V_{\mathcal{L}} := \mathcal{L} \otimes_O (\mathfrak{g} \otimes K)$ carries a nondegenerate symmetric bilinear form (see 4.3.3) and $L := \mathcal{L} \otimes \mathfrak{g} \subset V$ is a Lagrangian c-lattice. Set $M = M_{\mathcal{L}} := \mathrm{Cl}(V)/\,\mathrm{Cl}(V)L$; this is an irreducible $(\mathbb{Z}/2\mathbb{Z})$-graded discrete module over $\mathrm{Cl}(V)$. We have the line bundle $\mathcal{P}_M$ on the ind-scheme $\mathrm{Lagr}(V)$ of Lagrangian c-lattices in V (see 4.3.2). We claim that

$$\lambda_{\mathcal{L}} = \varphi^* \mathcal{P}_{M_{\mathcal{L}}} \tag{239}$$

where the morphism*) $\varphi : G(K)/G(O) \to \mathrm{Lagr}(V)$ is defined by $\varphi(g) := gLg^{-1}$; in other words

(240) the fiber of $\lambda_{\mathcal{L}}$ over $g \in G(K)/G(O)$ is $M^{gLg^{-1}} := \{m \in M_{\mathcal{L}} | (gLg^{-1}) \cdot m = 0\}$.

Indeed, the central extension (212) is opposite to the one induced from (193) and therefore the action of $\widetilde{O}(V)$ on $\mathcal{P}_{M_{\mathcal{L}}}$ (see 4.3.2) induces an action of $\widetilde{G(K)}_{\mathcal{L}}$ on $\varphi^*\mathcal{P}_{M_{\mathcal{L}}}$ such that $c \in \mathbb{G}_m \subset \widetilde{G(K)}_{\mathcal{L}}$ acts as multiplication by c^{-1}; besides, the fiber of $\varphi^*\mathcal{P}_{M_{\mathcal{L}}}$ over $\overline{e}$ is $\mathbb{C}$.

Clearly the isomorphism (239) is functorial in $\mathcal{L} \in \omega^{1/2}(O)$.

4.6.12. *Remarks*

(i) The line bundle $\mathcal{P}_M$ from 4.3.2 is $(\mathbb{Z}/2\mathbb{Z})$-graded. So both sides of (239) are $(\mathbb{Z}/2\mathbb{Z})$-graded. The gradings of both sides of (239) are induced by the action of $\mu_2 = \mathrm{Aut}\,\mathcal{L}$ (to prove this for the r.h.s. notice that the $(\mathbb{Z}/2\mathbb{Z})$-grading on $\mathrm{Cl}(V)$ is induced by the natural action of μ_2 on V). Therefore (239) is a *graded* isomorphism.

(ii) According to 4.6.10 $-1 \in \mu_2 = \mathrm{Aut}\,\mathcal{L}$ acts on the r.h.s. of (239) as multiplication by $(-1)^p$ where p is the parity function (230). This also follows from the equality $\chi = \theta$ (see the proof of Lemma 4.3.4) and Remark (ii) at the end of 4.3.4.

4.6.13. We should think about super-aspects, in particular: what is the inverse of a 1-dimensional superspace? (maybe this should be formulated in an arbitrary Picard category; there may be troubles if it is not STRICTLY commutative).

---

*)It is easy to show that φ is a closed embedding and its image is the ind-scheme of $\Lambda \in \mathrm{Lagr}(V)$ such that $O\Lambda = \Lambda$ and $\mathcal{L}^{-1} \otimes_O \Lambda$ is a Lie subalgebra of $\mathfrak{g} \otimes K$.

Consider a $G(O)$-orbit $\mathrm{Orb}_\chi \subset \mathcal{GR}$, $\chi \in P_+({}^L G)$ (see 4.5.8). We will compute $\lambda_{\mathcal{L},\chi} :=$ the restriction of $\lambda_{\mathcal{L}}$ to Orb_χ, $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$. By 4.6.4 $\lambda_{\mathcal{L},\chi}$ is $G(O)$-equivariant. The orbit Orb_χ is $\mathrm{Aut}^0 O$-invariant and by 4.6.7 $\lambda_{\mathcal{L},\chi}$ is $\mathrm{Aut}^0_Z O$-equivariant where $\mathrm{Aut}^0_Z O$ is the preimage of $\mathrm{Aut}^0 O$ in $\mathrm{Aut}_Z O$ (see (233)). Finally $\lambda_{\mathcal{L},\chi}$ is $\mathbb{Z}/2\mathbb{Z}$-graded (but in fact $\lambda_{\mathcal{L},\chi}$ is even or odd depending on χ; besides, the $\mathbb{Z}/2\mathbb{Z}$-grading can be reconstructed from the action of $Z \subset \mathrm{Aut}^0_Z O$.....). The groups $G(O)$ and $\mathrm{Aut}^0_Z O$ also act on the canonical sheaf $\omega_{\mathrm{Orb}_\chi}$ ($\mathrm{Aut}^0_Z O$ acts via $\mathrm{Aut}^0 O$). In 4.6.17-4.6.19 (???) we will construct a canonical isomorphism

$$\lambda_{\mathcal{L},\chi} \xrightarrow{\sim} \omega_{\mathrm{Orb}_\chi} \otimes (\mathfrak{d}_{\mathcal{L},\chi})^{-1} \tag{241}$$

for a certain 1-dimensional vector space $\mathfrak{d}_{\mathcal{L},\chi}$. This space is equipped with an action of $G(O)$ and $\mathrm{Aut}^0_Z O$ and (241) is equivariant with respect to these groups.

4.6.14. Let us define $\mathfrak{d}_{\mathcal{L},\chi}$. Of course the action of $G(O)$ on $\mathfrak{d}_{\mathcal{L},\chi}$ is defined to be trivial ($G(O)$ has no nontrivial characters). So we have to construct for each χ a functor

$$Z\,\mathrm{tors}_\theta(O) \to \{\mathrm{Aut}^0_Z O\text{-mod}\}, \quad \mathcal{L} \mapsto \mathfrak{d}_{\mathcal{L},\chi} \tag{242}$$

where $\{\mathrm{Aut}^0_Z O\text{-mod}\}$ denotes the category of $\mathrm{Aut}^0_Z O$-modules. First let us define a functor

$$\omega^{1/2}(O) \to \{\mathrm{Aut}^0_Z O\text{-mod}\}, \quad \mathcal{L} \mapsto \mathfrak{d}_{\mathcal{L},\chi} \tag{243}$$

For $\mathcal{L} \in \omega^{1/2}(O)$ set

$$\mathfrak{d}_{\mathcal{L},\chi} := (\mathcal{L}_0)^{\otimes d(\chi)} \tag{244}$$

where $\mathcal{L}_0$ is the fiber of $\mathcal{L}$ over the closed point $0 \in \mathrm{Spec}\, O$ and

$$d(\chi) := (\chi, 2\rho) = \dim \mathrm{Orb}_\chi \tag{245}$$

Define the representation of $\operatorname{Aut}_Z^0 O$ in $\mathfrak{d}_{\mathcal{L},\chi}$ as follows: $\operatorname{Aut}_2^0 O = \operatorname{Aut}^0(O, \mathcal{L})$ acts in the obvious way and $Z \subset \operatorname{Aut}_Z^0 O$ acts via

$$\chi_Z : Z \to \mathbb{G}_m \tag{246}$$

where χ_Z is the restriction of $\chi \in P^+({}^L G)$ to $Z \subset {}^L G$ (these two actions are compatible because the composition of χ_Z and the morphism (56) maps $-1 \in \mu_2$ to $(-1)^{(\chi, 2\rho)}$).

So we have constructed (243). $\omega^{1/2}(O)$ is a μ_2-category in the sense of 3.4.4, $\{\operatorname{Aut}_Z^0 O\text{-mod}\}$ is a Z-category, and (243) is a μ_2-functor (the μ_2-structure on $\{\operatorname{Aut}_Z^0 O\}$ comes from the morphism (56) or, equivalently, from the canonical embedding $\mu_2 \to \operatorname{Aut}_2^0 O$). So (243) induces a Z-functor $Z \operatorname{tors}_\theta(O) = \omega^{1/2}(O) \otimes_{\mu_2} Z \to \{\operatorname{Aut}_Z^0 O\text{-mod}\}$. This is the definition of (242).

4.6.15. Clearly $\operatorname{Lie} \operatorname{Aut}_Z^0 O = \operatorname{Der}^0 O$ acts on the one-dimensional space $\mathfrak{d}_{\mathcal{L},\chi}$ as follows:

$$L_0 \mapsto (\chi, \rho) = -\frac{1}{2} \dim \operatorname{Orb}_\chi \;, \quad L_n \mapsto 0 \text{ for } n > 0 \tag{247}$$

As usual, $L_n := -t^{n+1} \frac{d}{dt} \in \operatorname{Der}^0 O$.

4.6.16. *Remark.* The definition of $\mathfrak{d}_{\mathcal{L},\chi}$ from 4.6.14 can be reformulated as follows. Using the equivalence $Z \operatorname{tors}_\theta(O) \xrightarrow{\sim} \tilde{Z} \operatorname{tors}_\omega(O)$ from 3.4.5 we interpret $\mathcal{L} \in Z \operatorname{tors}_\theta(O)$ in terms of (59) as a lifting of the $\mathbb{G}_m$-torsor ω_O to a $\tilde{Z}$-torsor. We have the canonical morphism $\tilde{Z} \to {}^L H$ from (62) where ${}^L H$ is *the* Cartan torus of ${}^L G$ or, which is the same, ${}^L H$ is a Cartan subgroup of ${}^L G$ with a fixed Borel subgroup containing it. Denote by $\chi_{\tilde{Z}}$ the composition of $\tilde{Z} \to {}^L H$ and $\chi : {}^L H \to \mathbb{G}_m$. The $\tilde{Z}$-torsor $\mathcal{L}$ on $\operatorname{Spec} O$ and the 1-dimensional representation $\chi_{\tilde{Z}} : \tilde{Z} \to \mathbb{G}_m$ define a line bundle $\mathfrak{d}_{\mathcal{L},\chi}^O$ on $\operatorname{Spec} O$. According to 4.6.6 $\operatorname{Aut}_Z O = \operatorname{Aut}(O, \mathcal{L})$, so the action of $\operatorname{Aut} O$ on $\operatorname{Spec} O$ lifts to a canonical action of $\operatorname{Aut}_Z O$ on $\mathfrak{d}_{\mathcal{L},\chi}^O$. Therefore $\operatorname{Aut}_Z^0 O$ acts on the fiber of $\mathfrak{d}_{\mathcal{L},\chi}^O$ at $0 \in \operatorname{Spec} O$. The reader can easily identify this fiber with the $\mathfrak{d}_{\mathcal{L},\chi}$ from 4.6.14.

4.6.17. Let us construct the isomorphism (241) for $\mathcal{L} \in \omega^{1/2}(O)$. We use the Cliffordian description of $\lambda_{\mathcal{L}}$. Just as in 4.6.11 we set $V = V_{\mathcal{L}} := \mathcal{L} \otimes_O (\mathfrak{g} \otimes K)$, $L := \mathcal{L} \otimes \mathfrak{g} \subset V$, $M = M_{\mathcal{L}} := \mathrm{Cl}(V)/\mathrm{Cl}(V)L$. For $x \in \mathcal{GR} = G(K)/G(O)$ set $L_x := gLg^{-1}$ where g is a preimage of x in $G(K)$. By (240) the fiber of $\lambda_{\mathcal{L}}$ at x equals

$$M^{L_x} := \{m \in M_{\mathcal{L}} | L_x \cdot m = 0\} \tag{248}$$

Suppose that $x \in \mathrm{Orb}_\chi$. Since Orb_χ is the $G(O)$-orbit of x the tangent space to Orb_χ at x is $(\mathfrak{g} \otimes O)/((\mathfrak{g} \otimes O) \cap g(\mathfrak{g} \otimes O)g^{-1}) = \mathcal{L}^{-1} \otimes_O (L/(L \cap L_x))$ where $g \in G(K)$ is a preimage of x. So the fiber of $\omega^{-1}_{\mathrm{Orb}_\chi}$ at x equals $(\mathcal{L}_0)^{\otimes - d(\chi)} \otimes \det(L/(L \cap L_x))$ where $d(\chi) = \dim \mathrm{Orb}_\chi$. Taking (244) into account we see that the fiber of the r.h.s. of (241) at x equals

$$(\det(L/(L \cap L_x)))^{-1} \tag{249}$$

So it remains to construct an isomorphism

$$\det(L/(L \cap L_x)) \otimes M^{L_x} \xrightarrow{\sim} \mathbb{C} \tag{250}$$

4.6.18. *Lemma.* Consider a Tate space V equipped with a nondegenerate symmetric bilinear form. Let $L, \Lambda \subset V$ be Lagrangian c-lattices and M an irreducible discrete module over the Clifford algebra $\mathrm{Cl}(V)$. Consider the operator

$$\bigwedge^d L \otimes M \to M \tag{251}$$

induced by the natural map $\bigwedge^d L \to \bigwedge^d V \to \mathrm{Cl}(V)$. If $d = \dim L/(L \cap \Lambda)$ then (251) induces an isomorphism

$$\bigwedge^d (L/(L \cap \Lambda)) \otimes M^\Lambda \xrightarrow{\sim} M^L \tag{252}$$

The proof is reduced to the case where $\dim V < \infty$ and $V = L \oplus \Lambda$.

4.6.19. We define (250) to be the isomorphism (252) for $\Lambda = L_x$ (in the situation of 4.6.17 $M^L = \mathbb{C}$). So for $\mathcal{L} \in \omega^{1/2}(O)$ we have constructed the isomorphism (241), which is equivariant with respect to $G(O)$ and $\operatorname{Aut}^0_2 O = \operatorname{Aut}^0(O, \mathcal{L})$.

Denote by C_χ the category of line bundles on Orb_χ. Both sides of (241) are μ_2-functors $\omega^{1/2}(O) \to C_\chi$ extended to Z-functors

$$Z \operatorname{tors}_\theta(O) = \omega^{1/2}(O) \otimes_{\mu_2} Z \to C_\chi$$

(the Z-structure on C_χ is defined by the character of Z inverse to (246)); for the l.h.s of (241) this follows from 4.6.8. Clearly (241) is an isomorphism of functors $\omega^{1/2}(O) \to C_\chi$. Therefore (241) is an isomorphism of functors $Z \operatorname{tors}_\theta(O) \to C_\chi$. The isomorphism (241) is $\operatorname{Aut}^0_Z O$-equivariant because it is $\operatorname{Aut}^0_2 O$-equivariant and Z-equivariant.

4.6.20. Recall that $\lambda_{\mathcal{L}}$ depends on the choice of a nondegenerate invariant bilinear form on $\mathfrak{g}$ (see 4.6.3 and 4.4.7). As explained in the footnote to 4.4.7 there is a more canonical version of $\lambda_{\mathcal{L}}$. In the case where G is simple this version $\lambda^{\mathrm{can}}_{\mathcal{L}}$ depends on the choice of $\beta^{1/2}$ where β is the line of invariant bilinear forms on $\mathfrak{g}$ (cf. 4.4.5); $\lambda^{\mathrm{can}}_{\mathcal{L}}$ comes from the version of (212) obtained by using $SO(\mathfrak{g} \otimes \beta^{1/2})$ instead of $SO(\mathfrak{g})$. It is easy to see that the $(\mathbb{Z}/2\mathbb{Z})$-grading on $\lambda^{\mathrm{can}}_{\mathcal{L}}$, corresponding to the action of $-1 \in \operatorname{Aut} \beta^{1/2}$ coincides with the grading from 4.6.10. The "canonical" version of (241) is an isomorphism

$$\lambda^{\mathrm{can}}_{\mathcal{L},\chi} \xrightarrow{\sim} \omega_{\mathrm{Orb}_\chi} \otimes (\mathfrak{d}_{\mathcal{L},\chi})^{-1} \otimes (\beta^{1/2})^{\otimes -d(\chi)} \tag{253}$$

where $d(\chi)$ is defined by (245). Details are left to the reader.

5. Hecke eigen-$\mathcal{D}$-modules

5.1. Construction of $\mathcal{D}$-modules.

5.1.1. In this subsection we construct a family of $\mathcal{D}$-modules on Bun_G parametrized by $\mathcal{O}\mathfrak{p}_{^L G}(X)$, i.e., the stack of $^L G$-opers on X.

Denote by Z the center of $^L G$. According to formula (57) from 3.4.3 we must associate to $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$ a family of $\mathcal{D}$-modules on Bun_G parametrized by $\mathcal{O}\mathfrak{p}_{^L\mathfrak{g}}(X)$. In 4.4.3 we defined $\lambda_\mathcal{L} \in \mu_\infty\,\mathrm{tors}_\theta(\mathrm{Bun}_G)$. $\lambda_\mathcal{L}$ is a line bundle on Bun_G equipped with an isomorphism $\lambda_\mathcal{L}^{\otimes 2n} \xrightarrow{\sim} (\omega^\sharp_{\mathrm{Bun}_G})^{\otimes n}$ for some $n \neq 0$ (see 4.0.1). So $\lambda_\mathcal{L}$ is a $\mathcal{D}'$-module. Therefore $M_\mathcal{L} := \lambda_\mathcal{L}^{-1} \bigotimes_{\mathcal{O}_{\mathrm{Bun}_G}} \mathcal{D}'$ is a left $\mathcal{D}$-module on Bun_G. According to 3.3.2 and 2.7.4 there is a canonical morphism of algebras $h_X\varphi_X : A_{^L\mathfrak{g}}(X) \to \Gamma(\mathrm{Bun}_G, D')$. So the right action of $\Gamma(\mathrm{Bun}_G, D')$ on $\mathcal{D}'$ yields an $A_{^L\mathfrak{g}}(X)$-module structure on $M_\mathcal{L}$. Therefore we may consider $M_\mathcal{L}$ as a family of left $\mathcal{D}$-modules on Bun_G parametrized by $\mathrm{Spec}\, A_{^L\mathfrak{g}}(X) = \mathcal{O}\mathfrak{p}_{^L\mathfrak{g}}(X)$.

So we have constructed a family of left $\mathcal{D}$-modules on Bun_G parametrized by $\mathcal{O}\mathfrak{p}_{^L G}(X)$. For an $^L G$-oper $\mathfrak{F}$ the corresponding $\mathcal{D}$-module $M_\mathfrak{F}$ is $M_\mathcal{L}/\mathfrak{m}_\mathfrak{F} M_\mathcal{L} = \lambda_\mathcal{L}^{-1} \otimes \mathcal{D}'/\mathcal{D}'\mathfrak{m}_\mathfrak{F}$ where $\mathcal{L}$ is the image of $\mathfrak{F}$ in $Z\,\mathrm{tors}_\theta(X)$ and $\mathfrak{m}_\mathfrak{F} \subset A_{^L\mathfrak{g}}(X)$ is the maximal ideal of the $^L\mathfrak{g}$-oper corresponding to $\mathfrak{F}$.

5.1.2. *Proposition.*

(i) For every $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$ $M_\mathcal{L}$ is flat over $A_{^L\mathfrak{g}}(X)$.

(ii) For every $^L G$-oper $\mathfrak{F}$ the $\mathcal{D}$-module $M_\mathfrak{F}$ is holonomic. Its singular support coincides as a cycle with the zero fiber of Hitchin's fibration.

Proof. According to 2.2.4 (iii) $\mathrm{gr}\,\mathcal{D}'$ is flat*) over $\mathrm{gr}\, A_{^L\mathfrak{g}}(X)$. So $\mathcal{D}'$ is flat over $A_{^L\mathfrak{g}}(X)$. This implies i) and the equality $\mathrm{gr}(\mathcal{D}'/\mathcal{D}'I) = \mathrm{gr}\,\mathcal{D}'/(\mathrm{gr}\,\mathcal{D}' \cdot \mathrm{gr}\, I)$ for any ideal $I \subset A_{^L\mathfrak{g}}(X)$. If I is maximal we obtain ii). □

---

*)This means that if $f : S \to \mathrm{Bun}_G$ is smooth and S is affine $\Gamma(S, f^*\,\mathrm{gr}\,\mathcal{D}')$ is a *free* module over $\mathrm{gr}\, A_{^L\mathfrak{g}}(X)$ (a flat $\mathbb{Z}_+$-graded module over a $\mathbb{Z}_+$-graded ring A with $A_0 = \mathbb{C}$ is free).

5.2. **Main theorems I: an introduction.**

5.2.1. Our main global theorem 5.2.6 asserts that the $\mathcal{D}$-module $M_{\mathfrak{F}}$ is an eigenmodule of the *Hecke functors*. In order to define them we introduce the *big Hecke stack* $\mathcal{H}$ecke. The groupoid of S-points $\mathcal{H}\mathrm{ecke}(S)$ consists of quadruples $(\mathcal{F}_1, \mathcal{F}_2, x, \alpha)$ where $\mathcal{F}_1$, $\mathcal{F}_2$ are G-torsors on $X \times S$, $x \in X(S)$, and $\alpha : \mathcal{F}_1|_U \xrightarrow{\sim} \mathcal{F}_2|_U$ is an isomorphism over the complement U to the graph of x. One has the obvious projection $p_{1,2,X} = (p_1, p_2, p_X) : \mathcal{H}\mathrm{ecke} \to \mathrm{Bun}_G \times \mathrm{Bun}_G \times X$.

The stack $\mathcal{H}$ecke is ind-algebraic and the projections p_i, $p_{i,X}$ are ind-proper. Precisely, there is an increasing family of closed algebraic substacks $\mathcal{H}\mathrm{ecke}_1 \subset \mathcal{H}\mathrm{ecke}_2 \subset \cdots \subset \mathcal{H}\mathrm{ecke}$ such that $\mathcal{H}\mathrm{ecke} = \bigcup \mathcal{H}\mathrm{ecke}_a$ and $p_i : \mathcal{H}\mathrm{ecke}_a \to \mathrm{Bun}_G$, $p_{i,X} : \mathcal{H}\mathrm{ecke}_a \to \mathrm{Bun}_G \times X$ are proper morphisms.

5.2.2. *Remarks.* (i) The composition of α's makes $\mathcal{H}$ecke an X-family of groupoids on Bun_G.

(ii) $\mathcal{H}$ecke is a family of twisted affine Grassmannians over $\mathrm{Bun}_G \times X$. Precisely, for $(\mathcal{F}_2, x) \in \mathrm{Bun}_G \times X$ the fiber $\mathcal{H}\mathrm{ecke}_{(\mathcal{F}_2,x)} := p_{2,X}^{-1}(\mathcal{F}_2, x)$ is canonically isomorphic to the affine Grassmannian $\mathcal{GR}_x := G(K_x)/G(O_x)$ twisted by the $G(O_x)$-torsor $\mathcal{F}_2(O_x)$ (with respect to the left $G(O_x)$-action). In the case where $\mathcal{F}_2$ is the trivial bundle we described this isomorphism in 4.5.2. In the general case the construction is similar: for fixed $\gamma_2 \in \mathcal{F}_2(O_x)$ we assign to $(\mathcal{F}_1, \mathcal{F}_2, x, \alpha)$ the image of $\gamma_2/\alpha(\gamma_1)$ in $G(K_x)/G(O_x)$ where γ_1 is any element of $\mathcal{F}_1(O_x)$ and $\gamma_2/\alpha(\gamma_1)$ denotes the element $g \in G(K_x)$ such that $g\alpha(\gamma_1) = \gamma_2$; by 2.3.4 the morphism $\mathcal{H}\mathrm{ecke}_{(\mathcal{F}_2,x)} \to G(K_x)/G(O_x)$ is an isomorphism.

5.2.3. The set of conjugacy classes of morphisms $\nu : \mathbb{G}_m \to G$ can be canonically identified with the set $P_+({}^L G)$ of dominant weights of ${}^L G$. Recall that $G(O_x)$-orbits in $\mathcal{GR}_x = G(K_x)/G(O_x)$ are labeled by $\chi \in P_+({}^L G)$; by definition, Orb_χ is the orbit of the image of $\nu(t_x) \in G(K_x)$ in $\mathcal{GR}_x$ where $\nu : \mathbb{G}_m \to G$ is of class χ and $t_x \in O_x$ is a uniformizer.

According to 5.2.2 (ii) the stratification of $\mathcal{GR}_x$ by Orb_χ yields a stratification of the stack $\mathcal{H}$ecke by substacks $\mathcal{H}\mathrm{ecke}_\chi$, $\chi \in P_+({}^LG)$. The $\mathbb{C}$-points of $\mathcal{H}\mathrm{ecke}_\chi$ are quadruples $(\mathcal{F}_1, \mathcal{F}_2, x, \alpha)$ such that for some $\gamma_i \in \mathcal{F}_i(O_x)$ and a formal parameter t_x at x one has $\gamma_2 = \nu(t_x)\alpha(\gamma_1)$ where $\nu : \mathbb{G}_m \to G$ is of class χ. The involution $(\mathcal{F}_1, \mathcal{F}_2, x, \alpha) \mapsto (\mathcal{F}_2, \mathcal{F}_1, x, \alpha^{-1})$ identifies $\mathcal{H}\mathrm{ecke}_\chi$ with $\mathcal{H}\mathrm{ecke}_{\chi^\circ}$ where χ° is the dual weight. So the fibers of $p_{2,X} : \mathcal{H}\mathrm{ecke}_\chi \to \mathrm{Bun}_G \times X$ are twisted forms of Orb_χ while the fibers of $p_{1,X} : \mathcal{H}\mathrm{ecke}_\chi \to \mathrm{Bun}_G \times X$ are twisted forms of $\mathrm{Orb}_{\chi^\circ}$.

For every χ the stack $\mathcal{H}\mathrm{ecke}_\chi$ is smooth over $\mathrm{Bun}_G \times X$. Usually its closure $\overline{\mathcal{H}\mathrm{ecke}_\chi}$ is not smooth.

Remarks. (i) According to 4.5.12 $\overline{\mathcal{H}\mathrm{ecke}_\chi}$ is the union of the strata $\mathcal{H}\mathrm{ecke}_{\chi'}$, $\chi' \leq \chi$.

(ii) If $G = GL(n)$ then our labeling of strata coincides with the "natural" one. Namely, let V_1, V_2 be the vector bundles corresponding to $\mathfrak{F}_1, \mathfrak{F}_2$. Then $\mathcal{H}\mathrm{ecke}_\chi$ consists of all collections (V_1, V_2, x, α) such that for certain bases of V_i's on the formal neighbourhood of x the matrix of α equals t_x^χ.

5.2.4. Let us define the Hecke functors $T_\chi^i : \mathcal{M}(\mathrm{Bun}_G) \to \mathcal{M}(\mathrm{Bun}_G \times X)$ where $\mathcal{M}$ denotes the category of $\mathcal{D}$-modules, $\chi \in P_+({}^LG)$, $i \in \mathbb{Z}$.

For $\chi \in P_+({}^LG)$, $M \in \mathcal{M}(\mathrm{Bun}_G)$ denote by $p_{1\chi}^\star M$ the minimal (= Goresky–MacPherson) extension to $\overline{\mathcal{H}\mathrm{ecke}_\chi}$ of the pullback of M by the smooth projection $p_{1\chi} : \mathcal{H}\mathrm{ecke}_\chi \to \mathrm{Bun}_G$, $p_{1\chi} := p_1|_{\mathcal{H}\mathrm{ecke}_\chi}$. Notice that the fibration $p_{1X} : \overline{\mathcal{H}\mathrm{ecke}_\chi} \to \mathrm{Bun}_G \times X$ is locally trivial (see 5.2.2 (ii), 5.2.3), so the choice of a local trivialization identifies $p_{1\chi}^\star M$ (locally) with the external tensor product of M and the "intersection cohomology" $\mathcal{D}$-module on the closure of the corresponding $G(O)$-orbit*) on the affine Grassmannian.

Define the Hecke functors $T_\chi^i : \mathcal{M}(\mathrm{Bun}_G) \to \mathcal{M}(\mathrm{Bun}_G \times X)$ by

$$T_\chi^i = H^i(p_{2,X})_* p_{1\chi}^\star \tag{254}$$

---

*) This orbit is $\mathrm{Orb}_{\chi^\circ}$ where χ° is the dual weight, see 5.2.3.

where $H^i(p_{2,X})_*$ is the cohomological pushforward functor for the projection $p_{2,X} : \overline{\mathcal{H}\mathrm{ecke}_\chi} \to \mathrm{Bun}_G \times X$.

Remark. For a representable quasi-compact morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks of locally finite type the definition of $H^i f_* : \mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathcal{Y})$ is clear. Indeed, in the case of schemes one has a definition of $H^i f_*$ and one knows that $H^i f_*$ commutes with smooth base change.

5.2.5. For $\chi \in P_+(^L G)$ we denote by V^χ the irreducible $^L G$-module of highest weight χ with marked highest vector. If $\mathfrak{F}$ is an $^L G$-oper on X (or, more generally, an $^L G$-bundle with a connection) denote by $V^\chi_{\mathfrak{F}}$ the $\mathfrak{F}$-twist of V^χ; this is a smooth $\mathcal{D}$-module on X.

5.2.6. *Main Global Theorem.* Let $\mathfrak{F}$ be an $^L G$-oper on X and $M_{\mathfrak{F}}$ the $\mathcal{D}$-module on Bun_G defined in 5.1.1. Then $T^i_\chi M_{\mathfrak{F}} = 0$ for $i \neq 0$ and there is a canonical isomorphism of $\mathcal{D}$-modules on $\mathrm{Bun}_G \times X$

$$T^0_\chi M_{\mathfrak{F}} \xrightarrow{\sim} M_{\mathfrak{F}} \boxtimes V^\chi_{\mathfrak{F}} . \tag{255}$$

The isomorphisms (255) are compatible with composition of Hecke correspondences and tensor products of V^χ. For the precise statement see 5.4.3. All this means that $M_{\mathfrak{F}}$ is a *Hecke eigen-$\mathcal{D}$-module of eigenvalue* $\mathfrak{F}$.

5.2.7. Laumon defined (see §§5.3 and 4.3.3 from [La87]) a conjectural "Langlands transform" K_E of an irreducible local system E on X (K_E does exist if rank $E \leq 2$). K_E is a holonomic $\mathcal{D}$-module on Bun_{GL_n}, $n = \mathrm{rank} E$, and at least for $n = 2$ its singular support is the zero fiber of Hitchin's fibration (see §5.5 from [La87]). Besides K_E has regular singularities and its restriction to each connected component of Bun_{GL_n} is irreducible. If E is an SL_n local system then K_E lives on Bun_{PGL_n}.

Taking in account 5.1.2 and 5.2.6 it is natural to conjecture that for $G = PGL_n$ the $\mathcal{D}$-module $M_{\mathfrak{F}}$ from 5.1.1 equals $K_{\mathfrak{F}}$ (some results in this direction can be found in [Fr]). It would also be interesting to find out (for any G) whether $M_{\mathfrak{F}}$ has regular singularities and whether its restrictions to connected components of Bun_G are irreducible.

5.2.8. It is convenient and important to rewrite 5.2.6 in terms of the $\mathcal{D}$-modules $M_{\mathcal{L}}$ from 5.1.1, $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$. According to (57) $\mathcal{L} \in Z\,\mathrm{tors}_\theta(X)$ defines a family $\mathfrak{F}_{\mathcal{L}}$ of ${}^LG$-opers on X parametrized by $\mathrm{Spec}\, A_{{}^L\mathfrak{g}}(X)$. Thus $\mathfrak{F}_{\mathcal{L}}$ is an ${}^LG$-torsor on $X \times \mathrm{Spec}\, A_{{}^L\mathfrak{g}}(X)$ equipped with a connection along X. For $\chi \in P_+({}^LG)$ the $\mathfrak{F}_{\mathcal{L}}$-twist of V^χ is a vector bundle on $X \times \mathrm{Spec}\, A_{{}^L\mathfrak{g}}(X)$ equipped with a connection along X. We consider it as a $\mathcal{D}$-module $V^\chi_{\mathcal{L}}$ on X equipped with an action of $A_{{}^L\mathfrak{g}}(X)$.

Now consider the $\mathcal{D}$-module $M_{\mathcal{L}}$ on Bun_G (sec 5.1.1); $A_{{}^L\mathfrak{g}}(X)$ acts on it. It is easy to see (use 5.1.2 (i)) that 5.2.6 is a consequence of the following theorem.

5.2.9. *Theorem.* There is a canonical isomorphism of $\mathcal{D}$-modules on $\mathrm{Bun}_G \times X$

$$T^0_\chi M_{\mathcal{L}} \widetilde{\to} M_{\mathcal{L}} \underset{A_{{}^L\mathfrak{g}}(X)}{\boxtimes} V^\chi_{\mathcal{L}} \tag{256}$$

compatible with the action of $A_{{}^L\mathfrak{g}}(X)$, and $T^i_\chi M_{\mathcal{L}} = 0$ for $i \neq 0$.

5.2.10. We will deduce the above global theorem from its local version which we are going to explain now. Consider the affine Grassmannian $\mathcal{GR} := G(K)/G(O)$ where $O := \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$. This is an ind-proper ind-scheme. Thus we have the "abstract" category $\mathcal{M}(\mathcal{GR})$ of $\mathcal{D}$-modules on $\mathcal{GR}$ defined as $\varinjlim \mathcal{M}(Y)$ where Y runs over the set of all closed subschemes $Y \subset \mathcal{GR}$.

We are not able to represent $\mathcal{GR}$ as a union of an increasing sequence of smooth subschemes. However $\mathcal{GR}$ is a *formally smooth* ind-scheme. This permits to treat $\mathcal{D}$-modules on $\mathcal{GR}$ as "concrete" objects in the same way as if $\mathcal{GR}$ were a smooth finite dimensional variety, i.e., to identify them with certain sheaves of $\mathcal{O}$-modules equipped with some extra structure. Namely, assume we have an $\mathcal{O}$-module P on $\mathcal{GR}$ such that each local section of P

is supported on some subscheme of $\mathcal{GR}$. Then one easily defines what is a continuous *right* action of $\operatorname{Der}\mathcal{O}_{\mathcal{GR}}$ on P. Such P equipped with such an action is the same as a $\mathcal{D}$-module on $\mathcal{GR}$ (we also assume an appropriate quasi-coherency condition). Details can be found in ???.

5.2.11. *Remark.* We see that it is the *right* $\mathcal{D}$-modules that make sense as sheaves in this infinite dimensional setting. The reason for this is quite finite dimensional. Indeed, if $i : Y \hookrightarrow Z$ is a closed embedding of smooth manifolds and M is a $\mathcal{D}$-module on Y then in order to identify M with a subsheaf of i_*M one needs to consider *right* $\mathcal{D}$-modules.

5.2.12. According to 3.4.3 one has the groupoid $Z\,\mathrm{tors}_\theta(O)$, which is the local analog of $Z\,\mathrm{tors}_\theta(X)$. A choice of $\mathcal{L} \in Z\,\mathrm{tors}_\theta(O)$ (which essentially amounts to that of square root of ω_O) defines the "local" Pfaffian line bundle $\lambda_{\mathcal{L}}^{\mathrm{loc}}$ on $\mathcal{GR}$ (see 4.6). The action of $\mathfrak{g} \otimes K$ on $\mathcal{GR}$ by left infinitesimal translations lifts to the action of the central extension $\widetilde{\mathfrak{g} \otimes K}$ from 2.5.1 on $\lambda_{\mathcal{L}}^{\mathrm{loc}}$ such that $\mathbf{1} \in \mathbb{C} \subset \widetilde{\mathfrak{g} \otimes K}$ acts as multiplication by -1 (see 4.6.5). This yields an antihomomorphism $\overline{U}' \to \Gamma(\mathcal{GR}, \mathcal{D}')$ where $\overline{U}' = \overline{U}'(\mathfrak{g} \otimes K)$ is the completed twisted universal enveloping algebra defined in 2.9.4 and $\Gamma(\mathcal{GR}, \mathcal{D}')$ is the ring of $\lambda_{\mathcal{L}}^{\mathrm{loc}}$-twisted differential operators on $\mathcal{GR}$. Hence for any $\mathcal{D}$-module M on $\mathcal{GR}$ the algebra $\overline{U}'$ acts on $M\lambda_{\mathcal{L}}^{-1} := M \otimes_{\mathcal{O}_{\mathcal{GR}}} (\lambda_{\mathcal{L}}^{\mathrm{loc}})^{\otimes -1}$. So $\Gamma(\mathcal{GR}, M\lambda_{\mathcal{L}}^{-1})$ is a (left) $\overline{U}'$-module.

For example, consider the $\mathcal{D}$-module I_1 of δ-functions at the distinguished point of $\mathcal{GR}$. The $\overline{U}'$-module $\Gamma(\mathcal{GR}, I_1\lambda_{\mathcal{L}}^{-1})$ is the vacuum module Vac'.

5.2.13. Recall (see 4.5.8) that $\mathcal{GR}$ is stratified by $G(O)$-orbits Orb_χ labeled by $\chi \in P_+({}^LG)$. Denote by I_χ the irreducible "intersection cohomology" $\mathcal{D}$-module on $\mathcal{GR}$ that corresponds to $\overline{\mathrm{Orb}_\chi}$.

Here is the first part of our main local theorem.

5.2.14. *Theorem.* The $\overline{U}'$-module $\Gamma(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1})$ is isomorphic to a sum of several copies of Vac', and $H^i(\mathcal{GR}, I_\chi\lambda_{\mathcal{L}}^{-1}) = 0$ for $i > 0$.

Remark. This theorem means (see 5.4.8, 5.4.10) that the Harish-Chandra module Vac' is an eigenmodule of the Harish-Chandra version of the Hecke functors from 7.8.2, 7.14.1.

5.2.15. The group $\mathrm{Aut}\, O$ acts on $\mathcal{GR}$, and the action of its Lie algebra $\mathrm{Der}\, O$ lifts to $\lambda_{\mathcal{L}}^{\mathrm{loc}}$ (see 4.6.7). The second part of our theorem describes the action of $\mathrm{Der}\, O$ on $\Gamma(\mathcal{GR}, I_\chi \lambda_{\mathcal{L}}^{-1})$.

Consider the scheme of local ${}^L\mathfrak{g}$-opers $\mathcal{O}\mathfrak{p}_{^L\mathfrak{g}}(O) = \mathrm{Spec}\, A_{^L\mathfrak{g}}(O)$ from 3.2.1. Write A instead of $A_{^L\mathfrak{g}}(O)$. Just as in 5.2.8 $\mathcal{L}$ defines a family of ${}^LG$-opers on $\mathrm{Spec}\, O$ parametrized by $\mathrm{Spec}\, A$. This family defines an ${}^LG$-torsor $\mathfrak{F}_A$ over $\mathrm{Spec}\, A$ equipped with an action of $\mathrm{Der}\, O$ compatible with its action on A; see 3.5.4*). The $\mathfrak{F}_A$-twist of the ${}^LG$-module V^χ is a vector bundle over $\mathrm{Spec}\, A$. Denote by $V^\chi_{\mathcal{L}A}$ the A-module of its sections; $\mathrm{Der}\, O$ acts on it.

5.2.16. *Theorem.* There is a canonical isomorphism of $\overline{U}'$-modules

$$\Gamma(\mathcal{GR}, I_\chi \lambda_{\mathcal{L}}^{-1}) \simeq Vac' \otimes_A V^\chi_{\mathcal{L}A} \tag{257}$$

compatible with the action of $\mathrm{Der}\, O$.

Here we use the A-module structure on Vac' that comes from the Feigin–Frenkel isomorphism (80).

5.2.17. A few words about the proofs. The global theorem follows from the local one by an easy local-to-global argument similar to that used in 2.8. The proof of the local theorem is based on the interplay of the following two key structures:

(i) The *Satake equivalence* ([Gi95], [MV]) between the tensor category of representations of ${}^LG$ and the category of $\mathcal{D}$-modules on $\mathcal{GR}$ generated by I_χ's equipped with the "convolution" tensor structure.

(ii) The *"renormalized" enveloping algebra* $U^\natural$. The morphism of algebras $\overline{U}' \to \Gamma(\mathcal{GR}, \mathcal{D}')$ is neither injective (it kills the annihilator I of Vac' in

---

*)In 3.5.4 we used the notation $\mathfrak{F}^0_G$ instead of $\mathfrak{F}_A$ and we considered the "particular" case where $\mathcal{L}$ is a square root of ω_O.

the center $\mathfrak{Z}$ of $\overline{U}'$) nor surjective (its image does not contain $\operatorname{Der} O$). We decompose it as $\overline{U}' \to U^\natural \to \Gamma(\mathcal{GR}, \mathcal{D}')$ where $U^\natural$ is obtained by "adding" to $\overline{U}'/I\overline{U}'$ the algebroid I/I^2 from 3.6.5 (the commutation relations between $\mathfrak{z}_{\mathfrak{g}}(O) = \mathfrak{Z}/I \subset \overline{U}'/I\overline{U}'$ and I/I^2 come from the algebroid structure on I/I^2, they are almost of Heisenberg type). The vacuum representation Vac' is irreducible as an $U^\natural$-module; the same is true for $\Gamma(\mathcal{GR}, I_\chi \lambda^{-1})$, $\chi \in P_+({}^L G)$.

5.2.18. Here is the idea of the proof of 5.2.16 (we assume 5.2.14). Set $\mathfrak{z} := \mathfrak{z}_{\mathfrak{g}}(O)$. Consider the $\mathfrak{z}$-modules $V^\chi_{\mathcal{L}\mathfrak{z}} := \operatorname{Hom}_{\overline{U}'}(Vac', \Gamma(\mathcal{GR}, I_\chi \lambda_{\mathcal{L}}^{-1}))$, so $\Gamma(\mathcal{GR}, I_\chi \lambda_{\mathcal{L}}^{-1}) = Vac' \underset{\mathfrak{z}}{\otimes} V^\chi_{\mathcal{L}\mathfrak{z}}$. Some Tannakian formalism joint with Satake equivalence yields a canonical ${}^L G$-torsor $\mathfrak{F}_{\mathfrak{z}}$ over $\operatorname{Spec} \mathfrak{z}$ such that $V^\chi_{\mathcal{L}\mathfrak{z}}$ are $\mathfrak{F}$-twists of V^χ. The $U^\natural$-module structure on $\Gamma(\mathcal{GR}, I_\chi \lambda_{\mathcal{L}}^{-1})$ defines the action of the Lie algebroid I/I^2 on $\mathfrak{F}_{\mathfrak{z}}$. Some extra geometric considerations define a canonical B-structure on $\mathfrak{F}_{\mathfrak{z}}$, which satisfies the "oper" property with respect to the action of $\operatorname{Der} O \subset I/I^2$. Now the results of 3.5, 3.6 yield a canonical identification $(\operatorname{Spec} \mathfrak{z}, \mathfrak{F}_{\mathfrak{z}}) \overset{\sim}{\rightarrow} (\operatorname{Spec} A, \mathfrak{F}_A)$ such that $A \overset{\sim}{\rightarrow} \mathfrak{z}$ is the Feigin–Frenkel isomorphism, and we are done.

5.2.19. DO WE NEED IT???

Here is a direct construction of M that does not appeal to twisted $\mathcal{D}$-modules. For $x \in X$ consider the scheme $\operatorname{Bun}_{G,\bar{x}}$ (see 2.3.1). For $\mathcal{L} \in Z\operatorname{tors}_\theta(X)$ denote by $\lambda_{\mathcal{L},\bar{x}}$ the pull-back of the line bundle $\lambda_{\mathcal{L}}$ to $\operatorname{Bun}_{G,\bar{x}}$. Let $\widetilde{\mathfrak{g} \otimes K_x}$ be the central extension of $\mathfrak{g} \otimes K_x$ from 2.5.1, so the $\mathfrak{g} \otimes K_x$-action on $\operatorname{Bun}_{G,\bar{x}}$ lifts canonically to a $\widetilde{\mathfrak{g} \otimes K_x}$-action on $\lambda_{\mathcal{L},\bar{x}}$ such that $\mathbf{1} \in \mathbb{C}$ acts as identity (see 4.4.12). Denote by $\operatorname{Bun}_{G,\mathcal{L},\bar{x}}$ the space of the $\mathbb{G}_m$-torsor over $\operatorname{Bun}_{G,\bar{x}}$ that corresponds to $\lambda_{\mathcal{L},\bar{x}}$. We have a Harish-Chandra pair $(\widetilde{\mathfrak{g} \otimes K_x}, \mathbb{G}_m \times G(O_x)), \operatorname{Lie} \mathbb{G}_m = \mathbb{C} \subset \widetilde{\mathfrak{g} \otimes K_x}$. The $\widetilde{\mathfrak{g} \otimes K_x}$-action on $\operatorname{Bun}_{G,\mathcal{L},\bar{x}}$ extends to the action of this pair in the obvious way.

Note that $\operatorname{Bun}_G = \mathbb{G}_m \times G(O_x) \setminus \operatorname{Bun}_{G,\mathcal{L},\bar{x}}$. Therefore by 1.2.4 and 1.2.6 we have the functor $\Delta_{\mathcal{L}} : \left(\widetilde{\mathfrak{g} \otimes K_x}, \mathbb{G}_m \times G(O_x)\right) \bmod \to \mathcal{M}^\ell(\operatorname{Bun}_G)$.

Consider the projection $\mathbb{G}_m \times G(O_x) \to \mathbb{G}_m$ as a character; let $Vac^{\sim}$ be the corresponding induced Harish-Chandra module. One has

$$M_{\mathcal{L}} = \Delta_{\mathcal{L}}(Vac^{\sim}). \tag{258}$$

Let us identify the $A_{{}^L\mathfrak{g}}(X)$-module structure on $M_{\mathcal{L}}$. The action of $\mathrm{End}(Vac^{\sim}) = \mathfrak{z}_{\mathfrak{g}}(O_x)$ on $\Delta_{\mathcal{L}}(Vac^{\sim})$ identifies, via Feigin-Frenkel's isomorphism φ_{O_x} (see 3.2.2) with an $A_{{}^L\mathfrak{g}}(O_x)$-action. This action factors through the quotient $A_{{}^L\mathfrak{g}}(X)$.

5.3. **The Satake equivalence.** We recall the basic facts and constructions, and fix notation. For details and proofs see [MV]. The authors of [MV] use perverse sheaves; we use $\mathcal{D}$-modules.

5.3.1. Consider the affine (or loop) Grassmannian $\mathcal{GR} = G(K)/G(O)$ (as usual $K = \mathbb{C}((t))$, $O = \mathbb{C}[[t]]$); this is a formally smooth ind-projective ind-scheme (see 4.5.1). It carries the stratification by $G(O)$-orbits Orb_χ, $\chi \in P_+({}^LG)$ (see 4.5.8). Each stratum is $\mathrm{Aut}^0 O$-invariant.

In 4.5.10 we introduced the notion of parity of a connected component of $\mathcal{GR}$. According to 4.5.11

(259) All the strata of an even (resp. odd) component of $\mathcal{GR}$ have even (resp. odd) dimension.

5.3.2. *Lemma.*

(i) Each stratum Orb_χ is connected and simply connected.
(ii) Any smooth $\mathcal{D}$-module on Orb_χ is constant.
(iii) Orb_χ has cohomology only in even degrees.

Proof. Denote by Stab_x the stabilizer of $x \in \mathcal{GR}$ in $G(O)$. The image of Stab_x in $G(O/tO) = G$ is a parabolic subgroup P_x and the morphism $G(O)/\mathrm{Stab}_x \to G/P_x$ is a locally trivial fibration whose fibers are isomorphic to an affine space. Now (i) and (iii) are clear. Notice that $\overline{\mathrm{Orb}_\chi}$ is projective and according to (259) $\overline{\mathrm{Orb}_\chi} \setminus \mathrm{Orb}_\chi$ has codimension ≥ 2. So by

Deligne's theorem*) a smooth $\mathcal{D}$-module on Orb_χ has regular singularities and therefore (ii) follows from (i).

□

Denote by $\mathcal{P}$ the category of coherent (or, equivalently, holonomic) $\mathcal{D}$-modules on $\mathcal{GR}$ smooth along our stratification.

5.3.3. *Proposition.*

(i) The category $\mathcal{P}$ is semisimple.

(ii) If $M \in \mathcal{P}$ is supported on an even (resp. odd) component then $H^a_{DR}(\mathcal{GR}, M) = 0$ if a is odd (resp. even).

Proof. Denote by I_χ the intersection cohomology perverse sheaf of $\mathbb{C}$-vector spaces on $\overline{\mathrm{Orb}_\chi}$. Denote by $\mathcal{GR}_{(\chi)}$ the connected component of $\mathcal{GR}$ containing Orb_χ and by $p(\chi)$ the parity of $\mathcal{GR}_{(\chi)}$. According to Lusztig (Theorem 11c from [Lu82]) I_χ has the following property: the cohomology sheaves $H^i(I_\chi)$ are zero unless $i \mod 2 = p(\chi)$. Denote by C the category of all objects of $D^b(\mathcal{GR}_{(\chi)})$ having this property and smooth along our stratification. It follows from (259) and 5.3.2 (iii) that for any $M, N \in C$ one has $H^i(\mathcal{GR}_{(\chi)}, M) = 0$ unless $i \mod 2 = p(\chi)$ and $\mathrm{Ext}^i(M, N^*) = 0$ for odd i (here N^* is the Verdier dual of N). In particular $H^i(\mathcal{GR}, I_\chi) = 0$ unless $i \mod 2 = p(\chi)$ and $\mathrm{Ext}^1(I_{\chi_1}, I_{\chi_2}) = 0$. Using 5.3.2 (ii) one gets the Proposition. □

5.3.4. According to 5.3.2 (ii) the simple objects of $\mathcal{P}$ are "intersection cohomology" $\mathcal{D}$-modules I_χ of the strata Orb_χ. Thus 5.3.3 (i) implies that any object of $\mathcal{P}$ has a structure of $G(O)$-equivariant or $\mathrm{Aut}^0 O \ltimes G(O)$-equivariant $\mathcal{D}$-module. Such structure is unique and any morphism is compatible with it (since our groups are connected). We see that

*) Instead of using Deligne's theorem one can notice that for any vector bundle on Orb_χ its analytic sections are algebraic. Applying this to horizontal analytic sections of a vector bundle on Orb_χ equipped with an integrable connection one sees that (ii) follows from (i).