$\mathcal{P}$ coincides with the category of $G(O)$-equivariant or $\mathrm{Aut}^0 O \ltimes G(O)$-equivariant coherent $\mathcal{D}$-modules on $\mathcal{GR}$.

Remark. The existence of $G(O)$-equivariant structure follows also directly from the facts that $G(O)$ is connected and $\mathrm{Hom}(G(O), \mathbb{G}_m) = 0$ (and 5.3.2 (ii)); one needs not to evoke 5.3.3 (i) and therefore Lusztig's theorem (which is a deep result).

5.3.5. The category $\mathcal{P}$ carries a canonical tensor structure. There are two ways to describe it: the "convolution" construction (see 5.3.5 - 5.3.9) and the "fusion" construction (presented, after certain preliminaries of 5.3.10 - 5.3.12, in 5.3.13 - 5.3.16); for the equivalence of these definitions see 5.3.17. We begin with the convolution picture *). We have to define the convolution product functor $\circledast : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$, the associativity constraint for $\circledast$, and the commutativity constraint.

According to [MV] the functor $\circledast$ is defined as follows. Denote by $G(K) \times_{G(O)} \mathcal{GR}$ the quotient of $G(K) \times \mathcal{GR}$ by $G(O)$ where $u \in G(O)$ acts on $G(K) \times \mathcal{GR}$ by $(g, x) \mapsto (gu^{-1}, ux)$. The morphism $p : G(K) \times_{G(O)} \mathcal{GR} \to G(K)/G(O) = \mathcal{GR}$ defined by $(g, x) \mapsto g \bmod G(O)$ is the locally tivial fibration with fiber $\mathcal{GR}$ associated to the principal $G(O)$-bundle $G(K) \to \mathcal{GR}$ and the action of $G(O)$ on $\mathcal{GR}$. So $G(K) \times_{G(O)} \mathcal{GR}$ is a twisted form of $\mathcal{GR} \times \mathcal{GR}$. Let $M, N \in \mathcal{P}$. Using the $G(O)$-equivariant structure on M one defines a $\mathcal{D}$-module $M \boxtimes' N$ on $G(K) \times_{G(O)} \mathcal{GR}$, which is a "twisted form" of $M \boxtimes N$. Then

$$M \circledast N = m_*(M \boxtimes' N) \tag{260}$$

where $m : G(K) \times_{G(O)} \mathcal{GR} \to \mathcal{GR}$ comes from the action of $G(K)$ on $\mathcal{GR}$.

5.3.6. *Miraculous Theorem.* ([Gi95], [MV]) If $M, N \in \mathcal{P}$ then $M \circledast N \in \mathcal{P}$. $\square$

---

*) What follows is an algebraic version of Ginzburg's topological construction [Gi95]; we leave it to the interested reader to identify the two constructions.

Remark. The nontrivial statement is that $M \circledast N$ is a $\mathcal{D}$-module (not merely an object of the derived category). Since this $\mathcal{D}$-module is coherent and $G(O)$-equivariant it belongs to $\mathcal{P}$.

So we have defined $\circledast : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$. The associativity constraint for $\circledast$ is defined in the obvious way. The commutativity constraint will be defined in 5.3.8.

5.3.7. *Remarks.* (i) Suppose that $G(K)$ is replaced by an ind-affine group ind-scheme $\mathcal{G}$ and $G(O)$ by its closed group subscheme $\mathcal{K}$; assume that $\mathcal{G}/\mathcal{K}$ is an ind-scheme of ind-finite type. The construction of $\circledast : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ from 5.3.5 is based on the miracle 5.3.6. In general there is no convolution on the category of $\mathcal{K}$-equivariant $\mathcal{D}$-modules on $\mathcal{G}/\mathcal{K}$ and one has to consider a certain derived category $\mathcal{H}$ (the *Hecke monoidal category;* see 7.6.1 and 7.11.17). This is a triangulated category with a t-structure whose core is the category of $\mathcal{K}$-equivariant $\mathcal{D}$-modules on $\mathcal{G}/\mathcal{K}$; in general $\circledast : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is not t-exact and there is no commutativity constraint for $\circledast$. In the case of $(G(K), G(O))$ the functor $\circledast$ is t-exact by 5.3.6 and the core of $\mathcal{H}$ is the category of ind-objects of $\mathcal{P}$.

(ii) The construction of $\mathcal{H}$ mentioned above is a part of the "Hecke pattern" developed in §7. Later we will see that this pattern is useful (or maybe indispensable) even in the miraculously good situation of $(G(K), G(O))$.

5.3.8. Let us define the commutativity constraint for $\circledast$. Let $\theta : G \to G$ be an automorphism that sends any dominant weight to its dual. The anti-automorphism $\theta'(g) := \theta(g)^{-1}$ of G yields an anti-automorphism $\theta'_{\mathcal{H}}$ of the monoidal category $\mathcal{H}$, so for any $M, N \in \mathcal{H}$ one has a canonical isomorphism $l_{M,N} : \theta'_{\mathcal{H}}(M \circledast N) \xrightarrow{\sim} \theta'_{\mathcal{H}}(N) \circledast \theta'_{\mathcal{H}}(M)$.

For any $M \in \mathcal{P} \subset \mathcal{H}$ there is a canonical isomorphism $e_M : M \xrightarrow{\sim} \theta'_{\mathcal{H}}(M)$. To define e_M it suffices, according to 5.3.3 (i), to consider the case $M = I_\chi$. The action of θ' on $G(K)$ preserves the stratification $G(K)_\chi$ by the double

$G(O)$-classes (here $G(K)_\chi$ is the preimage of $\mathrm{Orb}_\chi \subset G(K)/G(O)$). So we have the induced automorphism θ'_χ of $G(K)_\chi$. As an object of $\mathcal{H}$ our I_χ is the Ω-complex $\Omega_{G(K)_\chi}[\dim \mathrm{Orb}_\chi]$ on $G(K)$. Now e_{I_χ} is the action of θ'_χ on $\Omega_{G(K)_\chi}$.

For $M, N \in \mathcal{P}$ define

$$s : M \circledast N \simeq N \circledast M \tag{261}$$

as the composition

$$M \circledast N \simeq \theta'_{\mathcal{H}}(M \circledast N) \simeq \theta'_{\mathcal{H}}(N) \circledast \theta'_{\mathcal{H}}(M) \simeq N \circledast M$$

where the first arrow is the isomorphism e corresponding to $M \circledast N$ and the other arrows are $l_{M,N}$ and $e_N^{-1} \circledast e_M^{-1}$.

5.3.9. *Proposition.* s is a commutativity constraint for the convolution tensor product $\circledast$.

Proof. In 5.3.17 below we identify the convolution tensor product with the fusion tensor product in a way compatible with all the constraints. Since the latter data obviously define a tensor category structure on $\mathcal{P}$ we are done. $\square$

So we have defined the promised convolution tensor structure on $\mathcal{P}$.

5.3.10. The fusion description of the tensor structure on $\mathcal{P}$ *) is based on the important *chiral semigroup* structure on the "space" $\mathrm{GRAS} = \mathrm{GRAS}_G$ from 4.3.14. This structure may be described as follows.

(i) For a $\mathbb{C}$-algebra R and $S \in \Sigma(R)$ (we use notation from 4.3.11, so S is a subscheme of $X \otimes R$ finite and flat over $\mathrm{Spec}\, R$) one has a subset $\mathrm{GRAS}(R)_S \subset \mathrm{GRAS}(R)$ defined as the set of pairs $(\mathcal{F}, \gamma)$ where $\mathcal{F}$ is a G-torsor on $X \otimes R$, γ is a section of $\mathcal{F}$ over the complement to S.

(ii) If S is a disjoint union of subschemes S_i, $i \in I$, then one has a canonical identification

*) The construction apparently involves a curve X, but actually it is purely local.

$$\mathrm{GRAS}(R)_S \overset{\sim}{\to} \prod_i \mathrm{GRAS}(R)_{S_i} \tag{262}$$

Namely, we identify $(\mathcal{F},\gamma)$ with the collection $(\mathcal{F}_i,\gamma_i)$, $i \in I$, where $(\mathcal{F}_i,\gamma_i) \in \mathrm{GRAS}(R)_{S_i}$ coincides with $(\mathcal{F},\gamma)$ over the complement to the union of $S_{i'}$, $i' \neq i$.

The data (i), (ii) enjoy the following properties:

a. If for $S_1, S_2 \in \Sigma(R)$ one has $S_{1red} \subset S_{2red}$ then $\mathrm{GRAS}(R)_{S_1} \subset \mathrm{GRAS}(R)_{S_2}$. The union of $\mathrm{GRAS}(R)_S$, $S \in \Sigma(R)$, coincides with $\mathrm{GRAS}(R)$. So $\mathrm{GRAS}(R)_S$ form a filtration on $\mathrm{GRAS}(R)$. This filtration is functorial (with respect to R).

b. The isomorphisms (ii) are also functorial and compatible with subdivisions of I in the obvious manner.

c. The subfunctor $\mathcal{GR}_\Sigma \subset \Sigma \times \mathrm{GRAS}$ defined by

$$\mathcal{GR}_\Sigma(R) := \{(S,\mathcal{F},\gamma) | S \in \Sigma(R), (\mathcal{F},\gamma) \in \mathrm{GRAS}(R)_S\}$$

is an ind-scheme formally smooth over Σ.

Remark. Let us explain why $\mathcal{GR}_\Sigma = \mathcal{GR}_\Sigma^G$ is an ind-scheme for any affine algebraic group G. Moreover we will show that $\mathcal{GR}_\Sigma$ is of ind-finite type and if G is reductive then $\mathcal{GR}_\Sigma$ is ind-proper. First consider the case $G = GL_n$. Then $\mathcal{GR}_\Sigma$ is the direct limit of $\mathcal{GR}_{\Sigma,k}$ where $\mathcal{GR}_{\Sigma,k}$ parametrizes pairs consisting of a finite subscheme $D \subset X$ and a subsheaf $\mathcal{E} \subset \mathcal{O}_X^n(kD)$ such that $\mathcal{E} \supset \mathcal{O}_X^n(-kD)$. The morphism $\mathcal{GR}_{\Sigma,k} \to \Sigma$ is proper, so $\mathcal{GR}_\Sigma$ is ind-proper. As explained in the proof of Theorem 4.5.1, to reduce the general case to the case of GL_n it suffices to show that if $G \subset G'$ and G'/G is affine (resp. quasiaffine) then the morphism $\mathcal{GR}_\Sigma^G \to \mathcal{GR}_\Sigma^{G'}$ is a closed (resp. locally closed) embedding. This is easy.

5.3.11. For a finite set J we have the morphism $X^J \to \Sigma$ that assigns to $(x_j) \in X^J$ the subscheme $D \subset X$ corresponding to the divisor $\sum\limits_j x_j$. Denote by $\mathcal{GR}_{X^J}$ the fibered product of $\mathcal{GR}_\Sigma$ and X^J over Σ. So an R-point of

$\mathcal{GR}_{X^J}$ is a collection $((x_j), \mathcal{F}, \gamma)$ where $(x_j) \in X^J(R)$, $\mathcal{F}$ is a G-bundle on $X \otimes R$, and γ is a section of $\mathcal{F}$ over the complement to the union of the graphs of the x_j's. Our $\mathcal{GR}_{X^J}$ is a formally smooth ind-proper ind-scheme over X^J (see the Remark at the end of 5.3.10).

According to 4.5.2 there is a canonical isomorphism between the fiber of $\mathcal{GR}_X$ over $x \in X(\mathbb{C})$ and the ind-scheme $\mathcal{GR}_x := G(K_x)/G(O_x)$. So according to 5.3.10 (ii) the fiber of $\mathcal{GR}_{X^J}$ over $(x_j) \in X^J(\mathbb{C})$ equals $\prod\limits_{x \in S} \mathcal{GR}_x$ where S is the subset $\{x_j\} \subset X$.

The following description of $\mathcal{GR}_X$ will be of use. Consider the scheme $X^\wedge$ of "formal parameters" on X (its points are smooth morphisms $\operatorname{Spec} O \to X$, see 2.6.5). This is an $\operatorname{Aut}^0 O$-torsor over X; a choice of coordinate,i.e., étale $\mathbb{A}^1$-valued map, on an open $U \subset X$ defines a trivialization of $X^\wedge$ over U. Now $\mathcal{GR}_X$ is the $X^\wedge$-twist of $\mathcal{GR}$ (with respect to the $\operatorname{Aut}^0 O$-action on $\mathcal{GR}$).

The stratification of $\mathcal{GR}$ defines a stratification of $\mathcal{GR}_X$ by strata $\operatorname{Orb}_{\chi X}$ smooth over X.

5.3.12. For the future references let us list some of the compatibilities between $\mathcal{GR}_{X^J}$'s that follow directly from 5.3.10.

a. For a surjective map $\pi : J \twoheadrightarrow J'$ there is an obvious Cartesian diagram

$$
\begin{array}{ccc}
\mathcal{GR}_{X^{J'}} & \overset{\widetilde{\Delta}^{(\pi)}}{\hookrightarrow} & \mathcal{GR}_{X^J} \\
\downarrow & & \downarrow \\
X^{J'} & \overset{\Delta^{(\pi)}}{\hookrightarrow} & X^J
\end{array}
\tag{263}
$$

where $\Delta^{(\pi)}$ is the π-diagonal embedding. If $|J'| = 1$ we have $\Delta^{(J)} : X \hookrightarrow X^J$ and $\widetilde{\Delta}^{(J)} : \mathcal{GR}_X \hookrightarrow \mathcal{GR}_{X^J}$.

b. Let $\nu^{(J)} : U^{(J)} \hookrightarrow X^J$ be the complement to the diagonal divisor. By 5.3.10 (ii) the restrictions to $U^{(J)}$ of the X^J-ind-schemes $\mathcal{GR}_{X^J}$ and $(\mathcal{GR}_X)^J$

are canonically identified. Therefore we have a Cartesian diagram

$$
\begin{array}{ccc}
(\mathcal{GR}_X)^J\big|_{U^{(J)}} & \overset{\widetilde{\nu}^{(J)}}{\hookrightarrow} & \mathcal{GR}_{X^J} \\
\downarrow & & \downarrow \\
U^{(J)} & \overset{\nu^{(J)}}{\hookrightarrow} & X^J
\end{array}
\tag{264}
$$

5.3.13. Now we are ready to define the fusion tensor structure on $\mathcal{P}$. This amounts to a construction of tensor product functors [*)]

$$\underset{J}{\circledast} : \mathcal{P}^{\otimes J} \to \mathcal{P} \tag{265}$$

for any finite non-empty set J together with identifications

$$\underset{J}{\circledast} = \underset{J'}{\circledast}\Big(\underset{j'\in J'}{\otimes}\big(\underset{\pi^{-1}(j')}{\circledast} \big)\Big) \tag{266}$$

for any surjective map $J \overset{\pi}{\twoheadrightarrow} J'$.

The construction goes as follows.

5.3.14. Since any $M \in \mathcal{P}$ is $\operatorname{Aut}^0 O$-equivariant it defines a $\mathcal{D}$-module on $\mathcal{GR}_X$ (see the description of $\mathcal{GR}_X$ at the end of 5.3.11). Denote by $M_X \in D(\mathcal{GR}_X)(:= D\mathcal{M}(\mathcal{GR}_X))$ its shift by 1 in the derived category. In other words for any open U as above and a trivialization θ of $X^\wedge$ over U one has $M_U = \pi_\theta^! M$, where $M_U := M_X\big|_{\mathcal{GR}_U}$, $\pi_\theta : \mathcal{GR}_U \to \mathcal{GR}$ is the projection that corresponds to θ, and we glue these objects together using the $\operatorname{Aut}^0 O$-action on M. The functor $\mathcal{P} \to D(\mathcal{GR}_X)$, $M \mapsto M_X$, is fully faithful. Its essential image consists of (shifted by 1) $\mathcal{D}$-modules isomorphic to a direct sum of (finitely many) copies of "intersection cohomology" $\mathcal{D}$-modules $I_{\chi X}$ that correspond to the trivial local system on $\mathrm{Orb}_{\chi X}$.

Let now $\{M_j\}_{j\in J}$ be a collection of objects of $\mathcal{P}$. Using (264) one interprets $\boxtimes M_{jX}\big|_{U^{(J)}}$ as a $\mathcal{D}$-module on $\mathcal{GR}_{X^J}\big|_{U^{(J)}}$ shifted by $|J|$. Denote by $\circledast M_{jX} \in D(\mathcal{GR}_{X^J})$ its minimal (i.e., $\widetilde{\nu}^{(J)}_{!*}-$) extension to $\mathcal{GR}_{X^J}$. This is

[*)] Here $\mathcal{P}^{\otimes J}$ denotes the tensor product of J copies of $\mathcal{P}$ (since $\mathcal{P}$ is semisimple the definition of tensor product is clear).

a $\mathcal{D}$-module on $\mathcal{GR}_{X^J}$ shifted by $|J|$. Therefore we have defined a functor

$$\underset{J}{\boxtimes}\colon \mathcal{P}^{\otimes J} \to D(\mathcal{GR}_{X^J}), \quad \otimes M_j \mapsto \boxtimes M_{jX} \tag{267}$$

which is obviously fully faithful.

5.3.15. *Proposition.* ([MV])

For any $\pi : J \twoheadrightarrow J'$ the complex $\widetilde{\Delta}^{(\pi)!}(\boxtimes M_{jX}) \in D(\mathcal{GR}_X^{(J')})$ belongs to the essential image of $\underset{J'}{\boxtimes}$. □

5.3.16. We get a functor

$$\underset{\pi}{\circledast}\colon \mathcal{P}^{\otimes J} \to \mathcal{P}^{\otimes J'} \tag{268}$$

such that $\underset{J'}{\boxtimes}\underset{\pi}{\circledast} = \widetilde{\Delta}^{(\pi)!}\underset{J}{\boxtimes}$. In particular for $|J'| = 1$ we have the functor $\underset{J}{\circledast} : \mathcal{P}^{\otimes J} \to \mathcal{P}$ which is our tensor product functor (265). The obvious identification $\underset{\pi}{\circledast} = \underset{j' \in J'}{\otimes}(\underset{\pi^{-1}(j')}{\circledast})$ (look at our $\mathcal{D}$-modules over $U^{(J')}$) and the standard isomorphism $\Delta^{(J)!} = (\Delta^{(\pi)}\Delta^{(J')})^! = \Delta^{(J')!}\Delta^{(\pi)!}$ yield the compatibility isomorphisms (266). So $\mathcal{P}$ is a tensor category. It is easy to see that I_0 is a unit object in $\mathcal{P}$.

5.3.17. Let us identify the convolution and fusion tensor structures on $\mathcal{P}$. Below in this subsection we denote by $\circledast^c$ the convolution tensor product, and by $\circledast^f$ the fusion tensor product on $\mathcal{P}$. We have to construct for $M, N \in \mathcal{P}$ a canonical isomorphism $M \circledast^c N \overset{\sim}{\rightarrow} M \circledast^f N$ compatible with the associativity and commutativity constraints.*)

Let $\mathcal{GR}'_{X^2}$ be the ind-scheme over X^2 such that $\mathcal{GR}'_{X^2}(R)$ is the set of collections $(x_1, x_2, \mathcal{F}_1, \mathcal{F}_2, \gamma_1, \gamma_2)$ where $x_1, x_2 \in X(R)$, $\mathcal{F}_1, \mathcal{F}_2$ are G-torsors over $X \otimes R$, γ_1 is a section of $\mathcal{F}_1$ over the complement to the graph of x_1, γ_2 is an isomorphism $\mathcal{F}_1 \to \mathcal{F}_2$ over the complement to the graph of x_2. We have the projection $q : \mathcal{GR}'_{X^2} \to \mathcal{GR}_{X^2}$ that sends

*) The construction is borrowed from [MV] where it is written in more details; however the commutativity constraint 5.3.8 was not considered there.

the above data to $((x_1, x_2), \mathcal{F}_2, \gamma_2\gamma_1)$. This projection is ind-proper; over $U := X^2 \setminus \{\text{the diagonal}\}$ it is an isomorphism.*)

Denote by $M_X \boxtimes' N_X \in D(\mathcal{GR}'_{X^2})$ the minimal extension to $\mathcal{GR}'_{X^2}$ of $M_X \boxtimes N_X\big|_U$. This is a $\mathcal{D}$-module on $\mathcal{GR}'_{X^2}$ shifted by 2. According to [MV] the obvious identification over U extends (uniquely) to a canonical isomorphism

$$q_*(M_X \boxtimes' N_X) \xrightarrow{\sim} M_X \boxtimes N_X \tag{269}$$

Now $\mathcal{GR}'_{X^2}$ is a twisted form of $(\mathcal{GR}_X)^2$. Indeed, a trivialization of $\mathcal{F}_1$ on the formal neighbourhood of x_2 yields an identification of the data $(\mathcal{F}_2, \gamma_2)$ above with $\mathcal{GR}_x$. These trivializations together with formal parameters at x_2 form an $\mathrm{Aut}^0 O \ltimes G(O)$-torsor over $\mathcal{GR}_X \times X$, and $\mathcal{GR}'_{X^2}$ identifies with the corresponding twist of $\mathcal{GR}$. So $M_X \boxtimes' N_X$ is the "twisted form" of $M_X \boxtimes N$. Restricting this picture to the diagonal $X \hookrightarrow X \times X$ we see that the pull-back of $q : \mathcal{GR}'_{X^2} \to \mathcal{GR}_{X^2}$ to X coincides with the $X^\wedge$-twist of the morphism $m : G(K) \times_{G(O)} \mathcal{GR} \to \mathcal{GR}$ from (260) and the pull-back of $M_X \boxtimes' N_X$ to the preimage of X in $\mathcal{GR}'_{X^2}$ equals $(M \boxtimes' N)_X$ where $M \boxtimes' N$ has the same meaning as in (260). Comparing (269) and (260) (and using the base change isomorphism) we get the desired canonical isomorphism $M \circledast^c N \xrightarrow{\sim} M \circledast^f N$.

Its compatibility with the associativity constraints comes from the similar picture over X^3. WRITE DOWN THE COMAT WITH COM CONSTRAINTS (use Bun_G and $Hecke$)!

5.3.18. For $M \in \mathcal{P}$ set $h^{\cdot}(M) := H^{\cdot}_{DR}(\mathcal{GR}, M)$. This is a $\mathbb{Z}$-graded vector space; denote by $h^\varepsilon(M)$ the corresponding $\mathbb{Z}/2\mathbb{Z}$-graded vector space.

Consider the projection $p : \mathcal{GR}_X \to X$. The $\mathcal{D}$-modules $H^a p_*(M_X)$ on X are constant, i.e., isomorphic to a sum of copies of ω_X (recall that we play

*) Over the diagonal the fibers of q are isomorphic to $\mathcal{GR}$; more precisely, the closed embedding $\mathcal{GR}'_{X^2} \to (\mathcal{GR}_X) \times_X (\mathcal{GR}_{X^2})$ defined by $(x_1, x_2, \mathcal{F}_1, \mathcal{F}_2, \gamma_1, \gamma_2) \mapsto (x_1, x_2, \mathcal{F}_1, \gamma_1, \mathcal{F}_2, \gamma_2\gamma_1)$ becomes an isomorphism when restricted to the diagonal $X \hookrightarrow X^2$. So the maximal open subset over which q is an isomorphism has the form $\mathcal{GR}_{X^2} \setminus Z$ where Z has codimension 1; this is an infinite-dimensional phenomenon.

with right $\mathcal{D}$-modules). The corresponding fiber is $h^{\cdot}(M)$: for any $x \in X$ one has $H^{\cdot} i_x^! p_*(M_X) = h^{\cdot}(M)$ (here i_x is the embedding $\{x\} \hookrightarrow X$).

5.3.19. *Proposition.* ([MV])

For any collection $\{M_j\}_{j\in J}$ of objects of $\mathcal{P}$ the $\mathcal{D}$-modules $H^a p_*^{(J)}(\boxtimes M_{jX})$ on X^J are constant. $\square$

For any $(x_j) \in X^J$ one has

$$H^{\cdot} i^!_{(x_j)} p_*^{(J)}(\boxtimes M_{jX}) = \otimes h^{\cdot}(M_j)\,. \tag{270}$$

This is clear from 5.3.18 for $(x_j) \in U^{(J)}$; then use 5.3.19.

5.3.20. For $(x_j) \in X \subset X^J$ (270) yields a canonical isomorphism $h^{\cdot}(\circledast M_j) = \otimes h^{\cdot}(M_j)$ which is obviously compatible with "constraints" (266). We see that

$$h^{\cdot} : \mathcal{P} \to \mathrm{Vect}^{\cdot}\,, \qquad h^{\varepsilon} : \mathcal{P} \to \mathrm{Vect}^{\varepsilon} \tag{271}$$

are *tensor functors.* Here $\mathrm{Vect}^{\cdot}$ is the tensor category of $\mathbb{Z}$-graded vector spaces with the *"super"* commutativity constraint, $\mathrm{Vect}^{\varepsilon}$ is the analogous tensor category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces.

5.3.21. One may twist the tensor structure on $\mathcal{P}$ to get rid of super vector spaces. To do this note that the objects of $\mathcal{P}$ carry a canonical $\mathbb{Z}/2\mathbb{Z}$-grading ε by parity of the components of support (see 4.5.10). This grading is compatible with $\circledast$.

Denote by $\mathcal{P}^{\natural}$ the full subcategory of even objects in $\mathcal{P}^{\varepsilon} := \mathcal{P} \otimes \mathrm{Vect}^{\varepsilon}$ (with respect to tensor product of the $\mathbb{Z}/2\mathbb{Z}$-gradings). This is a tensor subcategory in $\mathcal{P}^{\varepsilon}$. The "forgetting of the grading" functor $o_{\varepsilon} : \mathrm{Vect}^{\varepsilon} \to \mathrm{Vect}$ yields an equivalence $\mathcal{P}^{\natural} \xrightarrow{\sim} \mathcal{P}$. This is an equivalence of monoidal categories (i.e., it is compatible with the tensor products and associativity constraints); the commutativity constraints $A \otimes B \xrightarrow{\sim} B \otimes A$ for $\mathcal{P}$ and $\mathcal{P}^{\natural}$ differ by $(-1)^{p(A)p(B)}$.

The functor h^ε is compatible with the $\mathbb{Z}/2\mathbb{Z}$-gradings by 5.3.3 (ii). Therefore it defines a tensor functor

$$h:\,\mathcal{P}^\natural \to \mathrm{Vect}\,. \tag{272}$$

Note that h carries a canonical $\mathbb{Z}$-grading which we denote also by $h^{\cdot}$ by abuse of notation. So $h^{\cdot}$ is a tensor functor on $\mathcal{P}^\natural$ with values in the tensor category of graded vector spaces equipped with the plain (*not* super) commutativity constraint.

5.3.22. According to [MV] (WHAT ABOUT GINZBURG ??) the tensor category $\mathcal{P}^\natural$ is rigid, i.e., each object has a dual in the sense of §2.1.2 from [Del91] (the dual objects are explicitly constructed in [MV]). The tensor functor (272) is $\mathbb{C}$-linear and exact,[*)] so it is a fiber functor in the sense of [Del91]. Therefore by the general Tannakian formalism (272) induces an equivalence between the tensor categories $\mathcal{P}^\natural$ and $\mathrm{Rep}(\mathrm{Aut}^\otimes h)$ where $\mathrm{Aut}^\otimes h$ denotes the group scheme of tensor automorphisms of h and Rep means the category of finite-dimensional representations. According to [MV] there is an isomorphism $\varkappa:\,{}^LG \xrightarrow{\sim} \mathrm{Aut}^\otimes h$, so we may rewrite the above equivalence as

$$h:\,\mathcal{P}^\natural \xrightarrow{\sim} \mathrm{Rep}\,{}^LG\,. \tag{273}$$

Here ${}^LG$ is the Langlands dual group, i.e., it is a semisimple group together with a fixed Cartan torus ${}^LH \subset {}^LG$, an identification of the corresponding root datum with the dual to the root datum of G, and a collection of fixed non-zero vectors $y_\alpha \in ({}^L\mathfrak{g})^\alpha$ for simple negative roots α.

5.3.23. We are going to define a *canonical* isomorphism

$$\varkappa:\,{}^LG \xrightarrow{\sim} \mathrm{Aut}^\otimes h \tag{274}$$

[*)] Exactness is clear since $\mathcal{P}^\natural$ is semisimple. Mirković and Vilonen [MV] have to prove exactness because they want their proofs to work for perverse sheaves over arbitrary commutative rings.

by listing some properties of the action of ${}^LG$ on h, which determine $\varkappa$ uniquely.

(i) Denote by

$$t \mapsto t^{2\rho} \tag{275}$$

the morphism $\mathbb{G}_m \to {}^LH$ corresponding to the weight 2ρ of G. Then $t^{2\rho}$ acts on h^a as multiplication by t^{-a} (so the action of the 1-parameter subgroup (275) corresponds to the grading $h^{\cdot}$ of h).

It follows from (i) that the action of ${}^LH$ on h preserves the grading of h.

(ii) For any $\chi \in P_+({}^LG)$ the group ${}^LH$ acts on $h^{\min}(I_\chi) = h^{-\dim \mathrm{Orb}_\chi}(I_\chi)$ by the character χ.

This means that the highest weight of the irreducible ${}^LG$-module $h(I_\chi)$ equals χ.

Remark. Since $\dim \mathrm{Orb}_\chi = \langle \chi, 2\rho \rangle$ there is no contradiction between (i) and (ii).

The properties (i) and (ii) can be found in [MV]. They uniquely determine the restriction of (274) to ${}^LH$. So (274) is determined by (i) and (ii) up to ${}^LH$-conjugation. We normalize (274) by the following property.

(iii) Let $c \in (\mathrm{Sym}^2 \mathfrak{g}^*)^G$ be an invariant bilinear form on $\mathfrak{g}$ (or on $[\mathfrak{g}, \mathfrak{g}]$ in the reductive case???). Set

$$f_c := \frac{???}{2} \sum_\alpha c(\alpha, \alpha) y_\alpha \in {}^L\mathfrak{g} \tag{276}$$

(the expression $c(\alpha, \alpha)$ makes sense because $\alpha \in ({}^L\mathfrak{h})^* = \mathfrak{h} \subset \mathfrak{g}$). Then the Lie algebra element f_c acts on $h(M) = H^{\cdot}_{DR}(\mathcal{GR}, M)$, $M \in \mathcal{P}^\natural = \mathcal{P}$, as multiplication by $\nu(c)$ where

$$\nu : (\mathrm{Sym}^2 \mathfrak{g}^*)^G \to H^2_{DR}(\mathcal{GR}) \tag{277}$$

is the standard morphism whose definition will be reminded in 5.3.24.

Remark. (iii) is formulated by V.Ginzburg [Gi95] in a slightly different form. In fact, he describes in a similar way the action on h of the whole centralizer of f_c in ${}^L\mathfrak{g}$.

5.3.24. In this subsection (which can be skipped by the reader) we define the canonical morphism (277). We use the folowing ad hoc definition: for any ind-scheme Z one has $H^a_{DR}(Z) := \varprojlim H^a(Y, \Omega_Y)$ where Y runs over the set of all closed subschemes of Z and Ω_Y is the de Rham complex of Y (in the most naive sense). To define ν let us assume for simplicity (simplicity twice?? BAD STYLE) that G is semisimple *). Then the projection $G(K) \to \mathcal{GR}$ induces an isomorphism $H^2_{DR}(\mathcal{GR}) \approx H^2_{DR}(G(K))$ (indeed, this projection is a $G(O)$-torsor, $G(O)$ is connected, and $H^1_{DR}(G(O)) = H^2_{DR}(G(O)) = 0$). Now our c defines the Kac-Moody cocycle $u, v \mapsto \mathrm{Res}_{t=0}\, c(du, v)$ on $\mathfrak{g} \otimes K$. Let ω_c be the corresponding right invariant closed 2-form on $G(K)$. The image of its class by the inverse map to the above isomorphism is $\nu(c) \in H^2_{DR}(\mathcal{GR})$. WHAT ABOUT THE SIGN???

Remark. In 5.3.23(iii) we used the action of $H^{\cdot}_{DR}(\mathcal{GR})$ on $H^{\cdot}_{DR}(\mathcal{GR}, M)$ where M is a $\mathcal{D}$-module on $\mathcal{GR}$. It is defined as follows. Consider the $\Omega^!$-complex ΩM (see 7.11.13). Then $H^{\cdot}_{DR}(\mathcal{GR}, M) = \varinjlim H^{\cdot}(Y, \Omega M_{(Y)})$ where Y runs over the set of all subschemes of $\mathcal{GR}$. Now $\Omega M_{(Y)}$ is an Ω-complex on Y, so $H^{\cdot}(Y, \Omega_Y)$ acts on $H^{\cdot}(Y, \Omega M_{(Y)})$. Therefore $H^{\cdot}_{DR}(\mathcal{GR})$ acts on $H^{\cdot}_{DR}(\mathcal{GR}, M)$.

5.3.25. The brief characterization of the canonical isomorphism (274) given in 5.3.23 is enough for our purposes. Those who want to understand (274) better may read ???-??? and [MV].

5.3.26.

Remark. Recall (see 4.5.9) that the connected components of $\mathcal{GR}$ are labeled by elements of $Z({}^LG)^\vee$ where $Z({}^LG)^\vee$ is the group of characters of the center $Z({}^LG) \subset {}^LG$. The connected component of $\mathcal{GR}$ corresponding

*) We leave it to the reader to define ν for arbitrary G.

to $\zeta \in Z({}^LG)^\vee$ will be denoted by $\mathcal{GR}_\zeta$. The support decomposition $D(\mathcal{GR}) = \prod D(\mathcal{GR}_\zeta)$, $\mathcal{P} = \oplus \mathcal{P}_\zeta$ defines a $Z({}^LG)^\vee$-grading, i.e., a $Z({}^LG)$-action, on h. This action coincides with the one induced by the ${}^LG$-action.

In the rest of the section we explain how the above constructions are compatible with passage to a Levi subgroup of ${}^LG$. When this subgroup is ${}^LH \subset {}^LG$ this amounts to an explicit description of the action of ${}^LH$ on the fiber functor h due to Mirković – Vilonen.

5.3.27. Let $P \subset G$ be a parabolic subgroup, $N_P \subset P$ its unipotent radical, $F := P/N_P$ the Levi group. The Cartan tori of F and G are identified in the obvious way, and the root datum for F is a subset of that for G. So ${}^LF$ is a Levi subgroup of ${}^LG$ for the standard torus ${}^LH \subset {}^LF \subset {}^LG$. Thus $Z({}^LG) \subset Z({}^LF)$.

We are going to define a canonical tensor functor

(278)
$$r^\natural_{\mathcal{P}} : \mathcal{P}^\natural_G \to \mathcal{P}^\natural_F$$

which corresponds, via the equivalences h_G, h_F, to the obvious restriction functor $r^{GF} : \operatorname{Rep} {}^LG \to \operatorname{Rep} {}^LF$.

5.3.28. The diagram $G \hookleftarrow P \twoheadrightarrow F$ yields the morphisms of the corresponding affine Grassmanians

(279)
$$\mathcal{GR}^G \xleftarrow{i} \mathcal{GR}^P \xrightarrow{\pi} \mathcal{GR}^F .$$

Here π is a formally smooth ind-affine surjective projection. Its fibers are $N_P(K)$-orbits. Hence π yields a bijection between the sets of connected components of $\mathcal{GR}^P$ and $\mathcal{GR}^F$. For any $\zeta \in Z({}^LF)^\vee$ let $\mathcal{GR}^P_\zeta$ be the corresponding component. Then the restriction $i_\zeta : \mathcal{GR}^P_\zeta \hookrightarrow \mathcal{GR}^G$ of i is a locally closed embedding; its image lies in $\mathcal{GR}^G_{\overline{\zeta}}$ where $\overline{\zeta} := \zeta|_{Z({}^LG)}$. The ind-schemes $\mathcal{GR}^P_\zeta$ form a stratification of $\mathcal{GR}^G$ (i.e., for any closed subscheme $Y \subset \mathcal{GR}^G$ the intersections $Y_\zeta := Y \cap \mathcal{GR}^P_\zeta$ form a stratification of Y).

Set $\rho_{GF} := \rho_G - \rho_F \in \mathfrak{h}^*$. Since $2\rho_{GF}$ is a character of F (the determinant of the adjoint action on n_P) we may consider it as a one-parameter subgroup of $Z({}^L F) \subset {}^L H$. So for any ζ as above one has an integer $\langle \zeta, 2\rho_{GF} \rangle$. Let $\mathcal{GR}^F_n$ be the union of components $\mathcal{GR}^F_\zeta$ with $\langle \zeta, 2\rho_{GF} \rangle = n$. We have the corresponding decomposition $D(\mathcal{GR}^F) = \prod D(\mathcal{GR}^F_n)$, $\mathcal{P}^F = \oplus \mathcal{P}^F_n$. Set $\mathcal{P}^{F'} = \oplus \mathcal{P}^F_n[-n] \subset D(\mathcal{GR}^F)$. As in 5.3.18 for $M \in \mathcal{P}^{F'}$ we set $h^{\cdot}_F(M) = H^{\cdot}(\mathcal{GR}^F, M) \in \mathrm{Vect}^{\cdot}$.

5.3.29. *Proposition.*

(i) The functor $r^{GF}_D := \pi_* i^! : D(\mathcal{GR}^G) \to D(\mathcal{GR}^F)$ sends $\mathcal{P}^G$ to $\mathcal{P}^{F'}$, so we have

$$r^{GF}_{\mathcal{P}} : \mathcal{P}^G \to \mathcal{P}^{F'} . \tag{280}$$

(ii) There is a canonical identification of functors

$$h^{\cdot}_G = h^{\cdot}_F r^{GF}_{\mathcal{P}} : \mathcal{P}^G \to \mathrm{Vect}^{\cdot} . \tag{281}$$

Proof. Assume first that $P = B$ is a Borel subgroup. Then $F = H$ and $\mathcal{GR}^H_{\mathrm{red}} = ({}^L H)^\vee$, so $\mathcal{D}$-modules on $\mathcal{GR}^H$ are the same as $({}^L H)^\vee$-graded vector spaces, i.e., ${}^L H$-modules. The strata $\mathcal{GR}^B_\zeta$ are just $N_B(K)$-orbits on $\mathcal{GR}^G$. Thus 5.3.29 is just the key theorem of [MV].

Recall that the identification (281) is constructed as follows (see [MV]). Let $\overline{\mathcal{GR}}{}^B_n \subset \mathcal{GR}^G$ be the closure of $\mathcal{GR}^B_n := \pi^{-1}(\mathcal{GR}^H_n)$ in $\mathcal{GR}^G$. Then $\overline{\mathcal{GR}}{}^B_{\cdot}$ is a decreasing filtration on $\mathcal{GR}^G$. For any $M \in \mathcal{P}^G$ the obvious morphisms $h^n_H r^{GH}_{\mathcal{P}}(M) = H^n(\mathcal{GR}^B_n, i^! M) \longleftarrow H^n_{\overline{\mathcal{GR}}{}^B_n}(\mathcal{GR}^G, M) \longrightarrow H^n(\mathcal{GR}^G, M) = h^n_G(M)$ are isomorphisms. Their composition is (281).

Now let P be any parabolic subgroup. Choose a Borel subgroup $B \subset P$, so $B_F := B/N_P \cap B$ is a Borel subgroup of F. Consider the functors $r^{GH}_D : D(\mathcal{GR}^G) \to D(\mathcal{GR}^H)$, $r^{FH}_D : D(\mathcal{GR}^F) \to D(\mathcal{GR}^H)$. By base change one has a canonical identification of functors $r^{GH}_D = r^{FH}_D r^{GF}_D$. Let $\mathcal{P}^{H'} \subset D(\mathcal{GR}^H)$ be the category defined by $B \subset G$, so we know that $r^{GH}_D(\mathcal{P}^G) \subset \mathcal{P}^{H'}$ and (since $\rho_{GF} = \rho_G - \rho_F$) one has $r^{FH}_D(\mathcal{P}^{F'}) \subset \mathcal{P}^{H'}$.

The functor $r_{\mathcal{P}}^{GH} : \mathcal{P}^{F'} \to \mathcal{P}^{H'}$ is faithful (since up to shift if coincides with $h_F^{\cdot}$). Hence an object $T \in D(\mathcal{GR}^F)$ such that all H^iT are in $\mathcal{P}^F$ belongs to $\mathcal{P}^{F'}$ if and only if $r_D^{FH}(T) \subset \mathcal{P}^{H'}$. Applying this remark to $T = r_D^{GF}(M)$, $M \in \mathcal{P}^G$, we see that $r_D^{GF}(M) \in \mathcal{P}^{F'}$, which is 5.3.29 (i). We also know that $h_G^{\cdot}(M) = h_H^{\cdot}(r_{\mathcal{P}}^{GH}(M)) = h_H^{\cdot}(r_{\mathcal{P}}^{FH}(M)) = h_F^{\cdot} r^{GF}(M)$ which is the identification 5.3.29 (ii). We leave it to the reader check that it does not depend on the auxiliary choice of a Borel subgroup $B \subset P$. □

5.3.30. The category $\mathcal{P}^{F'}$ has a canonical tensor structure (defined by the same constructions that were used for $\mathcal{P}^F$). The functor $r_{\mathcal{P}}^{GF} : \mathcal{P}^G \to \mathcal{P}^{F'}$ is a tensor functor in a canonical manner. Indeed, (279) are morphisms of chiral semi-groups, so we may consider the corresponding functors $r_D^{GF} := \pi_* i^! : D(\mathcal{GR}_{X^J}^G) \to D(\mathcal{GR}_{X^J}^F)$. We leave it to the reader to check (hint: use 5.3.19) that for $M_j \in \mathcal{P}^G$ this functor sends $\boxtimes M_j$ to $\boxtimes r_D^{GF}(M_j)$ (see 5.3.14 for notation). Since (by base change) it also commutes with the functors $\widetilde{\Delta}^{(J)!}$ we get the desired tensor product compatibilities. As in 5.3.19 we see that (281) is an isomorphism of tensor functors.

Finally let us replace, as in 5.3.21, the tensor category $\mathcal{P}^G$ by $\mathcal{P}^{G\natural}$. Since $\rho_{GF} = \rho_G - \rho_F$ we see that $r_{\mathcal{P}}^{GF}$ yields a tensor functor $r^{GF} : \mathcal{P}^{G\natural} \to \mathcal{P}^{F\natural}$ compatible with the fiber functors h_G, h_F. It defines a morphism $r : \mathrm{Aut}^{\otimes} h_F \to \mathrm{Aut}^{\otimes} h_G$.

5.3.31. *Lemma.* The morphism $\varkappa_G^{-1} r \varkappa_F : {}^LF \to {}^LG$ coincides with the canonical embedding from 5.3.27. □

5.4. **Main Theorems II: from local to global.** In this section we give the precise version of the main theorems from 5.2 and show that the local main theorem implies the global one. We use in essential way the "Hecke pattern" from Chapter 7. To understand what is going on it is necessary (and almost sufficient) to read 7.1.1 and 7.9.1.

5.4.1. We start with the definition of Hecke eigen-$\mathcal{D}$-module. Consider the pair $(G(K), G(O))$ equipped with the action of $\mathrm{Aut}\, O$. Let $\mathcal{H}$ be the corresponding $(\operatorname{Der} O, \operatorname{Aut}^0 O)$-equivariant Hecke category as defined in 7.9.2*). Since any object of $\mathcal{P}$ is an $\operatorname{Aut} O$-equivariant $\mathcal{D}$-module in a canonical way*) our $\mathcal{P}$ is a full subcategory of $\mathcal{H}$. It follows from the definitions that the embedding $\mathcal{P} \to \mathcal{H}$ is a monoidal functor.

Consider the canonical $\operatorname{Aut} O$-structure $X^\wedge$ on X (see 2.6.5) and the scheme $M^\wedge$ over $X^\wedge$ defined in 2.8.3; it carries a canonical action of $\operatorname{Aut} O \ltimes G(K)$ (see 2.8.3 - 2.8.4). The quotient stack $(\operatorname{Aut}^0 O \ltimes G(O)) \setminus M^\wedge$ equals $\operatorname{Bun}_G \times X$. We arrive to the setting of 7.9.1, 7.9.4*). Thus $\mathcal{H}$ acts on $D(\operatorname{Bun}_G \times X)$. Therefore $D(\operatorname{Bun}_G \times X)$ is a $\mathcal{P}$-Module. Identifying the monoidal category $\mathcal{P}$ *) with $\operatorname{Rep} {}^L G$ via the Satake equivalence (273) one gets a canonical Action of $\operatorname{Rep} {}^L G$ on $D(\operatorname{Bun}_G \times X)$ called the Hecke Action. We denote it by $\circledast$.

Note that $D(\operatorname{Bun}_G \times X)$ also carries an obvious Action of the tensor category $\operatorname{Vect}^\nabla(X)$ of vector bundles with connection on X (or, in fact, of the larger tensor category of torsion free left $\mathcal{D}$-modules on X) which we denote by $\otimes$. It commutes with the Hecke Action, so $D(\operatorname{Bun}_G \times X)$ is a $(\operatorname{Rep} {}^L G, \operatorname{Vect}^\nabla(X))$-biModule.

Let $\mathfrak{F}$ be an ${}^L G$-bundle with a connection on X. It yields a tensor functor $\operatorname{Rep} {}^L G \to \operatorname{Vect}^\nabla(X)$, $V \to V_{\mathfrak{F}}$, hence the corresponding Action of $\operatorname{Rep} {}^L G$ on $D(\operatorname{Bun}_G \times X)$.

5.4.2. Let M be a $\mathcal{D}$-module on Bun_G. Let $M_{(X)} \in \mathcal{M}(\operatorname{Bun}_G \times X)$ be the pull-back of M. Assume that for any $V \in \operatorname{Rep} {}^L G$ we are given a natural isomorphism $\alpha_V : V \circledast M_{(X)} \xrightarrow{\sim} M_{(X)} \otimes V_{\mathfrak{F}}$ (so, in particular, $V \circledast M_{(X)}$ is a $\mathcal{D}$-module, and not merely an object of the derived category). We say that the α_V's define a *Hecke $\mathcal{F}$-eigenmodule structure* on M if for any

*) Our $(G(K), G(O))$, $(\operatorname{Der} O, \operatorname{Aut}^0 O)$ are (G, K), $(\mathfrak{l}, P)$ of 7.9.2.

*) According to 5.3.4 any object of $\mathcal{P}$ carries a unique strong $\operatorname{Aut}^0 O$-action which is the same as a strong $\operatorname{Aut} O$-action.

*) Our $X^\wedge$ and $M^\wedge$ are $X^\wedge$ and $Y^\wedge$ of 7.9.4.

*) In this section (except Remarks 5.4.6) we use only the monoidal structure on $\mathcal{P}$ (the commutativity constraint plays no role). So we may identify $\mathcal{P}$ with $\mathcal{P}^\natural$.

$V_1, V_2 \in \operatorname{Rep}{}^L G$ one has $\alpha_{V_1 \otimes V_2} = \alpha_{V_1} \circ (V_1 \circledast \alpha_{V_2})$. We call such (M, α_V), or simply M, a *Hecke* $\mathfrak{F}$*-eigenmodule.*

Remark. For any ${}^L G$-local system $\mathcal{F}$ on X one would like to define the triangulated category of Hecke $\mathfrak{F}$-eigenmodules*).

The following theorem is the precise version of Theorem 5.2.6.

5.4.3. *Theorem.* For any ${}^L G$-oper $\mathfrak{F}$ the $\mathcal{D}$-module $M_{\mathfrak{F}}$ defined in 5.1.1 has a natural structure of Hecke $\mathfrak{F}$-eigenmodule.

We leave it to the reader to check that the functor T^i_χ coincides with $H^i V^\chi \circledast$ (see 5.2.4, 5.2.5 for notation). Thus Theorem 5.4.3 implies 5.2.6.

5.4.4. We need a version of 5.4.1-5.4.3 "with parameters". Let A be a commutative ring. Denote by $\mathcal{M}(\mathrm{Bun}_G \times X, A)$ the category of A-modules in $\mathcal{M}(\mathrm{Bun}_G \times X)$ (i.e., $\mathcal{D}$-modules with A-action). It has a derived version $D(\mathrm{Bun}_G \times X, A)$, which is a t-category with core $\mathcal{M}(\mathrm{Bun}_G \times X, A)$ (see 7.3.13). The category $D(\mathrm{Bun}_G \times X, A)$ carries, as in 5.4.1, the Hecke Action of $\operatorname{Rep}{}^L G$.

We also have the obvious Action of the tensor category of $A \otimes \mathcal{O}_X$-flat $A \otimes \mathcal{D}_X$-modules on $D(\mathrm{Bun}_G \times X, A)$ which commutes with the Hecke Action. Therefore any flat A-family $\mathcal{F}_A$ of ${}^L G$-bundles with connection on X yields an Action of $\operatorname{Rep}{}^L G$ on $D(\mathrm{Bun}_G \times X, A)$.

Now for $M \in \mathcal{M}(\mathrm{Bun}_G, A)$ one defines the notion of Hecke $\mathfrak{F}_A$-eigenmodule structure on M as in 5.4.2. The following theorem is the precise version of 5.2.9; by 5.1.2(i) it implies 5.4.3.

5.4.5. *Theorem.* The $\mathcal{D}$-module $M_{\mathcal{L}} \in \mathcal{M}(\mathrm{Bun}_G, A_{{}^L\mathfrak{g}}(X))$ defined in 5.1.1 has a canonical structure of Hecke $\mathfrak{F}_{\mathcal{L}}$-eigenmodule.

5.4.6. *Remarks.* (i) Sometimes (when you want to use the commutativity constraint, see, e.g., the next Remark or the next section) it is convenient to

*) Certainly, in the above definition of Hecke eigenmodule you may take for M any object of $D(\mathrm{Bun}_G)$ instead of just a $\mathcal{D}$-module. However in this generality the definition does not look reasonable (such objects do not form a triangulated category).

deal with the above notions in the setting of super $\mathcal{D}$-modules. Note that any $\mathcal{D}$-module M on Bun_G has a canonical $\mathbb{Z}/2\mathbb{Z}$-grading such that M is even or odd depending on whether M is supported on even or odd components of Bun_G. We denote this super $\mathcal{D}$-module by $M^\natural$. So $\natural$ identifies $\mathcal{M}(\mathrm{Bun}_G)$ with a full subcategory $\mathcal{M}(\mathrm{Bun}_G)^\natural$ of $\mathcal{M}(\mathrm{Bun}_G)^\varepsilon := \mathcal{M}(\mathrm{Bun}_G)\otimes \mathrm{Vect}^\varepsilon$. The same applies to $D(\mathrm{Bun}_G)$ and $D(\mathrm{Bun}_G\times X)$.

The Action of $\mathcal{P}$ on $D(\mathrm{Bun}_G\times X)$ yields an Action of $\mathcal{P}^\varepsilon$ on $D(\mathrm{Bun}_G\times X)^\varepsilon$. The Action of $\mathcal{P}^\natural\subset\mathcal{P}^\varepsilon$ preserves $D(\mathrm{Bun}_G\times X)^\natural$, as well as the $\mathrm{Vect}^\nabla(X)$-Action. Now one defines the notion of Hecke $\mathfrak{F}$-eigenobject of $\mathcal{M}(\mathrm{Bun}_G)^\natural$ exactly as in 5.4.2. This definition brings nothing new: a $\mathcal{D}$-module M is a Hecke $\mathfrak{F}$-eigenmodule if and only if $M^\natural$ is.

(ii) In the above definition of the $\mathcal{F}$-eigenmodule structure on $M\in\mathcal{M}(\mathrm{Bun}_G)$ we used the convolution construction of the tensor structure on $\mathcal{P}$. One may rewrite it instead using the fusion construction of $\circledast$ as follows. DOPISAT'!!!

5.4.7. Let us turn to the main local theorems from 5.2. We are in the setting of 5.2.12, so we fix $\mathcal{L}\in Z\,\mathrm{tors}_\theta(O)$, which defines the central extension $\widetilde{G(K)}=\widetilde{G(K)}_{\mathcal{L}}$ of $G(K)$ split over the group subscheme $G(O)$ (see 4.4.9). We have the corresponding category of twisted Harish-Chandra modules $\mathcal{M}(\mathfrak{g}\otimes K, G(O))'$ and the derived category $D(\mathfrak{g}\otimes K, G(O))'$ of Harish-Chandra complexes (see 7.8.1 and 7.14.1)[*)]. According to 7.8.2, 7.14.1, $D(\mathfrak{g}\otimes K, G(O))'$ carries a canonical Action $\circledast$ of the Hecke monoidal category $\mathcal{H}$ of the pair $(G(K),G(O))$. Since $\mathcal{P}$ is a monoidal subcategory of (the core of) $\mathcal{H}$ our $D(\mathfrak{g}\otimes K, G(O))'$ is a $\mathcal{P}$-Module.

Let $Vac'\in\mathcal{M}(\mathfrak{g}\otimes K, G(O))'$ be the twisted vacuum module.

5.4.8. *Theorem.* For any object $P\in\mathcal{P}$ the object $P\circledast Vac'\in D(\mathfrak{g}\otimes K, G(O))'$ is isomorphic to a direct sum of copies of Vac' [*)].

[*)] So $\mathbf{1}\in\mathbb{C}\subset\widetilde{\mathfrak{g}\otimes K}$ acts on the objects of these categories as identity.

[*)] In particular it is a single Harish-Chandra module, not merely a complex of those.

This theorem is equivalent to 5.2.14. Indeed, according to (335) of 7.8.5 and 7.14.1, there is a canonical identification of $(\widetilde{\mathfrak{g}\otimes K}, G(O))$-modules

$$H^i(P \circledast Vac') = H^i(\mathcal{GR}, P\lambda_{\mathcal{L}}^{-1}). \tag{282}$$

Here $P\lambda_{\mathcal{L}}^{-1} := P \otimes \lambda_{\mathcal{L}}^{-1}$. The interested reader may pass directly to the proof of this theorem, which can be found in ???.

5.4.9. We need to incorporate the $\operatorname{Aut} O$ symmetry in the above setting. Recall (see 4.6.6) that the action of $\operatorname{Aut} O$ on $G(K)$ lifts to the action of $\operatorname{Aut}_Z O$ on $\widetilde{G(K)}$ that preserves $G(O)$. So we are in the setting of 7.9.5[*)]. Let D_{HC} be the derived category of Harish-Chandra complexes as defined in 7.9.5. This is a t-category with core $\mathcal{M}_{HC}$ equal to the category of Harish-Chandra modules for the pair $(\operatorname{Der} O \ltimes \widetilde{\mathfrak{g}\otimes K}, \operatorname{Aut}_Z^0 O \ltimes G(O))$ (we assume that the center $\mathbb{C} \subset \widetilde{\mathfrak{g}\otimes K}$ acts in the standard way).

The $(\operatorname{Der} O, \operatorname{Aut}_Z^0 O)$-equivariant Hecke category for $(G(K), G(O))$ (see 7.9.2) contains the corresponding $(\operatorname{Der} O, \operatorname{Aut}^0 O)$-equivariant categories $\mathcal{H}$ and $\mathcal{H}^c$ as full monoidal subcategory. So, by 7.9.5, D_{HC} is an $\mathcal{H}$-Module. hence it is a $\mathcal{P}$-Module.

We will need to change slightly our setting. Let as usual $\mathfrak{Z}$ be the center of the completed twisted universal enveloping algebra of $\mathfrak{g}\otimes K$ and $\mathfrak{z}$ the endomorphism ring of the twisted vacuum module Vac'; we have the obvious morphism of algebras $e : \mathfrak{Z} \to \mathfrak{z}$. Let $D_{HC\mathfrak{z}}$ be the corresponding derived category of Harish-Chandra complexes as defined in 7.9.8 (see also 7.9.7(iii))[*)]. This is a t-category with core $\mathcal{M}_{HC\mathfrak{z}}$ equal to the category of Harish-Chandra modules killed by $\operatorname{Ker} e$.

Let $\mathcal{H}_{\mathfrak{z}}$ be the $\mathfrak{z}$-linear version of the $(\operatorname{Der} O, \operatorname{Aut}_Z^0 O)$-equivariant Hecke category for $(G(K), G(O))$ as defined in 7.9.7(i). According to 7.9.8 it acts on $D_{HC\mathfrak{z}}$. Due to the obvious monoidal functor $\mathcal{H} \to \mathcal{H}_{\mathfrak{z}}$ (see the Remark in

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[*)] Our $(\operatorname{Der} O, \operatorname{Aut}_Z^0 O)$ and $(\widetilde{G(K)}, G(O))$ are $(\mathfrak{l}, P)$ and (G', K) of 7.9.5.

[*)] Our $D_{HC\mathfrak{z}}$ is $D^e_{HC\,A}$ of 7.9.8. In 7.9.8 $\mathfrak{Z}$ denotes the set of $G(K)$-invariant elements of the center, but according to 3.7.7(ii) all elements of the center are $G(K)$-invariant.

7.9.7) $\mathcal{H}_{\mathfrak{z}}$ contains $\mathcal{P}$, so $D_{HC_{\mathfrak{z}}}$ is a $\mathcal{P}$-Module. As in 5.4.1 we will replace $\mathcal{P}$ by $\operatorname{Rep}{}^L G$ by means of the Satake equivalence and denote the corresponding Action of $\operatorname{Rep}{}^L G$ on $D_{HC_{\mathfrak{z}}}$ by $\circledast$. On the other hand $\mathcal{H}_{\mathfrak{z}}$ contains in its center the tensor category $\mathcal{M}(\operatorname{Aut}_Z O)_{\mathfrak{z}}^{fl}$ of flat $\mathfrak{z}$-modules equipped with $\operatorname{Aut}_Z O$-action (see 7.9.7(i)). The corresponding Action of $\mathcal{M}(\operatorname{Aut}_Z O)_{\mathfrak{z}}^{fl}$ on $D_{HC_{\mathfrak{z}}}$ is the obvious one: for $W \in \mathcal{M}(\operatorname{Aut}_Z O)_{\mathfrak{z}}^{fl}$, $V \in D_{HC_{\mathfrak{z}}}$ one has $W \circledast V = W \otimes V := W \underset{\mathfrak{z}}{\otimes} V$. Therefore $D_{HC_{\mathfrak{z}}}$ is a $(\operatorname{Rep}{}^L G, \mathcal{M}(\operatorname{Aut}_Z O)_{\mathfrak{z}}^{fl})$-biModule.

Let $\mathfrak{F}$ be an $\operatorname{Aut}_Z O$-equivariant ${}^L G$-torsor on $\operatorname{Spec}\mathfrak{z}$. It yields the tensor functor $\operatorname{Rep}{}^L G \to \mathcal{M}(\operatorname{Aut}_Z O)_{\mathfrak{z}}^{fl}$, $V \mapsto V_{\mathfrak{F}}$, hence the corresponding Action of $\operatorname{Rep}{}^L G$ on $D_{HC_{\mathfrak{z}}}$.

5.4.10. Let us repeat the definition from 5.4.2 in the present Harish-Chandra setting. Namely, a *Hecke $\mathfrak{F}$-eigenmodule* is a Harish-Chandra module $M \in \mathcal{M}_{HC_{\mathfrak{z}}}$ together with natural isomorphisms $\alpha_V : V \circledast M \stackrel{\sim}{\to} M \otimes V_{\mathfrak{F}}$, $V \in \operatorname{Rep}{}^L G$, such that for any $V_1, V_2 \in \operatorname{Rep}{}^L G$ one has $\alpha_{V_1 \otimes V_2} = \alpha_{V_1} \circ (V_1 \circledast \alpha_{V_2})$.

Now we can formulate the precise version of 5.2.16. As in 5.2.15, our $\mathcal{L} \in Z\operatorname{tors}_\theta(O)$ (see 5.4.7) defines an $\operatorname{Aut}_Z O$-equivariant*) ${}^L G$-torsor over the moduli scheme of local ${}^L\mathfrak{g}$-opers. Identifying this scheme with $\operatorname{Spec}\mathfrak{z}$ via the Feigin-Frenkel isomorphism (80) we get the corresonding $\operatorname{Aut}_Z O$-equivariant torsor $\mathfrak{F}_{\mathcal{L}}$ over $\operatorname{Spec}\mathfrak{z}$.

From now on we consider Vac' as an object of $\mathcal{M}_{HC_{\mathfrak{z}}}$ (with respect to the $\operatorname{Aut}_Z O$-action that fixes the vacuum vector).

5.4.11. *Theorem.* Vac' has a canonical structure of Hecke $\mathfrak{F}_{\mathcal{L}}$-eigenmodule.

This theorem implies 5.2.16. Indeed, the isomorphism (282) is $\operatorname{Aut}_Z O$-equivariant since $\operatorname{Aut}_Z O$ acts on both sides of (282) by transport of structure.

Where will it be proved???

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*) The action of $\operatorname{Aut}_Z O$ comes from the identification $\operatorname{Aut}_Z O = \operatorname{Aut}(O, \mathcal{L})$; see 4.6.6.

Now we may turn to the main result of this section.

5.4.12. *Theorem.* Theorem 5.4.11 implies 5.4.5.

Proof. We will show that an appropriate "localization functor" $L\Delta$ transforms the local picture into the global one $^{*)}$.

We need to modify slightly the setting of 5.4.1 to be able to use the "equivariant Hecke pattern" from 7.9. Recall that in the formulation of the global theorem 5.4.5 we fixed $\mathcal{L}^{\mathrm{glob}} \in Z\,\mathrm{tors}_\theta(X)$ (see 5.2.8), while in the local theorem 5.4.11 we used $\mathcal{L}^{\mathrm{loc}} \in Z\,\mathrm{tors}_\theta(O)$. Consider the schemes $X_Z^\wedge$ and $M_Z^\wedge$ from 4.4.15 corresponding to $\mathcal{L}^{\mathrm{glob}}$ and $\mathcal{L}^{\mathrm{loc}}$ (they are etale Z-coverings of the schemes $X^\wedge$ and $M^\wedge$ used in 5.4.1). Recall that $\mathrm{Aut}_Z\, O$ acts on $X_Z^\wedge$ and $\mathrm{Aut}_Z\, O \ltimes G(K)$ acts on $M_Z^\wedge$ (see 4.4.15). One has $\mathrm{Aut}_Z^0\, O \backslash X_Z^\wedge = X$, and the quotient stack $(\mathrm{Aut}_Z^0\, O \ltimes G(O)) \setminus M_Z^\wedge$ equals $\mathrm{Bun}_G \times X$. It is clear that in the construction of the Hecke Action on $D(\mathrm{Bun}_G \times X)$ in 5.4.1 we may replace $(M^\wedge, \mathrm{Aut}\, O \ltimes G(K))$ by $(M_Z^\wedge, \mathrm{Aut}_Z\, O \ltimes G(K))$.

As in 5.1.1 let $\lambda_{\mathcal{L}^{\mathrm{glob}}}$ be the Pfaffian line bundle on Bun_G that corresponds to $\mathcal{L}^{\mathrm{glob}}$. Denote by $\widehat{\lambda} = \widehat{\lambda}_{\mathcal{L}^{\mathrm{glob}}}$ its pull-back to $M_Z^\wedge$. The action of $\mathrm{Aut}_Z\, O \ltimes G(K)$ on $M_Z^\wedge$ lifts in a canonical way to an action on $\widehat{\lambda}$ of the central extension $\mathrm{Aut}_Z\, O \ltimes \widetilde{G(K)}$ (see 4.4.16). So we are in the setting of 7.9.6$^{*)}$, and therefore, one has the right t-exact localization functor

$$L\Delta : D_{HC} \to D(\mathrm{Bun}_G \times X)$$

One has also the corresponding picture in the setting of $\mathfrak{z}$-modules. Indeed, following 7.9.7(ii), consider the $\mathcal{D}_X$-algebra $\mathfrak{z}_X$ $^{*)}$ (which we already used in 2.7) and the corresponding category $D(\mathrm{Bun}_G \times X, \mathfrak{z}_X)$ which is the derived category of $\mathcal{D}$-modules on $\mathrm{Bun}_G \times X$ equipped with $\mathfrak{z}_X$-action (see 7.3.13).

$^{*)}$The reader may decide if there is a method in this madness.

$^{*)}$Sorry for a terrible discrepancy of notations: our $M_Z^\wedge$, $X^\wedge$, $\widehat{\lambda}$, $\mathrm{Der}\, O$, $\mathrm{Aut}_Z^0\, O$, $\widetilde{G(K)}$, $G(O)$ are $Y^\wedge$, $X^\wedge$, $\mathcal{L}^*$, $\mathfrak{l}$, P, G', K of 7.9.6.

$^{*)}$Any $\mathrm{Aut}\, O$-module V yields the $\mathcal{D}_X$-module V_X, see 2.6.6.

It carries a canonical Action of $\mathcal{H}_{\mathfrak{z}}$. One has a canonical localization functor

$$L\Delta_{\mathfrak{z}} : D_{HC_{\mathfrak{z}}} \to D(\mathrm{Bun}_G \times X, \mathfrak{z}_X)$$

which is a Morhism of $\mathcal{H}_{\mathfrak{z}}$-Modules. The above $L\Delta$'s are compatible (they commute with the forgetting of $\mathfrak{z}$-action).

Now our theorem is immediate consequence of the following facts:

(a) There is a natural identification

$$L\Delta(Vac') = \Delta(Vac') = M_{\mathcal{L}^{\mathrm{glob}}} \boxtimes \mathcal{O}_X \tag{283}$$

such that the $\mathfrak{z}_X$-action on $\Delta(Vac') = \Delta_{\mathfrak{z}}(Vac')$ coincides with the action of $\mathfrak{z}_X$ on $M_{\mathcal{L}^{\mathrm{glob}}} \boxtimes \mathcal{O}_X$ through the maximal constant quotient $\mathfrak{z}(X) \otimes \mathcal{O}_X = A_{^L\mathfrak{g}}(X) \otimes \mathcal{O}_X$ and the standard $A_{^L\mathfrak{g}}(X)$-module structure on $M_{\mathcal{L}^{\mathrm{glob}}}$. For a proof see 7.14.9 (and note that $\mathfrak{z}_X$ acts by transport of structure).

(b) The functor $L\Delta_{\mathfrak{z}}$ is a Morphism of $(\mathrm{Rep}\, {}^LG, \mathcal{M}(\mathrm{Aut}_Z\, O)^{fl}_{\mathfrak{z}})$-biModules. Indeed, this is a Morphism of $\mathcal{H}_{\mathfrak{z}}$-Modules.

(c) For any $W \in \mathcal{M}(\mathrm{Aut}_Z\, O)^{fl}_{\mathfrak{z}}$, $T \in D(\mathrm{Bun}_G \times X, \mathfrak{z}_X)$ one has $W \circledast T = W_X \underset{\mathfrak{z}_X}{\otimes} T$ where W_X is the $\mathfrak{z}_X$-module that corresponds to W.

For a proof see 7.9.7(i).

(d) For any $V \in \mathrm{Rep}\, {}^LG$ there is a canonical identification

$$(V_{\mathfrak{F}_{\mathcal{L}^{\mathrm{loc}}}})_X \underset{\mathfrak{z}_X}{\otimes} (\mathfrak{z}(X) \otimes \mathcal{O}_X) \overset{\sim}{\to} V_{\mathfrak{F}_{\mathcal{L}^{\mathrm{glob}}}}$$

compatible with tensor products of V's (here $\mathfrak{F}_{\mathcal{L}^{\mathrm{loc}}}$ is $\mathfrak{F}_{\mathcal{L}}$ from 5.4.10). $\square$

5.5. **The birth of opers.** *In this section we assume Theorem 5.4.8.* We first show that this theorem implies that Vac' is a Hecke $\mathcal{F}$-eigenmodule for some $\mathrm{Aut}_Z\, O$-equivariant ${}^LG$-torsor $\mathcal{F}$ on $\mathrm{Spec}\, \mathfrak{z}$. The main point of this section is that $\mathcal{F}$ comes naturally from an $\mathrm{Aut}_Z\, O$-equivariant $\mathfrak{z}$-family of local opers. Later we will see that the corresponding map from $\mathrm{Spec}\, \mathfrak{z}$ to the moduli of local opers coincides with the Feigin-Frenkel isomorphism, which yields the main local theorem.

5.5.1. For any $V \in \operatorname{Rep}{}^L G$ set

$$F_{\mathcal{H}}(V) := \operatorname{Hom}_{\widehat{\mathfrak{g}\otimes K}}(Vac', V \circledast Vac') = (V \circledast Vac')^{G(O)}. \tag{284}$$

This is an $\operatorname{Aut}_Z O$-equivariant $\mathfrak{z}$-module*). According to 5.4.8 it is a free $\mathfrak{z}$-module, so $F_{\mathcal{H}}(V) \in \mathcal{M}(\operatorname{Aut}_Z O)_{\mathfrak{z}}^{fl}$. One has a canonical isomorphism

$$V \circledast Vac' = Vac' \otimes F_{\mathcal{H}}(V). \tag{285}$$

Since the Action of $\mathcal{M}(\operatorname{Aut}_Z O)_{\mathfrak{z}}^{fl}$ commutes with the Hecke Action we get a canonical identification $F_{\mathcal{H}}(V_1 \otimes V_2) = F_{\mathcal{H}}(V_1) \otimes F_{\mathcal{H}}(V_2)$, which means that

$$F_{\mathcal{H}} : \operatorname{Rep}{}^L G \to \mathcal{M}(\operatorname{Aut}_Z O)_{\mathfrak{z}}^{fl} \tag{286}$$

is a monoidal functor.

5.5.2. *Lemma.* For any $V \in \operatorname{Rep}{}^L G$ the free $\mathfrak{z}$-module $F_{\mathcal{H}}(V)$ has finite rank.

Proof. Since $F_{\mathcal{H}}$ is a monoidal functor $F_{\mathcal{H}}(V^*)$ is dual to $F_{\mathcal{H}}(V)$ in the sense of monoidal categories (see 2.1.2 of [Del91]). If a free $\mathfrak{z}$-module has a dual in the sense of monoidal categories then its rank is finite. □

Let

$$F_{\mathfrak{F}_{\mathcal{L}}} : \operatorname{Rep}{}^L G \to \mathcal{M}(\operatorname{Aut}_Z O)_{\mathfrak{z}}^{fl} \tag{287}$$

be the tensor functor $F_{\mathfrak{F}_{\mathcal{L}}}(V) = V_{\mathfrak{F}_{\mathcal{L}}}$ (see 5.4.10).

Now our main local theorem 5.4.11 may be restated as follows.

5.5.3. *Theorem.* The monoidal functors $F_{\mathcal{H}}$ and $F_{\mathfrak{F}_{\mathcal{L}}}$ are canonically isomorphic.

We are going to show that $F_{\mathcal{H}}$ indeed comes from a *some* canonically defined family of local opers parametrized by $\operatorname{Spec}\mathfrak{z}$. First let us check that $F_{\mathcal{H}}$ indeed comes from an ${}^L G$-torsor on $\operatorname{Spec}\mathfrak{z}$.

*) The two $\mathfrak{z}$-module structures on $V \circledast Vac'$ coincide because the Hecke functors are $\mathfrak{z}$-linear.

5.5.4. *Proposition.* The monoidal functor $F_{\mathcal{H}}$ is a tensor functor, i.e., it is compatible with the commutativity constraints.

The proof has two steps. First we write down the compatibility isomorphism $F_{\mathcal{H}}(V_1) \otimes F_{\mathcal{H}}(V_2) \xrightarrow{\sim} F_{\mathcal{H}}(V_1 \otimes V_2)$ as convolution product of sections of (twisted) $\mathcal{D}$-modules (see 5.5.5, 5.5.6). Then, using the fusion picture for the convolution, we show that it is commutative (see **??**?).

5.5.5. Let us replace the tensor category of ${}^LG$-modules by that of $\mathcal{D}$-modules on the affine Grassmanian using the Satake equivalence h (see (273)). For $P \in \mathcal{P}^\natural$ we set $F_{\mathcal{H}}(P) := F_{\mathcal{H}}(hP)$. Thus (see (282))

$$F_{\mathcal{H}}(P) = \Gamma(\mathcal{GR}, P\lambda_{\mathcal{L}}^{-1})^{G(O)}. \tag{288}$$

Remark. Recall that P is a "super" $\mathcal{D}$-module and $\lambda_{\mathcal{L}}$ is a "super" line bundle. However their parities coincide (being equal to the parity of components of $\mathcal{GR}$), so $P\lambda_{\mathcal{L}}^{-1}$ is a plain even sheaf. These "super" subtleties will be relevant when we pass to the commutativity constraint.

To describe the compatibility isomorphism $F_{\mathcal{H}}(P_1) \otimes F_{\mathcal{H}}(P_2) \xrightarrow{\sim} F_{\mathcal{H}}(P_1 \circledast P_2)$ consider the integration morphism of $\mathcal{O}^!$-modules (we use notation of 5.3.5; for integration see 7.11.16 (**??**))

$$i_m : m.(P_1 \boxtimes' P_2) \to P_1 \circledast P_2. \tag{289}$$

The line bundle $\lambda_{\mathcal{L}}$ on $\mathcal{GR}$ is $G(O)$-equivariant and its pull-back by $m : G(K) \times_{G(O)} \mathcal{GR} \to \mathcal{GR}$ is identified canonically with the "twisted product" $\lambda_{\mathcal{L}} \boxtimes' \lambda_{\mathcal{L}}$*). So, twisting i_m by $\lambda_{\mathcal{L}}$, we get the morphism $m.((P_1\lambda_{\mathcal{L}}^{-1}) \boxtimes' (P_2\lambda_{\mathcal{L}}^{-1})) \to (P_1 \circledast P_2)\lambda_{\mathcal{L}}^{-1}$.

Passing to $G(O)$-invariant sections we get the *convolution map* (notice that $G(O)$-invariance permits to neglect the twist)

(290)
$$* : \Gamma(\mathcal{GR}, P_1\lambda_{\mathcal{L}}^{-1})^{G(O)} \otimes \Gamma(\mathcal{GR}, P_2\lambda_{\mathcal{L}}^{-1})^{G(O)} \to \Gamma(\mathcal{GR}, (P_1 \circledast P_2)\lambda_{\mathcal{L}}^{-1})^{G(O)}$$

*) This follows since, by definition, $\lambda_{\mathcal{L}}$ comes from a central extension of $G(K)$ equipped with a splitting over $G(O)$.

5.5.6. *Lemma.* The convolution map coincides with the compatibility isomorphism $F_{\mathcal{H}}(P_1)\otimes F_{\mathcal{H}}(P_2)\xrightarrow{\sim} F_{\mathcal{H}}(P_1\circledast P_2)$.

Proof. Consider the canonical isomorphism (the Action constraint) a : $P_1\circledast(P_2\circledast Vac')\xrightarrow{\sim}(P_1\circledast P_2)\circledast Vac'$. For $f\in \operatorname{Hom}(Vac',P_1\circledast Vac')$, $g\in \operatorname{Hom}(Vac',P_2\circledast Vac')$ the compatibility isomorphism sends $f\otimes g$ to $(P_1\circledast g)\circ f$.

$\square$

5.6. **The renormalized universal enveloping algebra.**

5.6.1. Let A be the completed universal enveloping algebra of $\widetilde{\mathfrak{g}\otimes K}$. According to 3.6.2 A is a flat algebra over $\mathbb{C}[h]$, $h:=\mathbf{1}-1$, and $A/hA=\overline{U}'$. The natural topology on A induces a topology on $A[h^{-1}]:=A\otimes_{\mathbb{C}[h]}\mathbb{C}[h,h^{-1}]$; in fact this is the inductive limit topology (represent $A[h^{-1}]$ as the inductive limit of $A\to A\to\dots$ where each arrow is multiplication by h).

Let $I\subset\mathfrak{Z}$ be the ideal from 3.6.5. Denote by J the preimage of $I\overline{U}'\subset\overline{U}'=A/hA$ in A ($I\overline{U}'$ is understood in the topological sense, i.e., $I\overline{U}'$ is the closed ideal of $\overline{U}'$ generated by I). J is a closed ideal of A containing hA. Denote by $A^\natural$ the union of the increasing sequence $A\subset h^{-1}J\subset h^{-2}J^2\subset\dots$ where J^k is understood in the topological sense. Finally set $U^\natural:=A^\natural/hA^\natural$.

$A^\natural$ is a topological algebra over $\mathbb{C}[h]$ (the topology is induced from $A[h^{-1}]$). So $U^\natural$ is a topological $\mathbb{C}$-algebra ($U^\natural$ is equipped with the quotient topology).

5.6.2. Set $Vac_A=A/A(\mathfrak{g}\otimes O)$ where $A(\mathfrak{g}\otimes O)$ denotes the closed left ideal of A generated by $\mathfrak{g}\otimes O$. I acts trivially on $Vac'=Vac_A/h\,Vac_A$. Since Vac_A is a flat $\mathbb{C}[h]$-module $A^\natural$ acts on Vac_A. Therefore $U^\natural$ acts on Vac'.

5.6.3. Denote by $U_0^\natural$ the image of A in $U^\natural$. $U_0^\natural$ is a subalgebra of $U^\natural$. We equip $U_0^\natural$ with the induced topology. The map $A\to U_0^\natural$ factors through $A/hA=\overline{U}'$ and actually through $\overline{U}'/I\overline{U}'$. So we get a surjective continuous homomorphism $f:\overline{U}'/I\overline{U}'\to U_0^\natural$. Probably f is a homeomorphism.

Anyway f induces a topological isomorphism $\mathfrak{z} = \mathfrak{Z}/I \xrightarrow{\sim} f(\mathfrak{z})$ (use the action of $U^\natural$ on Vac'). We will identify $\mathfrak{z}$ with $f(\mathfrak{z})$.

5.6.4. Let $J_I \subset A$ denote the preimage of $I \subset \overline{U}' = A/hA$. Denote by $U_1^\natural$ the image of $h^{-1}J_I$ in $U^\natural$. Equip $U_1^\natural$ with the topology induced from $U^\natural$. The topological algebra $U^\natural$ is generated by $U_1^\natural$.

5.6.5. *Lemma.*

(i) $U_1^\natural$ is a Lie subalgebra of $U^\natural$.

(ii) $U_0^\natural$ is an ideal of $U_1^\natural$.

(iii) $\mathfrak{z}U_1^\natural \subset U_1^\natural$, $U_1^\natural\mathfrak{z} \subset U_1^\natural$.

(iv) $[U_1^\natural, \mathfrak{z}] \subset \mathfrak{z}$.

Proof. We will use some properties of the Hayashi bracket $\{,\}$ defined in 3.6.2. (i) follows from the inclusion $[J_I, J_I] \subset hJ_I$, which is clear because $\{I, I\} \subset I$ (see 3.6.4 (i)). (ii) and (iii) are obvious. (iv) is clear because $\{I, \mathfrak{Z}\} \subset \{\mathfrak{Z}, \mathfrak{Z}\} \subset \mathfrak{Z}$. □

5.6.6. It follows from 5.6.5 that $U_1^\natural/U_0^\natural$ is a topological Lie algebroid over $\mathfrak{z}$. Multiplication by h^{-1} defines a map $J_I \to A^\natural$, which induces a Lie algebroid morphism

$$I/I^2 = J_I/(J_I^2 + hA) \to U_1^\natural/U_0^\natural \tag{291}$$

(see 3.6.5 for the definition of the algebroid structure on I/I^2). The morphism (291) is continuous and surjective. In fact it is a topological isomorphism (see ???).

5.6.7. Denote by $U_i^\flat$ the set of elements of $U_i^\natural$ annihilating the vacuum vector from Vac', $i = 0, 1$. Lemma 5.6.5 remains valid if $U_i^\natural$ is replaced by $U_i^\flat$, $i = 0, 1$. So $U_1^\flat/U_0^\flat$ is a topological Lie algebroid over $\mathfrak{z}$. The natural map $U_1^\flat/U_0^\flat \to U_1^\natural/U_0^\natural$ is a topological isomorphism. So (291) induces a surjective continuous Lie algebroid morphism

$$I/I^2 \to U_1^\flat/U_0^\flat\,. \tag{292}$$

5.6.8. Let V be a topological $U^\natural$-module (in the applications we have in mind V will be discrete). Then $V^{\mathfrak{g}\otimes O}$ is a (left) topological module over the Lie algebroid I/I^2. Indeed, first of all $V^{\mathfrak{g}\otimes O}$ is a $\mathfrak{z}$-module. Secondly, $V^{\mathfrak{g}\otimes O} = \{v \in V | U_0^\flat v = 0\}$, so the Lie algebra $U_1^\flat/U_0^\flat$ acts on $V^{\mathfrak{g}\otimes O}$. If $v \in V^{\mathfrak{g}\otimes O}$, $z \in \mathfrak{z}$, $a \in U_1^\flat/U_0^\flat$, then $a(zv) - z(av) = \partial_a(z)v$ where $\partial_a \in \operatorname{Der}\mathfrak{z}$ corresponds to a according to the algebroid structure on $U_1^\flat/U_0^\flat$. So $V^{\mathfrak{g}\otimes O}$ is a module over the algebroid $U_1^\flat/U_0^\flat$. Using (292) we see that $V^{\mathfrak{g}\otimes O}$ is a module over the Lie algebroid I/I^2.

5.6.9. According to (89) one has the continuous Lie algebra morphism $\operatorname{Der} O \to h^{-1}J_I \subset A[h^{-1}]$ such that $L_n \mapsto h^{-1}\widetilde{\mathfrak{L}}_n$, $n \geq -1$. It induces a continuous Lie algebra morphism $\operatorname{Der} O \to U_1^\flat \subset U^\natural$. On the other hand in 3.6.16 we defined a canonical morphism $\operatorname{Der} O \to I/I^2$. Clearly the diagram

$$\begin{array}{ccc} \operatorname{Der} O & \longrightarrow & U_1^\flat \\ \downarrow & & \downarrow \\ I/I^2 & \longrightarrow & U_1^\flat/U_0^\flat \end{array}$$

is commutative.

Remark. The morphism $\operatorname{Der} O \to U_1^\flat/U_0^\flat$ induces a homeomorphism of $\operatorname{Der} O$ onto its image. Since $U_1^\flat/U_0^\flat$ acts continuously on $\mathfrak{z} \subset U_0^\natural$ this follows from the analogous statement for the morphism $\operatorname{Der} O \to \operatorname{Der}\mathfrak{z}$, which is clear (look at the Sugawara elements of $\mathfrak{z}$).

5.6.10. Suppose we are in the situation of 5.6.8. According to 5.6.9 $\operatorname{Der} O$ acts on V via the morphism $\operatorname{Der} O \to U^\natural$, the subspace $V^{\mathfrak{g}\otimes O}$ is $\operatorname{Der} O$-invariant and the action of $\operatorname{Der} O$ on $V^{\mathfrak{g}\otimes O}$ coincides with the one that comes from the morphism $\operatorname{Der} O \to I/I^2$.

5.6.11. *Remark.* The definition of $\widetilde{\mathfrak{g}\otimes K}$ from 2.5.1 involves the "critical" scalar product c defined by (18). Suppose we consider the central extension $0 \to \mathbb{C} \to (\widetilde{\mathfrak{g}\otimes K})_\lambda \to \mathfrak{g}\otimes K \to 0$ corresponding to λc, $\lambda \in \mathbb{C}^*$. Denote by A_λ the completed universal enveloping algebra of $(\widetilde{\mathfrak{g}\otimes K})_\lambda$. The construction of $U^\natural$ and the map (291) remain valid if A and $h = \mathbf{1} - 1$ are replaced by

A_λ and $h_\lambda := \mathbf{1}_\lambda - \lambda^{-1}$, where $\mathbf{1}_\lambda$ denotes $1 \in \mathbb{C} \subset (\widetilde{\mathfrak{g} \otimes K})_\lambda$. Denote by $U^\natural_\lambda$ and f_λ the analogs of $U^\natural$ and (291) corresponding to λ. One can identify A_λ and $U^\natural_\lambda$ with A and $U^\natural$ using the canonical isomorphism $\widetilde{\mathfrak{g} \otimes K} \xrightarrow{\sim} (\widetilde{\mathfrak{g} \otimes K})_\lambda$ such that $\mathbf{1} \mapsto \lambda \cdot \mathbf{1}_\lambda$. Then f_λ *does* depend on λ: indeed, $f_\lambda = \lambda f_1$.

6. The Hecke property II

6.1.

6.2. Proof of Theorem 8.1.6.

6.2.1. *Lemma.* Let V be a non-zero $\overline{U}'$-module such that the representation of $\mathfrak{g}\otimes O$ on V is integrable, and the ideal $I\subset\mathfrak{Z}$ annihilates V. Then V has a non-zero $\mathfrak{g}\otimes O$-invariant vector.

Proof. Denote by $\mathfrak{m}$ the maximal ideal of O. The kernel of the morphism $G(O)\to G(O/\mathfrak{m})$ is pro-unipotent and its Lie algebra is $\mathfrak{g}\otimes\mathfrak{m}$. So $V^{\mathfrak{g}\otimes\mathfrak{m}}\neq 0$. Consider the Sugawara element $\mathcal{L}_0\in I$ (see 3.6.15, 3.6.16). A glance at (85) shows that $2\mathcal{L}_0$ acts on $V^{\mathfrak{g}\otimes\mathfrak{m}}$ as the Casimir of $\mathfrak{g}$. On the other hand, $\mathcal{L}_0 V=0$ because $\mathcal{L}_0\in I$. So the action of $\mathfrak{g}$ on $V^{\mathfrak{g}\otimes\mathfrak{m}}$ is trivial and $V^{\mathfrak{g}\otimes O}=V^{\mathfrak{g}\otimes\mathfrak{m}}\neq 0$. □

6.2.2. *Lemma.* Let N be a $\mathfrak{z}_{\mathfrak{g}}(O)$-module equipped with an action of the Lie algebroid I/I^2. Suppose that the action of $L_0\in\operatorname{Der}O\subset I/I^2$ on N is diagonalizable and the intersection of its spectrum with $c+\mathbb{Z}$ is bounded from below for every $c\in\mathbb{C}$. Then N is a free $\mathfrak{z}_{\mathfrak{g}}(O)$-module.

Proof. Using (80), (81), and 3.6.17 we can replace $\mathfrak{z}_{\mathfrak{g}}(O)$ by $A_{{}^L\mathfrak{g}}(O)$ and I/I^2 by $\mathfrak{a}_{{}^L\mathfrak{g}}$. By definition, $\mathfrak{a}_{{}^L\mathfrak{g}}$ is the algebroid of infinitesimal symmetries of $\mathfrak{F}^0_G$. In 3.5.6 we described a trivialization of $\mathfrak{F}^0_G$. The corresponding splitting $\operatorname{Der}A_{{}^L\mathfrak{g}}(O)\to\mathfrak{a}_{{}^L\mathfrak{g}}$ is $\operatorname{Der}^0 O$-equivariant (see (69) and (70); the point is that the r.h.s. of these formulas are constant as functions on $\operatorname{Spec}A_{{}^L\mathfrak{g}}(O)$). So N becomes a module over $\operatorname{Der}A_{{}^L\mathfrak{g}}(O)$ and the mapping $\operatorname{Der}A_{{}^L\mathfrak{g}}(O)\to\operatorname{End}N$ is $\operatorname{Der}^0 O$-equivariant. According to 3.5.6 $A_{{}^L\mathfrak{g}}(O)$ is the ring of polynomials in u_{jk}, $1\le j\le r$, $0\le k<\infty$, and $L_0u_{jk}=(d_j+k)u_{jk}$ for some $d_j>0$. So N is an L_0-graded module over the algebra generated by u_{jk} and $\frac{\partial}{\partial u_{jk}}$, $\deg(\frac{\partial}{\partial u_{jk}})=-\deg u_{jk}=-(d_j+k)\to-\infty$ when $k\to\infty$. Therefore every element of N is annihilated by almost all $\frac{\partial}{\partial u_{jk}}$ and by all monomials in the $\frac{\partial}{\partial u_{jk}}$ of sufficiently high degree. It is well known (see, e.g., Lemma 9.13 from [Kac90] or Theorem 3.5 from [Kac97]) that in this situation $N = A_{{}^L\mathfrak{g}}(O) \otimes N_0$ where N_0 is the space of $n \in N$ such that $\frac{\partial}{\partial u_{jk}} n = 0$ for all j and k. $\square$

6.2.3. Let us prove Theorem 8.1.6. According to 5.6.8 we can apply Lemma 6.2.2 to $N := V^{\mathfrak{g}\otimes O}$. So $N = \mathfrak{z}_{\mathfrak{g}}(O) \otimes W$ for some vector space W. We will show that the natural $\overline{U}'$-module morphism $f : Vac' \otimes W = Vac' \otimes_{\mathfrak{z}_{\mathfrak{g}}(O)} N \to V$ is an isomorphism. One has $(\operatorname{Ker} f)^{\mathfrak{g}\otimes O} = \operatorname{Ker} f \cap N = 0$, so by 6.2.1 $\operatorname{Ker} f = 0$. Suppose that $\operatorname{Coker} f \neq 0$. Then according to 6.2.1 there is a non-zero $\mathfrak{g} \otimes O$-invariant element of $\operatorname{Coker} f$, i.e., a non-zero $\overline{U}'$-module morphism $Vac' \to \operatorname{Coker} f$. It induces an extension $0 \to Vac' \otimes W \to P \to Vac' \to 0$ which does not split (the composition of a splitting $Vac' \to P$ and the natural morphism $P \to V$ would yield a $\mathfrak{g} \otimes O$-invariant vector of V not contained in N). So it remains to prove the following statement.

6.2.4. *Proposition.* Any extension of discrete $\overline{U}'$-modules $0 \to Vac' \otimes W \to P \to Vac' \to 0$ such that $IP = 0$ splits (here W is a vector space).

Proof. Let $p \in P$ belong to the preimage of the vacuum vector from Vac'. Then $(\mathfrak{g} \otimes O) \cdot p \subset Vac' \otimes W$. In fact $(\mathfrak{g} \otimes O) \cdot p \subset Vac' \otimes W_1$ for some finite-dimensional $W_1 \subset W$, so we can assume that $\dim W < \infty$. Moreover, since the functor Ext is additive we can assume that $W = \mathbb{C}$.

Let p be as above. Define $\varphi : \mathfrak{g} \otimes O \to Vac'$ by $\varphi(a) = ap$, so φ is a 1-cocycle and $\operatorname{Ker} \varphi$ is open. We must show that φ is a coboundary. One has the standard filtration $\overline{U}'_k$ of $\overline{U}'$. The induced filtration Vac'_k of Vac' is $(\mathfrak{g} \otimes O)$-invariant because the vacuum vector is annihilated by $\mathfrak{g} \otimes O$. So $\mathfrak{g} \otimes O$ acts on $\operatorname{gr} Vac'$. There is a k such that $\operatorname{Im} \varphi \subset Vac'_k$. Denote by ψ the composition of $\varphi : \mathfrak{g} \otimes O \to Vac'_k$ and $Vac'_k \to Vac'_k \,/\, Vac'_{k-1} \subset \operatorname{gr} Vac'$. So $\psi : \mathfrak{g} \otimes O \to \operatorname{gr} Vac'$ is a 1-cocycle and it suffices to show that ψ is a coboundary (then one can proceed by induction).

Denote by Vac^{cl} the space of polynomials on $\mathfrak{g}^* \otimes \omega_O$ (by definition, a polynomial on $\mathfrak{g}^* \otimes \omega_O$ is a function $\mathfrak{g}^* \otimes \omega_O \to \mathbb{C}$ that comes from a polynomial on the vector space $\mathfrak{g}^* \otimes (\omega_O/\mathfrak{m}^n\omega_O)$ for some n). According to 2.4.1 one has a canonical $\mathfrak{g} \otimes O$-equivariant identification $\operatorname{gr} Vac' = \operatorname{Sym}(\mathfrak{g} \otimes K/\mathfrak{g} \otimes O) = Vac^{cl}$ (the action of $\mathfrak{g} \otimes O$ on Vac^{cl} is induced by the natural action of $\mathfrak{g} \otimes O$ on $\mathfrak{g}^* \otimes \omega_O$). So we can consider ψ as a 1-cocycle $\mathfrak{g} \otimes O \to Vac^{cl}$. Define $\beta_\psi : (\mathfrak{g} \otimes O) \times (\mathfrak{g}^* \otimes \omega_O) \to \mathbb{C}$ by

$$\beta_\psi(a, \eta) := (\psi(a))(\eta)\,. \tag{293}$$

We say that $\eta \in \mathfrak{g}^* \otimes \omega_O$ is *regular* if the image of η in $\mathfrak{g}^* \otimes (\omega_O/\mathfrak{m}\omega_O)$ is regular.

Lemma. If $\eta \in \mathfrak{g}^* \otimes \omega_O$ is regular and $\mathfrak{c}(\eta)$ is the stabilizer of η in $\mathfrak{g} \otimes O$ then

$$\beta_\psi(a, \eta) = 0 \quad \text{for} \quad a \in \mathfrak{c}(\eta)\,. \tag{294}$$

Proof. We will use that $IP = 0$. Let $F \in \operatorname{Ker}(\mathfrak{Z}^{cl} \to \mathfrak{z}^{cl}_{\mathfrak{g}}(O))$, i.e., F is a $(\mathfrak{g} \otimes K)$-invariant polynomial function on $\mathfrak{g}^* \otimes \omega_K$ whose restriction to $\mathfrak{g}^* \otimes \omega_O$ is zero (see 2.9.8). Suppose that F is homogeneous of degree r. By 3.7.8 F is the symbol of some $z \in \mathfrak{Z}_r$. Since the image of F in $\mathfrak{z}^{cl}_{\mathfrak{g}}(O)$ is zero the image of z in $\mathfrak{z}_{\mathfrak{g}}(O)$ belongs to the $(r-1)$-th term of the filtration, so according to 2.9.5 it comes from some $z' \in \mathfrak{Z}_{r-1}$. Replacing z by $z - z'$ we can assume that $z \in I \cap \mathfrak{Z}_r$.

Since $I \subset \overline{U}' \cdot (\mathfrak{g} \otimes O)$ we can write z as

$$z = \sum_{i=1}^{\infty} u_i a_i\,, \quad a_i \in \mathfrak{g} \otimes O\,, \quad u_i \in \overline{U}'\,, \quad a_i \to 0 \quad \text{for} \quad i \to \infty\,. \tag{295}$$

It follows from the Poincaré – Birkhoff – Witt theorem that the decomposition (295) can be chosen so that $u_i \in \overline{U}'_{r-1}$ for all i. Rewrite the equality $zp = 0$ as

$$\sum_i u_i \varphi(a_i) = 0\,. \tag{296}$$

Denote by $\widetilde{u}_i$ the image of u_i in $\overline{U}'_{r-1}/\overline{U}'_{r-2}$. (295) and (296) imply that

$$F = \sum_i \widetilde{u}_i a_i \,, \tag{297}$$

$$\sum_i \overline{u}_i \psi(a_i) = 0 \tag{298}$$

where $a_i \in \mathfrak{g} \otimes O$ is considered as a linear function on $\mathfrak{g}^* \otimes \omega_K$ and $\overline{u}_i$ is the restriction of $\widetilde{u}_i$ to $\mathfrak{g}^* \otimes \omega_O$. Denote by $\underline{d}F$ the restriction of the differential of F to $\mathfrak{g}^* \otimes \omega_O$. Since F vanishes on $\mathfrak{g}^* \otimes \omega_O$ we have $\underline{d}F \in \mathit{Vac}^{cl} \,\widehat{\otimes}(\mathfrak{g} \otimes O)$ where $\widehat{\otimes}$ is the completed tensor product. According to (297) $\underline{d}F = \sum\limits_i \overline{u}_i \otimes a_i$, so we can rewrite (298) as

$$\mu(\underline{d}F) = 0 \tag{299}$$

where μ is the composition of $\mathrm{id} \otimes \psi : \ \mathit{Vac}^{cl} \,\widehat{\otimes}(\mathfrak{g} \otimes O) \to \mathit{Vac}^{cl} \otimes \mathit{Vac}^{cl}$ and the multiplication map $\mathit{Vac}^{cl} \otimes \mathit{Vac}^{cl} \to \mathit{Vac}^{cl}$.

Now set

$$F(\eta) = \operatorname{Res} f(\eta)\nu \,, \quad \nu \in \omega_O^{\otimes(1-r)} \tag{300}$$

where f is a homogeneous invariant polynomial on $\mathfrak{g}^*$ of degree r. In this case (299) can be rewritten as

$$\beta_\psi(A_f(\eta)\nu, \eta) = 0 \tag{301}$$

where β_ψ is defined by (293) and A_f is the differential of f considered as a polynomial map $\mathfrak{g}^* \to \mathfrak{g}$ (so $A_f(\eta) \in \mathfrak{g} \otimes \omega_O^{\otimes(r-1)}$, $A_f(\eta)\nu \in \mathfrak{g} \otimes O$). Since f is invariant $A_f(l)$ belongs to the stabilizer of $l \in \mathfrak{g}^*$ and if l is regular the elements $A_f(l)$ for all invariant f generate the stabilizer. So the lemma follows from (301) $\square$

To prove the Proposition it remains to show that any 1-cocycle ψ : $\mathfrak{g} \otimes O \to \mathit{Vac}^{cl}$ with open kernel such that the function (293) satisfies (294) is a coboundary.

Lemma. Let K be a connected affine algebraic group with $\mathrm{Hom}(K, \mathbb{G}_m) = 0$, W a K-module, and ψ a 1-cocycle $\mathrm{Lie}\, K \to W$. Then ψ comes from a unique 1-cocycle $\Psi : K \to W$.

Proof. The uniqueness of Ψ is clear. The proof of existence is reduced to the case where K is unipotent (represent K as a semidirect product of a semisimple subgroup K_{ss} and a unipotent normal subgroup; then notice that the restriction of ψ to $\mathrm{Lie}\, K_{ss}$ is a coboundary and reduce to the case where this restriction is zero). Let $\widetilde{K}$ denote the semidirect product of K and W. A 1-cocycle $K \to W$ is the same as a morphism $K \to \widetilde{K}$ such that the composition $K \to \widetilde{K} \to K$ equals id. A 1-cocycle $\mathrm{Lie}\, K \to W$ has a similar interpretation. So we can use the fact that the functor $\mathrm{Lie} : \{\text{unipotent groups}\} \to \{\text{nilpotent Lie algebras}\}$ is an equivalence. $\square$

So our 1-cocycle $\psi : \mathfrak{g} \otimes O \to \mathit{Vac}^{cl}$ comes from a 1-cocycle $\Psi : G(O) \to \mathit{Vac}^{cl}$ where $G(O)$ is considered as a group scheme. Define $B_\Psi : G(O) \times (\mathfrak{g}^* \otimes \omega_O) \to \mathbb{C}$ by $B_\Psi(g, \eta) = (\Psi(g))(\eta)$.

Lemma. If $\eta \in \mathfrak{g}^* \otimes \omega_O$ is regular and $C(\eta)$ is the stabilizer of η in $G(O)$ then

$$B_\Psi(g, \eta) = 0 \quad \text{for} \quad g \in C(\eta)\,. \tag{302}$$

Proof. For fixed η the map $g \mapsto B_\Psi(g, \eta)$ is a morphism of group schemes $f : C(\eta) \to \mathbb{G}_a$. According to (294) the differential of f equals 0. So $f = 0$ (even if $C(\eta)$ is not connected $\mathrm{Hom}(\pi_0(C(\eta)), \mathbb{G}_a) = 0$ because $\pi_0(C(\eta))$ is finite; but in fact if G is the adjoint group, which can be assumed without loss of generality, then $C(\eta)$ is connected). $\square$

The fact that Ψ is a cocycle means that

$$B_\Psi(g_1 g_2, \eta) = B_\Psi(g_1, \eta) + B_\Psi(g_2, g_1^{-1} \eta g_1)\,. \tag{303}$$

We have to prove that B_Ψ is a coboundary, i.e.,

$$B_\Psi(g, \eta) = f(g^{-1} \eta g) - f(\eta) \tag{304}$$

for some polynomial function $f : \mathfrak{g}^* \otimes \omega_O \to \mathbb{C}$. Denote by $\mathfrak{g}^*_{\mathrm{reg}}$ the set of regular elements of $\mathfrak{g}^*$ and by $(\mathfrak{g}^* \otimes \omega_O)_{\mathrm{reg}}$ the set of regular elements of $\mathfrak{g}^* \otimes \omega_O$ (i.e., the preimage of $\mathfrak{g}^*_{\mathrm{reg}}$ in $\mathfrak{g}^* \otimes \omega_O$). Since $\operatorname{codim}(\mathfrak{g}^* \setminus \mathfrak{g}^*_{\mathrm{reg}}) > 1$ it is enough to construct f as a regular function on $(\mathfrak{g}^* \otimes \omega_O)_{\mathrm{reg}}$.

Let C have the same meaning as in 2.2.1. The morphism $\mathfrak{g}^*_{\mathrm{reg}} \to C$ is smooth and surjective, G acts transitively on its fibers, and Kostant constructed in [Ko63] a subscheme $\mathrm{Kos} \subset \mathfrak{g}^*_{\mathrm{reg}}$ such that $\mathrm{Kos} \to C$ is an isomorphism. If $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ using an invariant scalar product on $\mathfrak{g}$ then $\mathrm{Kos} = i\left(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right) + V$ where i and V have the same meaning as in 3.1.9. Define $\mathrm{Kos}_O \subset \mathfrak{g}^* \otimes \omega_O$ by $\mathrm{Kos}_O := i\left(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\right) \cdot dt + V \otimes \omega_O$.

The equation (304) has a unique solution f that vanishes on Kos_O. The restriction of f to $(\mathfrak{g}^* \otimes \omega_O)_{\mathrm{reg}}$ is defined by

$$f(g^{-1}\eta g) = B_\Psi(g, \eta) \quad \text{for} \quad \eta \in \mathrm{Kos}\,, \ g \in G(O)\,. \tag{305}$$

Here f is well-defined since (as follows from (302) and (303)) one has $B_\Psi(g_1 g, \eta) = B_\Psi(g, \eta)$ for $\eta \in (\mathfrak{g}^* \otimes \omega_O)_{\mathrm{reg}}\,,\ g_1 \in C(\eta)$. Now (303) implies that the function f defined by (305) satisfies (304) □

Remark. At the end of the proof we used Kostant's global section of the fibration $(\mathfrak{g}^* \otimes \omega_O)_{\mathrm{reg}} \to \mathrm{Hitch}_{\mathfrak{g}}(O)$ (see 2.4.1 for the definition of $\mathrm{Hitch}_{\mathfrak{g}}(O)$). Instead one could use *local* sections and the equality $H^1(\mathrm{Hitch}_{\mathfrak{g}}(O), \mathcal{O}) = 0$, which is obvious because $\mathrm{Hitch}_{\mathfrak{g}}(O)$ is affine.

6.2.5. Proposition 6.2.4 seems to be related with [F91] (see, e.g., the Propositions in the lower parts of pages 97 and 98 of [F91]). Maybe a modification of the methods of [F91] would yield Proposition 6.2.4 and much more.

7. Appendix: $\mathcal{D}$-module theory on algebraic stacks and Hecke patterns

7.1. Introduction.

7.1.1. The principal goal of this section is to present a general Hecke format which is used in the proof of our main Theorem. Its (untwisted) finite-dimensional version looks as follows. Let G be an algebraic group, $K \subset G$ an algebraic subgroup, $\mathfrak{g}$ the Lie algebra of G, and Y a smooth variety with G-action. Denote by $\mathcal{H} := D(K \setminus G/K)$ the $\mathcal{D}$-module derived category of the stack $K \setminus G/K$. One has the similar derived category $D(K \setminus Y)$ and the derived category $D(\mathfrak{g}, K)$ of the category $\mathcal{M}(\mathfrak{g}, K)$ of $(\mathfrak{g}, K)$-modules. Then we have the following "Hecke pattern":

(a) $\mathcal{H}$ is a monoidal triangulated category,
(b) $D(K \setminus Y)$ is an $\mathcal{H}$-Module,
(c) $D(\mathfrak{g}, K)$ is an $\mathcal{H}$-Module,
(d) the standard functors

$$L\Delta : D(\mathfrak{g}, K) \longrightarrow D(K \setminus Y), \qquad R\Gamma : D(K \setminus Y) \longrightarrow D(\mathfrak{g}, K)$$

are Morphisms of $\mathcal{H}$-Modules.

Here $L\Delta$, $R\Gamma$ are derived versions of the functors Δ, Γ from 1.2.4. The tensor product on $\mathcal{H}$ and $\mathcal{H}$-Actions from (b) and (c) are appropriate "convolution" functors $\circledast$. For example, consider the case $K = \{1\}$. Denote by δ_g the $\mathcal{D}$-module of δ-functions at $g \in G$. One has $\delta_{g_1} \circledast \delta_{g_2} = \delta_{g_1 g_2}$. For a $\mathcal{D}$-module M on Y $\delta_g \circledast M$ is the g-translation of M, and for a $\mathfrak{g}$-module V $\delta_g \circledast V$ is V equipped with the $\mathfrak{g}$-action turned by Ad_g. The $\mathcal{D}$-module structure on M identifies canonically $\delta_g \circledast M$ for infinitely close g's; similarly, the $\mathfrak{g}$-action on V identifies such $\delta_g \circledast V$'s. This allows to define the convolution functors for an arbitrary $\mathcal{D}$-module on G.

7.1.2. The accurate construction of Hecke functors requires some $\mathcal{D}$-module formalism for stacks. For example, one needs a definition of the $\mathcal{D}$-module derived category $D(\mathcal{Y})$ of a smooth stack $\mathcal{Y}$ (it might not coincide with the derived category of the category of $\mathcal{D}$-modules on $\mathcal{Y}$!). There seems to be no reference available (except in the specific case when $\mathcal{Y}$ is an orbit stack, i.e., the quotient of a smooth variety by an affine group action, that was treated in [BL], [Gi87] in a way not too convenient for the Hecke functor applications), so we have to supply some general nonsense to keep afloat.

We start in 7.2, following Kapranov [Kap91] and Saito [Sa89], with a canonical equivalence between the derived category of $\mathcal{D}$-modules and that of Ω-modules (here Ω is the DG algebra of differential forms) which identifies a $\mathcal{D}$-module with its de Rham complex. When you deal with stacks, Ω-modules are easier to handle: the reason is that Ω is a sheaf of rings on the smooth topology while $\mathcal{D}$ is not. In the important special case of a stack for which the diagonal morphism is affine this super*) format is especially convenient. Here one may define (see 7.3) the $\mathcal{D}$-module derived category directly using "global" Ω-complexes. In 7.5, after a general homological algebra digression of 7.4, we give a "local" definition of the $\mathcal{D}$-module derived category that works for arbitrary smooth stacks. In 7.6 parts (a), (b) of the Hecke pattern are explained; we also show that for an orbit stack its $\mathcal{D}$-module derived category is equivalent to the equivariant derived category from [BL], [Gi87]. In 7.7 we describe a similar super format for Harish-Chandra modules; as a bonus we get in 7.7.12 a simple proof of the principal result of [BL]. The Harish-Chandra parts (c), (d) of the Hecke pattern are treated in 7.8. A version with extra symmetries and parameters needed in the main body of the article is presented in 7.9. Before passing to an infinite-dimensional setting we discuss in 7.10 a crystalline approach to $\mathcal{D}$-modules which is especially convenient when you deal with singular spaces (we owe this section to discussions with J.Bernstein back in 1980). Sections 7.11 and 7.12 contain some basic material about ind-schemes, Mittag-Leffler modules,

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*) A mathematician's abbreviation of Mary Poppins' coinage "supercalifragelistic-expialidocious".

and $\mathcal{D}$-modules on formally smooth ind-schemes. Section 7.13 is a review of BRST reduction. The infinite-dimensional rendering of parts (c), (d) of the Hecke pattern is in 7.14. Finally in 7.15 we show that positively twisted $\mathcal{D}$-modules on affine flag varieties are essentially the same as representations of affine Kac-Moody Lie algebras of less than critical level. In the particular case of $\mathcal{D}$-modules smooth along the Schubert stratification, similar result was found by Kashiwara and Tanisaki [KT95] (the authors of [KT95] do not use the language of $\mathcal{D}$-modules on ind-schemes). We also identify the corresponding de Rham and BRST cohomology groups.

Our exposition of $\mathcal{D}$-module theory is quite incomplete; basically we treat the subjects that are used in the main body of the paper. The exceptions are sections 7.4, 7.5 (the stack Bun_G fits into the formalism of 7.3), 7.10 (the singular spaces that we encounter are strata on affine Grassmannians, so one may use 7.11), and 7.15 (included for the mere fun of the reader).

Recall that $\mathcal{M}^\ell(X)$ (resp. $\mathcal{M}^r(X)$) denotes the category of left (resp. right) $\mathcal{D}$-modules on a smooth variety X; we often identify these categories and denote them by $\mathcal{M}(X)$. If F is a complex then we denote by $F^{\boldsymbol{\cdot}}$ the corresponding graded object (with the differential forgotten).

7.2. $\mathcal{D}$- and Ω-modules.

7.2.1. Let X be a smooth algebraic variety $^{*)}$. Denote by Ω_X the DG algebra of differential forms on X. Then (X, Ω_X) is a DG ringed space, so we have the category of Ω_X-complexes (:= DG Ω_X-modules). An Ω_X-complex $F = (F^{\boldsymbol{\cdot}}, d)$ is quasi-coherent if F^i are quasi-coherent $\mathcal{O}_X$-modules; quasi-coherent Ω_X-complexes will usually be called Ω-*complexes on* X. Denote

$^{*)}$or, more generally, a smooth quasi-compact algebraic space over $\mathbb{C}$ such that the diagonal morphism $X \to X \times X$ is affine. The constructions and statements of this section (but 7.2.10) are local, so they make sense for any smooth algebraic space. The condition on X is needed to ensure that the derived categories we define satisfy an appropriate local-to-global (descent) property. We discuss this in the more general setting of stacks in 7.5.

the DG category of Ω-complexes on X by $C(X,\Omega)$. This is a tensor DG category.

Remark. For an Ω_X-complex F the differential $d : F^{\cdot} \to F^{\cdot+1}$ is a differential operator of order ≤ 1 with symbol equal to the product map $\Omega^1_X \otimes F^{\cdot} \to F^{\cdot+1}$. We see that the $\Omega^{\cdot}_X$-module structure on $F^{\cdot}$ can be reconstructed from the $\mathcal{O}_X$-module structure and d. In fact, forgetting the $\Omega^{\geq 1}_X$-action identifies $C(X,\Omega)$ with the category of complexes $(F^{\cdot},d)$ where $F^{\cdot}$ are quasi-coherent $\mathcal{O}_X$-modules, d are differential operators of order ≤ 1.

7.2.2. Let $C(X,\mathcal{D}) := C(\mathcal{M}^r(X))$ be the DG category of complexes of right $\mathcal{D}$-modules on X (*right $\mathcal{D}$-complexes*, or just *$\mathcal{D}$-complexes* for short), and $K(X,\mathcal{D})$ the corresponding homotopy category. We have a pair of adjoint DG functors

$$\mathcal{D} : C(X,\Omega) \longrightarrow C(X,\mathcal{D}), \quad \Omega : C(X,\mathcal{D}) \longrightarrow C(X,\Omega) \tag{306}$$

defined as follows. Denote by DR_X the de Rham complex of $\mathcal{D}_X$ considered as a left $\mathcal{D}$-module, so $DR^{\cdot}_X = \Omega^{\cdot}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$. This is an Ω-complex equipped with the right action of $\mathcal{D}_X$. Now for an Ω-complex F and a right $\mathcal{D}$-complex M one has

$$\mathcal{D}F = F \underset{\Omega_X}{\otimes} DR_X, \quad \Omega M := Hom_{\mathcal{D}_X}(DR_X, M). \tag{307}$$

The adjunction property is clear.

7.2.3. *Remarks.* (i) One has $\mathcal{D}F^{\cdot} = F^{\cdot} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X = \mathrm{Diff}(\mathcal{O}, F^{\cdot})$; the differential $d_{\mathcal{D}F} : \mathcal{D}F^{\cdot} \to \mathcal{D}F^{\cdot+1}$ sends a differential operator $a : \mathcal{O}_X \to F^{\cdot}$ to the composition $d{\cdot}a$. The Ω-complex ΩM, $(\Omega M)^i = \bigoplus_{a-b=i} M^a \otimes \Lambda^b \Theta_X$ is the de Rham complex of M.

(ii) The category $\mathcal{M}^{\ell}(X)$ of left $\mathcal{D}$-modules on X is a tensor category in the usual way (tensor product over $\mathcal{O}_X$), so the category of *left $\mathcal{D}$-complexes* $C(\mathcal{M}^{\ell}(X))$ is a tensor DG category. The DG functor Ω :

$C(\mathcal{M}^\ell(X)) \to C(X,\Omega)$ which assigns to a left $\mathcal{D}$-complex N its de Rham complex, $(\Omega N)^\cdot = \Omega^\cdot_X \underset{\mathcal{O}_X}{\otimes} N$, is a tensor functor.

(iii) The DG categories $C(X,\Omega)$ and $C(X,\mathcal{D})$ are Modules over the tensor DG category $C(\mathcal{M}^\ell(X))$. The functors $\mathcal{D}$ and Ω are Morphisms of $C(\mathcal{M}^\ell(X))$-Modules.

7.2.4. *Lemma.* For any $\mathcal{D}$-complex M the canonical morphism $\mathcal{D}\Omega M \to M$ is a quasi-isomorphism.

Proof. Set

$$V^i_j := \bigoplus_{\substack{a-b=i\\ b+c=j}} M^a \otimes \Lambda^b\Theta_X \otimes \mathcal{D}^{\leq c}_X \quad \subset \ (\mathcal{D}\Omega M)^i.$$

Then V_* is a increasing filtration of $\mathcal{D}\Omega M$ by $\mathcal{O}$-subcomplexes such that $V_0 \overset{\sim}{\to} M$ and V_i/V_{i-1} are acyclic for $i \geq 1$ (since V_i/V_{i-1} is the tensor product of M and the i-th Koszul complex for Θ_X). $\square$

7.2.5. For an Ω-complex F set $H^\cdot_{\mathcal{D}}F = H^\cdot\mathcal{D}F$. Thus $H_{\mathcal{D}}$ is a cohomology functor on $K(X,\Omega)$ with values in the abelian category $\mathcal{M}^r(X)$. A morphism of Ω-complexes $\phi : F_1 \to F_2$ is called $\mathcal{D}$-*quasi-isomorphism* if the morphism of $\mathcal{D}$-complexes $\mathcal{D}\phi : \mathcal{D}F_1 \to \mathcal{D}F_2$ is a quasi-isomorphism, i.e., $H^\cdot_{\mathcal{D}}F_1 \to H^\cdot_{\mathcal{D}}F_2$ is an isomorphism. We have the following simple properties (use 7.2.4 to prove (ii), (iii)):

(i) If ϕ is a $\mathcal{D}$-quasi-isomorphism, N is a left $\mathcal{D}$-module flat as an $\mathcal{O}$-module then $\phi \otimes id_N : F_1 \otimes N \to F_2 \otimes N$ is a $\mathcal{D}$-quasi-isomorphism.

(ii) The canonical morphism $\alpha_F : F \to \Omega\mathcal{D}F$ is a $\mathcal{D}$-quasi-isomorphism.

(iii) Ω sends quasi-isomorphisms to $\mathcal{D}$-quasi-isomorphisms.

The following lemma will not be used in the sequel; the reader may skip it. We say that a morphism of Ω-complexes $\phi : F_1 \to F_2$ is a *naive quasi-isomorphism* if it is a quasi-isomorphism of complexes of sheaves of vector spaces.

7.2.6. *Lemma.* (i) Any $\mathcal{D}$-quasi-isomorphism is a naive quasi-isomorphism.

(ii) A morphism ϕ as above is a $\mathcal{D}$-quasi-isomorphism if and only if for any bounded below complex A of locally free Ω-modules the morphism$\phi \otimes id_A : F_1 \otimes A \to F_2 \otimes A$ is a naive quasi-isomorphism.

(iii) Assume either that $\Omega^{\geq 1} F_i^{\cdot} = 0$ (i.e., the differential is $\mathcal{O}$-linear), or that $F_i^{\cdot}$ are bounded and $\mathcal{O}$-coherent. Then any naive quasi-isomorphism ϕ is a $\mathcal{D}$-quasi-isomorphism. For arbitrary Ω-complexes this may be not true.

Proof. (i) For any Ω-complex F the canonical morphism $\alpha_F : F \to \Omega\mathcal{D}F$ is a naive quasi-isomorphism. Since Ω sends quasi-isomorphisms of $\mathcal{D}$-complexes to naive quasi-isomorphisms we see that $\Omega(\mathcal{D}\phi)$ is a naive quasi-isomorphism. Now our statement follows from the fact that $\alpha_{F_2}\phi = \Omega((\mathcal{D}\phi)\alpha_{F_1}$.

(ii) To prove the "if" statement just take $A = DR_X$. Conversely, assume that ϕ is a $\mathcal{D}$-quasi-isomorphism. There is a bounded below increasing filtration A_i on A such that $\cup A_i = A$ and each $gr_i A$ is a locally free $\Omega_X^{\cdot}$-module with generators in degree i (set $A_i := \Omega_X \cdot A_{\leq i}$). So $\phi \otimes id_A$ is a naive quasi-isomorphism if all $\phi \otimes id_{gr_i A}$ are naive quasi-isomorphisms. Thus we may assume that A is a locally free $\Omega_X^{\cdot}$-module with generators in fixed degree, say 0, i.e., $A = \Omega N$ where N is a left $\mathcal{D}$-module locally free as an $\mathcal{O}$-module. Then $\phi \otimes id_A = \phi \otimes id_N$, and we are done by (i) from 7.2.5.

(iii) The $\mathcal{O}$-linear case is obvious (since in this situation $\mathcal{D}F = F \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$). The $\mathcal{O}$-coherent case follows from the Sublemma below applied to $\mathcal{D}\phi$ (notice that because of property (ii) from 7.2.5 the fiber of $\mathcal{D}F$ at x coincides with $R\Gamma_x(X, F)$).

Sublemma. Let $\psi : M_1 \to M_2$ be a morphism of finite complexes of coherent $\mathcal{D}$-modules on X. Assume that for any $x \in X(\mathbb{C})$ the corresponding morphism of fibers[*)] $M_{1x} \to M_{2x}$ is a quasi-isomorphism. Then ψ is a quasi-isomorphism.

Proof of Sublemma. Set $C = Cone(\psi)$; denote by Y the support of $H^{\cdot}(C)$. Assume that ψ is not a quasi-isomorphism, i.e., Y is not empty. Restricting

[*)] Certainly here we consider the $\mathcal{O}$-moduli fibers in the usual derived category sense.

X if necessary we may assume that Y is a smooth subvariety of X and the coherent $\mathcal{D}_Y$-modules $P^{\cdot} := i_Y^! H^{\cdot}(C) = H^{\cdot} i_Y^!(C)$ are free as $\mathcal{O}_Y$-modules. Since for $x \in Y$ one has $H^{\cdot}(C_x) = P_x^{\cdot+n}$ where n is codimension of Y in X we see that $P^{\cdot} = 0$ which is a contradiction.

To get an example of a naive quasi-isomorphism which is not a $\mathcal{D}$-quasi-isomorphism it suffice to find a non-zero $\mathcal{D}$-module M such that ΩM is an acyclic complex of sheaves. Take M to be a constant sheaf of $\mathcal{D}_X$-modules equal to the field of fractions of the ring of differential operators (at the generic point of X). □

7.2.7. Since $H_{\mathcal{D}}$ is a cohomology functor, $\mathcal{D}$-quasi-isomorphisms form a localizing family in the homotopy category of $C(X,\Omega)$. Therefore the corresponding localization $D(X,\Omega)$ is a triangulated category (see [Ve]); we call it $\mathcal{D}$-*derived category* of Ω-complexes. The functors $\mathcal{D}$, Ω give rise to mutually inverse equivalences of triangulated categories

$$\mathcal{D}:\, D(X,\Omega) \longrightarrow D(X,\mathcal{D})\,, \qquad \Omega:\, D(X,\mathcal{D}) \longrightarrow D(X,\Omega)\,. \tag{308}$$

Here $D(X,\mathcal{D}) = D\mathcal{M}^r(X)$. We often denote these triangulated categories thus identified by $D(X)$. One may consider bounded derived categories as well.

Remark. For a bounded from below complex of injective $\mathcal{D}$-modules M the corresponding Ω-complex ΩM is injective. Thus the homotopy category $K^+(X,\Omega)$ has many injective objects.

7.2.8. Let $f : Y \to Z$ be a morphism of smooth varieties. It yields the morphism of DG ringed spaces $f_\Omega : (Y,\Omega_Y) \to (Z,\Omega_Z)$. Thus we have the corresponding DG functors $f_\Omega^{\cdot} : C(Z,\Omega) \to C(Y,\Omega)$, $f_{\cdot} = f_{\Omega\cdot} : C(Y,\Omega) \to C(Z,\Omega)$. Let us consider first the pull-back functor.

We have the usual pull-back functor for left $\mathcal{D}$-modules $f^\dagger : \mathcal{M}^\ell(Z) \to \mathcal{M}^\ell(Y)$, $f^\dagger(N) = \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_Z}{\otimes} f^{-1}N$. One has $\Omega f^\dagger(N) = f_\Omega^{\cdot}(\Omega N)$. One may replace left $\mathcal{D}$-modules by right ones[*)] and consider the corresponding functor $f^\dagger : \mathcal{M}^r(Z) \to \mathcal{M}^r(Y)$; then $f^\cdot_\Omega(\Omega M) = \Omega f^\dagger M[-\dim Y/Z]$.

If f is smooth then for any $F \in C(Z,\Omega)$ one has $H^\cdot_{\mathcal{D}} f^\cdot_\Omega F = f^\dagger H^{\cdot - \dim Y/Z}_{\mathcal{D}} F$. So $f^\cdot_\Omega$ preserves $\mathcal{D}$-quasi-isomorphisms and we have the functor $f^\cdot_\Omega : D(Z,\Omega) \to D(Y,\Omega)$. The adjunction morphism $\mathcal{D} f^\cdot_\Omega(\Omega M) \to f^\dagger M[-\dim U/X]$ is a quasi-isomorphism.

7.2.9. *Lemma.* Ω-complexes are local objects with respect to the smooth topology, i.e., the pull-back functors make $C(U,\Omega)$, $U \in X_{sm}$, a sheaf of DG categories on the smooth topology of X. The notion of $\mathcal{D}$-quasi-isomorphism is local on X_{sm}. □

7.2.10. Let us return to situation 7.2.8 and consider the DG functor $f_\cdot : C(Y,\Omega) \to C(Z,\Omega)$. The right derived functor $Rf_\cdot : D(Y,\Omega) \to D(Z,\Omega)$ is correctly defined. Indeed, let $U_\cdot$ be a (finite) affine covering (either étale or Zariski) of Y. For $F \in C(Y,\Omega)$ let $F \to \mathcal{C}(F)$ be the corresponding Čech resolution of F. Then[*)] $f_\cdot \mathcal{C}(F) \xrightarrow{\sim} Rf_\cdot F$.

We denote the corresponding functor $D(Y) \to D(Z)$ by f_*. It coincides with the usual $\mathcal{D}$-module push-forward functor. Indeed, for a $\mathcal{D}$-complex M on Y one has $\mathcal{D} f_\cdot \Omega M = f_\cdot(\Omega M \otimes f^\dagger \mathcal{D}_Z) = f_\cdot(\mathcal{D}(\Omega M) \underset{\mathcal{D}_Y}{\otimes} f^\dagger \mathcal{D}_Z)$. Since $f^\dagger \mathcal{D}_Z$ is a flat $\mathcal{O}_Y$-module and $\mathcal{D}(\Omega M)$ is a resolution of M we see that $\mathcal{D}(\Omega M) \underset{\mathcal{D}_Y}{\otimes} f^\dagger \mathcal{D}_Z = M \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} f^\dagger \mathcal{D}_Z$. Thus $f_* M = Rf_\cdot(M \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} f^\dagger \mathcal{D}_Z)$, q.e.d.

We leave it to the reader to check that $Rf_\cdot$ is compatible with composition of f's, i.e., that the canonical morphism $R(fg)_\cdot \to Rf_\cdot Rg_\cdot$ is an isomorphism[*)], and that this identification $(fg)_* = f_* g_*$ coincides with the standard identification from $\mathcal{D}$-module theory.

7.2.11. For a $\mathcal{D}$-complex M on Y denote by $M_{\mathcal{O}} \in D(Y,\mathcal{O})$ same M considered as a complex of $\mathcal{O}^!$-modules. One has a canonical *integration*

[*)] using the standard equivalence $\mathcal{M}^\ell(Z) \xrightarrow{\sim} \mathcal{M}(Z)$, $N \mapsto N \otimes \omega_Z$.

[*)] this follows, e.g., from Remark after 7.3.9.

[*)] see 7.3.10(ii) for a proof of this statement in a more general situation.

morphism

$$i_f : Rf\cdot(M_{\mathcal{O}}) \to (f_* M)_{\mathcal{O}} \tag{309}$$

in $D(Y, \mathcal{O})$ defined as follows. It suffice to define the morphism i_f : $f\cdot(M_{\mathcal{O}}) \to \mathcal{D}(f\cdot \Omega M)$. Now i_f is the composition

$$f\cdot(M_{\mathcal{O}}) \to [\mathcal{D}(f\cdot(M_{\mathcal{O}}))]_{\mathcal{O}} \to [\mathcal{D}(f\cdot \Omega M)]_{\mathcal{O}}$$

where the arrows come from the canonical morphisms $N \to (\mathcal{D}N)_{\mathcal{O}}$ (for $N = f\cdot(M_{\mathcal{O}})$) and $M_{\mathcal{O}} \to \Omega M$. In other words, i_f comes by applying $Rf\cdot$ to the obvious morphism $M_{\mathcal{O}} \to (M \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} f^{\dagger}\mathcal{D}_Z)_{\mathcal{O}}$.

We leave it to the reader to check that i_f is compatible with composition of f's.

7.3. **$\mathcal{D}$-module theory on smooth stacks I.** We establish the basic $\mathcal{D}$-module formalism for a smooth stack that satisfies condition (310) below. In 7.3.12 we modify the definitions so that one may drop the quasi-compactness assumption. The arbitrary smooth stacks will be treated in 7.5.

7.3.1. Let $\mathcal{Y}$ be a smooth quasi-compact algebraic stack. Assume that it satisfies the following condition[*)]:

(310) The diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is affine.

Equivalently, this means that there exist a smooth affine surjective morphism $U \to \mathcal{Y}$ such that U is an affine scheme. In other words, $\mathcal{Y}$ is a quotient of a smooth algebraic variety X modulo the action of a smooth groupoid Q[*)] such that the structure morphism $Q \to X \times X$ is affine.

Note that $\Omega(U)$, $U \in \mathcal{Y}_{sm}$, form a sheaf of DG algebras $\Omega_{\mathcal{Y}}$ on $\mathcal{Y}_{sm}$. An Ω-*complex on* $\mathcal{Y}$ is a DG $\Omega_{\mathcal{Y}}$-module which is quasi-coherent as an $\mathcal{O}_{\mathcal{Y}}$-module. We denote the DG category of Ω-complexes on $\mathcal{Y}$ by $C(\mathcal{Y}, \Omega)$.

[*)] This condition is needed to ensure that the category $D(\mathcal{Y})$ we define has right local-to-global properties, see 7.5.3. The constructions 7.3.1-7.3.3 make sense for any smooth algebraic stack.

[*)] $Q = X \underset{\mathcal{Y}}{\times} X$.

Remark. The categories $C(U,\Omega)$, $U\in\mathcal{Y}_{sm}$, form a sheaf of DG categories $\mathcal{C}(\mathcal{Y}_{sm},\Omega)$ on $\mathcal{Y}_{sm}$ (see 7.2.9), and an Ω-complex on $\mathcal{Y}$ is the same as a Cartesian section of $\mathcal{C}(\mathcal{Y}_{sm},\Omega)$. Equivalently, an Ω-complex on $\mathcal{Y}$ is the same as a Q-equivariant Ω-complex on X.

7.3.2. Recall that the categories of $\mathcal{D}$-modules $\mathcal{M}(U)$, $U\in\mathcal{Y}_{sm}$, form a sheaf of abelian categories on $\mathcal{Y}_{sm}$, and the category $\mathcal{M}(\mathcal{Y})$ of $\mathcal{D}$-modules on $\mathcal{Y}$ is the category of its Cartesian sections. By 7.2.8 there is a canonical cohomology functor $H^{\cdot}_{\mathcal{D}}: C(\mathcal{Y},\Omega)\to\mathcal{M}(\mathcal{Y})$, $H^{\cdot}_{\mathcal{D}}(F)_U := H^{\cdot+\dim U/\mathcal{Y}}_{\mathcal{D}}(F_U)$. A morphism of Ω-complexes is called a $\mathcal{D}$-quasi-isomorphism if it induces an isomorphism of $H^{\cdot}_{\mathcal{D}}$'s. Localizing the homotopy category of Ω-complexes by $\mathcal{D}$-quasi-isomorphisms we get a triangulated category $D(\mathcal{Y})=D(\mathcal{Y},\Omega)$. One has the corresponding bounded derived categories as well.

There is a fully faithful embedding $\mathcal{M}(\mathcal{Y})\hookrightarrow D(\mathcal{Y})$ which assigns to a $\mathcal{D}$-module M on $\mathcal{Y}$ its de Rham complex ΩM, $(\Omega M)_U:=\Omega M_U[-\dim U/\mathcal{Y}]$. One has $H^0_{\mathcal{D}}\Omega M=M$ and $H^a_{\mathcal{D}}\Omega M=0$ for $a\neq 0$. It is easy to see that Ω identifies $\mathcal{M}(\mathcal{Y})$ with the full subcategory of $D(\mathcal{Y})$ that consists of those Ω-complexes F that $H^a_{\mathcal{D}}(F)=0$ for $a\neq 0$.

7.3.3. *Example.* Denote by $\Omega\mathcal{D}_{\mathcal{Y}}$ the Ω-complex on $\mathcal{Y}$ defined by $\Omega\mathcal{D}_{\mathcal{Y}U}:=\Omega_{U/\mathcal{Y}}[\dim\mathcal{Y}]$. Note that $H^a_{\mathcal{D}}(\Omega\mathcal{D}_{\mathcal{Y}})=0$ for $a>0$. If $\mathcal{Y}$ is good then our Ω-complex belongs to the essential image of $\mathcal{M}(\mathcal{Y})$; the corresponding $\mathcal{D}$-module $\mathcal{D}_{\mathcal{Y}}=H^0_{\mathcal{D}}(\Omega\mathcal{D}_{\mathcal{Y}})$ coincides with the left $\mathcal{D}$-module $\mathcal{D}_{\mathcal{Y}}$ from 1.1.3. More generally, for any $\mathcal{O}$-module P on $\mathcal{Y}$ we have the Ω-complex $\Omega(\mathcal{D}_{\mathcal{Y}}\otimes P)$ with $\Omega(\mathcal{D}_{\mathcal{Y}}\otimes P)_U := \Omega_{U/\mathcal{Y}}\underset{\mathcal{O}_{\mathcal{Y}}}{\otimes}P_U[\dim\mathcal{Y}]$. If $\mathcal{Y}$ is good and P is locally free then our Ω-complex sits in $\mathcal{M}(\mathcal{Y})$ and equals to the left $\mathcal{D}$-module $\mathcal{D}_{\mathcal{Y}}\otimes P=\mathcal{D}_{\mathcal{Y}}\underset{\mathcal{O}_{\mathcal{Y}}}{\otimes}P$.

Denote by $D(\mathcal{Y})^{\geq 0}\subset D(\mathcal{Y})$ the full subcategory of Ω-complexes F such that $H^a_{\mathcal{D}}F=0$ for $a<0$; define $D(\mathcal{Y})^{\leq 0}$ in the similar way.

7.3.4. *Proposition.* This is a t-structure on $D(\mathcal{Y})$ with core $\mathcal{M}(\mathcal{Y})$ and cohomology functor $H_{\mathcal{D}}$.

This proposition follows immediately from Lemma 7.5.3 below. A different proof in the particular case where $\mathcal{Y}$ is an orbit stack may be found in 7.6.11.

7.3.5. *Remark.* Consider the functor $\Omega : C(\mathcal{M}(\mathcal{Y})) \to C(\mathcal{Y}, \Omega)$. For $M \in C(\mathcal{M}(\mathcal{Y}))$ one has $H^{\cdot}M = H^{\cdot}_{\mathcal{D}}(\Omega M)$, so Ω yields the t-exact functor $\Omega : D(\mathcal{M}(\mathcal{Y})) \to D(\mathcal{Y})$ which extends the "identity" equivalence between the cores. This functor is an equivalence of categories if $\mathcal{Y}$ is a Deligne-Mumford stack$^{*)}$, but not in general.

7.3.6. Let $f : \mathcal{Y} \to \mathcal{Z}$ be a morphism of smooth stacks that satisfy (310). It yields a morphism of DG ringed topologies $(\mathcal{Y}_{sm}, \Omega_{\mathcal{Y}}) \to (\mathcal{Z}_{sm}, \Omega_{\mathcal{Z}})$ hence a pair of adjoint DG functors

$$f^{\cdot}_{\Omega} : C(\mathcal{Z}, \Omega) \to C(\mathcal{Y}, \Omega), \;\; f_{\cdot} : C(\mathcal{Y}, \Omega) \to C(\mathcal{Z}, \Omega) \tag{311}$$

and the corresponding adjoint triangulated functors between the homotopy categories (since $\mathcal{Y}$ is quasi-compact $f_{\cdot}$ preserves quasi-coherency).

If f is smooth then $f^{\cdot}_{\Omega}$ preservres $\mathcal{D}$-quasi-isomorphisms, so it defines a t-exact functor $f^{\cdot} : D(\mathcal{Z}) \to D(\mathcal{Y})$. It is obviously compatible with composition of f's.

Let f be an arbitrary morphism. We define the push-forward functor $f_* : D^+(\mathcal{Y}) \to D^+(\mathcal{Z})$ as the right derived functor $Rf_{\cdot}$. We will show that f_* is correctly defined in 7.3.10 below. One needs for this a sufficient supply of "flabby" objects.

7.3.7. *Definition.* We say that an $\mathcal{O}$-module F on $\mathcal{Y}$ is *loose* if for any flat $\mathcal{O}$-module P on $\mathcal{Y}$ one has $H^a(\mathcal{Y}, P \otimes F) = 0$ for $a > 0$. An $\mathcal{O}$- or Ω-complex F is loose if each F^i is loose.

$^{*)}$which means that $\mathcal{Y}$ admits an etale covering by a variety. In this situation the functor $\mathcal{D} : C(\mathcal{Y}, \Omega) \to C(\mathcal{M}(\mathcal{Y}))$ makes obvious sense (which yields the inverse equivalence $D(\mathcal{M}(\mathcal{Y})) \to D(\mathcal{Y})$ as in 7.2.7.

7.3.8. *Lemma.* (i) For any Ω-complex F' on $\mathcal{Y}$ there exists a $\mathcal{D}$-quasi-isomorphism $F' \to F$ such that F is loose. If F' is bounded from below then we may choose F bounded from below.

(ii) Assume that f (see 7.3.6) is smooth and affine. Then $f^{\cdot}_{\Omega}, f_{\cdot}$ send loose Ω-complexes to loose ones.

(iii) If F_1, F_2 are loose Ω-complexes on stacks $\mathcal{Y}_1, \mathcal{Y}_2$ then $F_1 \boxtimes F_2$ is a loose Ω-complex on $\mathcal{Y}_1 \times \mathcal{Y}_2$.

Proof. (i) Since $\mathcal{Y}$ is quasi-compact, there exists a hypercovering $U_{\cdot}$ of $\mathcal{Y}$ such that U_a are affine schemes. Since the diagonal morphism for $\mathcal{Y}$ is affine, the projections $\pi_a : U_a \to \mathcal{Y}$ are affine. Take for F the Čech complex of F' for this hypercovering, so $F^i = \bigoplus\limits_{a\geq 0} \pi_{a\cdot}(F^{i-a}_{U_a})$.

(ii) Clear.

(iii) We may assume that F_i are loose $\mathcal{O}_{\mathcal{Y}_i}$-modules. Let P be a flat $\mathcal{O}$-module on $\mathcal{Y}_1 \times \mathcal{Y}_2$. Since F_1 is loose, one has $R^a p_{2\cdot}(P \otimes p_1^* F_1) = 0$ for $a > 0$ and $p_{2\cdot}(P \otimes p_1^* F_1)$ is a flat $\mathcal{O}$-module on $\mathcal{Y}_2$ (here $p_i : \mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_i$ are the projections). Thus $H^a(\mathcal{Y}_1 \times \mathcal{Y}_2, P \otimes (F_1 \boxtimes F_2)) = H^a(\mathcal{Y}_2, (p_{2\cdot}(P \otimes p_1^* F_1)) \otimes F_2)$ which vanishes for $a > 0$ since F_2 is loose. □

Let us return to the situation at the end of 7.3.6.

7.3.9. *Lemma.* If F is a loose Ω-complex on $\mathcal{Y}$ bounded from below then $f_{\cdot}F = Rf_{\cdot}F$.

Proof. It suffices to check that if our F is in addition $\mathcal{D}$-acyclic (i.e., satisfies condition $H^{\cdot}_{\mathcal{D}} F = 0$) then $f_{\cdot}F$ is also $\mathcal{D}$-acyclic (use 7.3.8(i)).

a. We may assume that $\mathcal{Z}$ is a smooth affine scheme Z. Indeed, the statement we want to check is local with respect to $\mathcal{Z}$. Replace $\mathcal{Z}$ by an affine $Z \in \mathcal{Z}_{sm}$, $\mathcal{Y}$ by $\mathcal{Y} \underset{\mathcal{Z}}{\times} Z$, and F by its pull-back to $\mathcal{Y} \underset{\mathcal{Z}}{\times} Z$. The new data satisfy all the conditions of the lemma.

b. We may assume that $\mathcal{Y}$ is a smooth affine scheme Y. Indeed, take $U_{\cdot}$ as in (i), and denote by A the Čech complex with terms $A^i = \bigoplus\limits_{a\geq 0} (f\pi_a)_{\cdot}(F^{i-a}_{U_a})$. This is an Ω-complex on Z. Since F is loose the obvious morphism $f_{\cdot}F \to A$

is a $\mathcal{D}$-quasi-isomorphism (use (310)). Note that A carries an obvious filtration with successive quotients $(f\pi_a).(F_{U_a})[-a]$. If we know that these are $\mathcal{D}$-acyclic, then A is $\mathcal{D}$-acyclic (use the fact that F is bounded from below), hence $f.F$ is $\mathcal{D}$-acyclic.

c. Let $i : Y \to Y \times Z$ be the graph embedding for f. Then $G := i.F$ is $\mathcal{D}$-acyclic. Since $f.F = p.G$ (here p is the projection $Y \times Z \to Z$) what we need to show is that $p.G$ is $\mathcal{D}$-acyclic. Let T be the relative de Rham complex for $\mathcal{D}G$ along the fibers of p. We are in a direct product situation so $p.T$ is a $\mathcal{D}$-complex on Z. There is an obvious morphism of $\mathcal{D}$-complexes $\mathcal{D}p.G \to p.T$ which is a quasi-isomorphism. Since $p.T$ is acyclic (T carries a filtration with successive quotients $\mathcal{D}G \otimes \Lambda\Theta_Y$, and $\mathcal{D}G$ is acyclic) we are done. □

Remark. If f is an affine morphism then for any $F \in C(\mathcal{Y}, \Omega)$ one has $f.F = Rf.F$. Indeed, the statement is local with respect to $\mathfrak{z}$, so we may assume that $\mathfrak{z}$ is an affine scheme. Then $\mathcal{Y}$ is an affine scheme, hence any complex on $\mathcal{Y}$ is loose; now use 7.3.9.

7.3.10. *Corollary.* (i) The functor $f_* := Rf. : D^+(\mathcal{Y}) \to D^+(\mathcal{Z})$ is correctly defined.

(ii) f_* is compatible with composition of f's, i.e., the canonical morphism $(f_1 f_2)_* \to f_{1*} f_{2*}$ is an isomorphism.

Proof. (i) Use 7.3.8(i) and 7.3.9.

(ii) $f.$ sends loose Ω-complexes to loose ones. □

7.3.11. *Remarks.* (i) The above lemmas are also true in the setting of $\mathcal{O}$-complexes.

(ii) Assume that the functor $f.$ on the category of $\mathcal{O}$-modules on $\mathcal{Y}$ has finite cohomology dimension (e.g., this happens when f is representable). Then $f_* := Rf.$ is well-defined for the derived categories of Ω-complexes with arbitrary boundary conditions. Indeed, 7.3.9 (together with its proof) remains valid for unbounded loose Ω-complexes.

(iii) If our stacks are smooth varieties then the above functor f_* is the standard push-forward functor of $\mathcal{D}$-module theory (see 7.2.10). In this situation lemma 7.3.9 (and its proof) remains valid if we assume only that the cohomology $H^a(U,F^i)$, $a>0$, vanish for any Zariski open U of Y such that $U\to Y$ is an affine morphism.

7.3.12. Let now $\mathcal{Y}$ be any smooth stack such that the diagonal morphism $\mathcal{Y}\to\mathcal{Y}\times\mathcal{Y}$ is affine (i.e., we drop the quasi-compactness assumption). Then the category of Ω-complexes on $\mathcal{Y}$ may be too small to define the right $\mathcal{D}$-module derived category. One extends the above formalism as follows.

To simplify the notations let us assume that $\mathcal{Y}$ admits a countable covering by quasi-compact opens. In other words $\mathcal{Y}$ is a union of an increasing sequence $\mathcal{Y}_1\subset\mathcal{Y}_2\subset\ldots$ of open quasi-comact substacks. An *Ω-complex on* $\mathcal{Y}.$ is a collection $F=(F_i,a_i)$ where F_i are Ω-complexes on $\mathcal{Y}_i$ and $a_i:F_{i+1}|_{\mathcal{Y}_i}\to F_i$ are morphisms of Ω-complexes which are $\mathcal{D}$-quasi-isomorphisms. Such Ω-complexes form a DG category $C(\mathcal{Y}.,\Omega)$, so we have the corresponding homotopy category $K(\mathcal{Y}.,\Omega)$. It carries the cohomology functor $H_{\mathcal{D}}$ with values in the abelian category $\mathcal{M}(\mathcal{Y})$ of $\mathcal{D}$-modules on $\mathcal{Y}$, $H_{\mathcal{D}}(F)|_{\mathcal{Y}_i}=H_{\mathcal{D}}(F_i)$.

We define $D(\mathcal{Y}.,\Omega)$ as the localization of $K(\mathcal{Y}.,\Omega)$ with respect to $\mathcal{D}$-quasi-isomorphisms. The triangulated categories $D(\mathcal{Y}.,\Omega)$ for different $\mathcal{Y}.$'s are canonically identified. Indeed, let $\mathcal{Y}'_j$ be another sequence of open substacks of $\mathcal{Y}$ as above. Choose an increasing function $j=j(i)$ such that $\mathcal{Y}_i\subset\mathcal{Y}'_{j(i)}$. Let us assign to an Ω-complex F' on $\mathcal{Y}'_.$ the Ω-complex F on $\mathcal{Y}.$, $F_i=F'_{j(i)}|_{\mathcal{Y}_i}$. This functor commutes with $H_{\mathcal{D}}$. The corresponding functor between the $\mathcal{D}$-derived categories does not depend (in the obvious sense) on the auxiliary choice of $j(i)$, and it is an equivalence of categories.

We see that the category $D(\mathcal{Y}.,\Omega)$ depends only on $\mathcal{Y}$, so we denote it by $D(\mathcal{Y},\Omega)$ or simply $D(\mathcal{Y})$. Our triangulated category carries the cohomology functor $H_{\mathcal{D}}:D(\mathcal{Y})\to\mathcal{M}(\mathcal{Y})$ and there is a canonical fully faithful embedding $\Omega : \mathcal{M}(\mathcal{Y}) \hookrightarrow D(\mathcal{Y})$ (see 7.3.2). Proposition 7.3.4 remains true; the proof follows from 7.5.4.

Let $f : \mathcal{Y} \to \mathcal{Z}$ be a morphism of smooth stacks that satisfy our assumption. If f is smooth then one defines the t-exact pull-back functor $f^{\cdot} : D(\mathcal{Z}) \to D(\mathcal{Y})$ in the obvious manner. If f is an arbitrary quasi-compact morphism then one has a canonical push-forward functor $f_* : D(\mathcal{Y})^+ \to D(\mathcal{Z})^+$. We define it after a short digression about loose Ω-complexes.

By definition, $F \in C(\mathcal{Y}_\cdot, \Omega)$ is loose if such are all $F_i \in C(\mathcal{Y}_i, \Omega)$. Lemma 7.3.8(i),(iii) remains true in our setting. This means that one may define the $\mathcal{D}$-derived category using only loose complexes. To prove 7.3.8(i) choose coverings $\pi_i : V_i \to \mathcal{Y}_i$ such that V_i is an affine scheme. Denote by U_i the disjoint union of V_j's, $1 \leq j \leq i$, and by $U_{i\cdot}$ the corresponding hypercovering of $\mathcal{Y}_i$, U_{ia} is the a-multiple fibered product of U_i over $\mathcal{Y}_i$. Now take any $F' \in C(\mathcal{Y}_\cdot, \Omega)$. Let F_i be the Čech complex of F'_i for the hypercovering $U_{i\cdot}$ (see the proof of 7.3.8(i)). Then F_i form an Ω-complex F on $\mathcal{Y}_\cdot$ in the obvious manner. This F is loose, and the obvious morphism $F' \to F$ is a $\mathcal{D}$-quasi-isomorphism, q.e.d.

Now let us define f_*. Let $\mathcal{Z}_i$ be a sequence of open quasi-compact substacks of $\mathcal{Z}$ as above. Then $\mathcal{Y}_i := f^{-1}\mathcal{Z}_i$ is the corresponding sequence for $\mathcal{Y}$. Let F be a bounded from below loose Ω-complex on $\mathcal{Y}_\cdot$. Then $(f_\cdot F)_i := f_\cdot(F_i)$ form an Ω-complex $f_\cdot F$ on $\mathcal{Z}_\cdot$ (use 7.3.9). The functor $f_\cdot$ preserves $\mathcal{D}$-quasi-isomorphisms (by 7.3.9). Our f_* is the corresponding functor between the $\mathcal{D}$-derived categories. Corollary 7.3.10(ii) together with its proof remains true.

Assume that in addition all the functors $f_{i\cdot} : \mathcal{M}(\mathcal{Y}_i, \mathcal{O}) \to \mathcal{M}(\mathcal{Z}_i, \mathcal{O})$ have finite cohomological dimension (e.g., this happens when f is representable). Then the functor f_* is correctly defined on the whole $D(\mathcal{Y})$. Indeed, let F be any loose Ω-complex on $\mathcal{Y}_\cdot$. Then $(f_\cdot F)_i := f_\cdot(F_i)$ form an Ω-complex $f_\cdot F$ on $\mathcal{Z}_\cdot$ (use 7.3.11(ii)). The functor $f_\cdot$ preserves $\mathcal{D}$-quasi-isomorphisms, and we define $f_* : D(\mathcal{Y}) \to D(\mathcal{Z})$ as the corresponding functor between the $\mathcal{D}$-derived categories.

7.3.13. *Remark.* Let A be a commutative algebra. Let $\mathcal{M}(\mathcal{Y}, A)$ be the abelian category of $\mathcal{D}$-modules on $\mathcal{Y}$ equipped with an action of A. One defines a t-category $D(\mathcal{Y}, A)$ with core $\mathcal{M}(\mathcal{Y}, A)$ as in 7.3.12 using Ω-complexes with A-action. The standard functors render to the A-linear setting without problems. More generally, let $\mathcal{A}_{\mathcal{Y}}$ be a commutative $\mathcal{D}$-algebra on $\mathcal{Y}$ (:= a commutative algebra in the tensor category $\mathcal{M}^\ell(\mathcal{Y})$). We have the abelian category $\mathcal{M}(\mathcal{Y}, \mathcal{A}_{\mathcal{Y}})$ of $\mathcal{A}_{\mathcal{Y}}$-modules and its derived version $D(\mathcal{Y}, \mathcal{A}_{\mathcal{Y}})$ defined as in 7.3.12 using Ω-complexes with $\mathcal{A}_{\mathcal{Y}}$-action.

7.4. **Descent for derived categories.** We explain a general homotopy inverse limit construction for derived categories. We need it to be able to formulate a "local" definition of the $\mathcal{D}$-module derived categories.

7.4.1. Denote by (Δ) the category of non-empty finite totally ordered sets $\Delta_n = [0, n]$ and increasing injections. Let $\mathcal{M}\cdot$ be a family of abelian categories cofibered over (Δ) such that for any morphism $\alpha : \Delta_n \hookrightarrow \Delta_m$ the corresponding functor $\alpha\cdot : \mathcal{M}_n \to \mathcal{M}_m$ is exact.

Denote by $\mathcal{M}_{\mathrm{tot}}$ the category of cocartesian sections of $\mathcal{M}\cdot$, so an object of $\mathcal{M}_{\mathrm{tot}}$ is a collection $M = \{M_n, \alpha^*\}$, $M_n \in \mathcal{M}_n$, $\alpha^* = \alpha^*_M : \alpha\cdot M_n \overset{\sim}{\rightarrow} M_m$ are isomorphisms such that $(\alpha\beta)^* = \alpha^*\alpha\cdot(\beta^*)$ (here $\beta : \Delta_l \hookrightarrow \Delta_n$). This is an abelian category. Note that $\mathcal{M}_{\mathrm{tot}}$ is compatible with duality: one has $(\mathcal{M}_{\mathrm{tot}})^\circ = (\mathcal{M}^\circ)_{\mathrm{tot}}$.

Our aim is to define a t-category $D_{\mathrm{tot}}(\mathcal{M}\cdot)$ with core $\mathcal{M}_{\mathrm{tot}}$ which satisfies the following key property:

(312) For any $M, N \in \mathcal{M}_{\mathrm{tot}}$ there is a canonical spectral sequence $E_r^{p,q}$ converging to $\mathrm{Ext}^{p+q}_{D_{\mathrm{tot}}(\mathcal{M}\cdot)}(N, M)$ with $E_1^{p,q} = \mathrm{Ext}^q_{\mathcal{M}_p}(N_p, M_p)$.

The construction of $D_{\mathrm{tot}}(\mathcal{M}\cdot)$ is compatible with duality.

7.4.2. Consider the category $\sec_+ = \sec_+(\mathcal{M}\cdot)$ whose objects are collections $M = (M_n, \alpha^*)$ where $M_n \in \mathcal{M}_n$, $\alpha^* = \alpha^*_M : \alpha\cdot M_n \to M_m$ are morphisms such that $(\alpha\beta)^* = \alpha^*\alpha\cdot(\beta^*)$, $\mathrm{id}^*_{\Delta_n} = \mathrm{id}_{M_n}$. This is an abelian category which contains $\mathcal{M}_{\mathrm{tot}}$ as a full subcategory closed under extensions. Define $\sec_- = \sec_-(\mathcal{M}\cdot)$ by duality: $\sec_-(\mathcal{M}\cdot) := (\sec_+(\mathcal{M}^\circ_\cdot))^\circ$, so an object of $\sec_-$ is a collection $N = (N_n, \alpha_*)$, $N_n \in \mathcal{M}_n$, $\alpha_* = \alpha^N_* : N_m \to \alpha\cdot N_n$.

Consider the DG categories $C\sec_\pm$ of complexes in $\sec_\pm$ and the corresponding homotopy categories $K\sec_\pm$. There are adjoint DG functors

$$c_+ : C\sec_- \longrightarrow C\sec_+ , \qquad c_- : C\sec_+ \longrightarrow C\sec_- \tag{313}$$

defined as follows. Take $M \in C\sec_+$. Then for any $m \geq 0$ we have a “cohomology type” coefficient system $\widetilde{M}_m$ on the simplex Δ_m with values in $C\mathcal{M}_m$. Namely, $\widetilde{M}_m$ assigns to a face $\alpha : \Delta_n \hookrightarrow \Delta_m$ the complex $\alpha\cdot M_n$, and if $\alpha' : \Delta_l \hookrightarrow \Delta_m$ is a face of α, i.e., $\alpha' = \alpha\beta$, then the corresponding connecting morphism $\alpha'_\cdot M_l \to \alpha\cdot M_n$ is $\alpha\cdot(\beta^*)$. Now $(c_-M)_m$ is the total cochain complex $C^\cdot(\Delta_m, \widetilde{M}_m)$ (so $c_-(M)^\cdot_m = \bigoplus_{\alpha\colon\Delta_n\to\Delta_m} \alpha\cdot M_n^{\cdot-n})$, $\alpha_*^{c_-(M)}$ are the obvious projections. One defines c_+ by duality.

To see that $c_\pm$ are adjoint consider for N, M as above the complex of abelian groups $Hom(N, M)$ with terms

$$Hom(N, M)^i = \prod_{a,n} \mathrm{Hom}(N_n^{a+n}, M_n^{a+i})$$

and the differential wich sends $f = (f_{a,n}) \in Hom(N, M)^i$ to df,

$$(df)_{a,n} = df_{a,n} - (-1)^{i+n} f_{a+1,n} d + \sum_{j=0,..,n} (-1)^j \alpha_j^* \alpha_j\cdot(f_{a+1,n-1})\alpha_{j*}.$$

Here $\alpha_j : \Delta_{n-1} \to \Delta_n$ is the j^{th} face embedding. Now the adjunction property follows from the obvious identification of complexes of homomorphisms

$$Hom(c_+N, M) \,\tilde{\rightarrow}\, Hom(N, M) \,\tilde{\leftarrow}\, Hom(N, c_-M) \tag{314}$$

7.4.3. *Remark.* Fix some $m \geq 0$. For $i = 0, .., m$ let $\nu_i : c_-(M)_m \to M_m$ be the composition of the projector $c_-(M)_m \to \alpha_{i\cdot} M_0$ and $\alpha_i^* : \alpha_{i\cdot} M_0 \to M_m$; here $\alpha_i : \Delta_0 \to \Delta_m$ is the i^{th} vertex. Now all the morphisms ν_i's are mutually homotopic (with canonical homotopies and "higher homotopies").

7.4.4. *Lemma.* The functors $c_\pm$ preserve quasi-isomorphisms. The adjunction morphisms $c_+ c_- M \to M$, $N \to c_- c_+ N$ are quasi-isomorphisms. □

We see that $c_\pm$ define mutually inverse equivalences between the derived categories $D\,\mathrm{sec}_\pm$. Let us denote these categories thus identified by $D\,\mathrm{sec}$. So $D\,\mathrm{sec}$ carries two t-structures with cores $\mathrm{sec}_\pm$ and cohomology functors $H_\pm : D\,\mathrm{sec} \to \mathrm{sec}_\pm$.

7.4.5. Let $C_{\mathrm{tot}\,+} \subset C\,\mathrm{sec}_+$ be the full subcategory of complexes M such that $H^i M \in \mathcal{M}_{\mathrm{tot}} \subset \mathrm{sec}_+$ for any i. In other words $M \in C\,\mathrm{sec}_+$ belongs to $C_{\mathrm{tot}\,+}$ if all the morphisms α_M^* are quasi-isomorphisms. Define $C_{\mathrm{tot}\,-} \subset C\,\mathrm{sec}_-$ in the similar way. Let $K_{\mathrm{tot}\,\pm} \subset K\,\mathrm{sec}_\pm$, $D_{\mathrm{tot}\,\pm} \subset D\,\mathrm{sec}_\pm$ be the corresponding homotopy and derived categories; these are triangulated categories.

The derived categories $D(\mathcal{M}_n)$ form a cofibered category over (Δ). Denote by D_{tot}^{fake} the category of its cocartesian sections (this is *not* a triangulated category!). The cohomology functors for $\mathcal{M}\cdot$ define a functor $H : D_{\mathrm{tot}}^{fake} \to \mathcal{M}_{\mathrm{tot}}$. One has an obvious functor $\epsilon_+ : D_{\mathrm{tot}\,+} \to D_{\mathrm{tot}}^{fake}$ which assigns to M the data $(M_n, \alpha *)$ considered as an object of D_{tot}^{fake}. There is a similar functor $\epsilon_- : D_{\mathrm{tot}} \to D_{\mathrm{tot}}^{fake}$.

7.4.6. *Lemma.* For any $M \in D_{\mathrm{tot}\,+}$ one has $c_- M \in D_{\mathrm{tot}\,-}$, and there is a unique isomorphism $\epsilon_-(c_- M) \overset{\sim}{\rightarrow} \epsilon_+(M)$ such that its 0^{th} component is id_{M_0}. One also has the dual statement with + and - interchanged.

Proof. Use 7.4.3. □

7.4.7. We see that the functors $c_\pm$ identify the triangulated categories $D_{\mathrm{tot}\,\pm}$. In other words, the subcategories $D_{\mathrm{tot}\,\pm} \subset D\,\mathrm{sec}$ coincide; this is the category $D_{tot} = D_{\mathrm{tot}}(\mathcal{M}\cdot)$ that was promised in 7.4.1. The functors $\epsilon_\pm$

are canonically identified, so we have the functor $\epsilon : D_{\mathrm{tot}} \to D_{\mathrm{tot}}^{fake}$. Note that $H_{\pm} = H\epsilon$, so we have a canonical cohomology functor $H : D_{\mathrm{tot}} \to \mathcal{M}_{\mathrm{tot}}$. This is a cohomology functor for a non-degenerate t-structure on D_{tot} with core $\mathcal{M}_{\mathrm{tot}}$. Note that the embedding $D_{\mathrm{tot}} \hookrightarrow D\,\mathrm{sec}$ is t-exact with respect to either of $\pm$ t-structures on $D\,\mathrm{sec}$; it identifies the core $\mathcal{M}_{\mathrm{tot}}$ with the intersection of cores sec_+ and sec_-.

7.4.8. Let us derive the spectral sequence (312) from 7.4.1. More generally, consider objects $N \in D^-\,\mathrm{sec}_- \subset D\,\mathrm{sec}$, $M \in D^+\,\mathrm{sec}_+ \subset D\,\mathrm{sec}$. Let us represent them by complexes $N \in K^-\,\mathrm{sec}_-$, $M \in K^+\,\mathrm{sec}_+$. Consider the complex $Hom(N, M)$ (see 7.4.2). It carries an obvious decreasing filtration $F^{\cdot}$ with $gr_F^n = Hom(N_n, M_n)[-n]$. Note that $Hom(N, M)$ is a bounded below complex and filtration $F^{\cdot}$ induces on each term $Hom(N, M)^i$ a finite filtration. We consider $Hom(N, M)$ as an object of the filtered derived category DF of such complexes. Let $RHom(N, \cdot)$ be the right derived functor of the functor $K^+\,\mathrm{sec}_+ \to \mathrm{DF}$, $M \to Hom(N, M)$. This functor is correctly defined, and the obvious morphism $gr_F^n RHom(N, M) \to RHom(N_n, M_n)[-n]$ is a quasi-isomorphism for any n. This follows from the fact that for any quasi-isomorphism $f : M_n \to I$ in $\mathcal{M}_n$ there exists a quasi-isomorphism $g : M \to J$ in $K^+\,\mathrm{sec}_+$ and a morphism $h : I \to J_n$ such that $g_n = hf$. Consider the spectral sequence $E_r^{p,q}$ of the filtered complex $RHom(N, M)$. It converges to $H^{\cdot} RHom(N, M)$, and $E_1^{p,q} = H^q RHom_{\mathcal{M}_p}(N_p, M_p)$.

7.4.9. *Remark.* Assume that the categories $\mathcal{M}_n$ have many injective objects. Then the category $K^+_{\mathrm{tot}-}$ has many injective objects (i.e., the functor $K^+_{\mathrm{tot}-} \to D^+_{\mathrm{tot}}$ admits a right adjoint functor). Indeed, if $I \in K^+_{\mathrm{tot}+}$ is a complex such that eacb I_n^a is an injective object of $\mathcal{M}_n$ then $c_- I$ is an injective object of $K^+_{\mathrm{tot}-}$, and any object in $K^+_{\mathrm{tot}-}$ is quasi-isomorphic to such I. Dually, if $\mathcal{M}_n$ have many projective objects then $K^-_{\mathrm{tot}+}$ has many projective objects.

7.4.10. This subsection will not be used in the sequel; the reader may skip it. One may define $D\sec$, hence D_{tot}, in a slightly different way which is convenient in some applications*). We define the category $hot_+ = hot_+(\mathcal{M}.)$ as follows. Its objects are families $A = (A_m)$, $A_m \in \mathcal{M}_m$. A morphism $f : A \to B$ is a collection (f_α) where for an arrow $\alpha : \Delta_n \to \Delta_m$ the corresponding f_α is a morphism $\alpha.A_n \to B_m$. The composition of morphisms is $(fg)_\alpha = \sum\limits_{\alpha=\beta\gamma} f_\beta \beta.(g_\gamma)$. This is an additive category. Set $hot_-(\mathcal{M}.) = (hot_+(\mathcal{M}.^\circ))^\circ$. We have the corresponding DG categories of complexes $Chot_\pm$.

One has a DG functor $t_+ : C\sec_+ \to Chot_+$ which sends $M \in C\sec_+$ to a complex $t_+M \in Chot_+$ with components $(t_+M)^a_m = M^{a-m}_m$ and the differential $d = d_{t_+M}$ such that $d_{\mathrm{id}_{\Delta_m}} = (-1)^m d^{a-m}_{M_m} : M^{a-m}_m \to M^{a-m+1}_m$, and for the i^{th} boundary map $\alpha_i : \Delta_m \hookrightarrow \Delta_{m+1}$ one has $d_{\alpha_i} = (-1)^i \alpha_i^* : \alpha_i.M^{a-m}_m \to M^{a-m}_{m+1}$, all other components of d are zero. For $l \in Hom(M_1, M_2)$ one has $t_+(l)_{\mathrm{id}_{\Delta_m}} = l_m$, the other components are zero.

Remark. The functor t_+ is faithful. One may consider objects of $Chot_+$ as "generalized complexes" in $\sec_+$ with extra higher homotopies.

One also has a DG functor $s_- : Chot_+ \to C\sec_-$ defined as follows. For $A \in Chot_+$ the complex s_-A has components $(s_-A)^a_m = \sum\limits_{\beta:\Delta_n\to\Delta_m} \beta.A^a_n$. The compatibility morphism $\alpha_* : (s_-A)^a_l \to \alpha.(s_-A)^a_m$ for $\alpha : \Delta_m \to \Delta_l$ has component $\gamma.A^a_k \to \alpha.\beta.A^a_n$ equal to $\mathrm{id}_{\gamma.A^a_k}$ if $k = n$, $\gamma = \alpha\beta$ and zero otherwise. A component $\gamma.A^a_k \to \alpha.\beta.A^a_n$ of the differential $d_{s_-A} : (s_-A)^a_m \to (s_-A)^{a+1}_m$ is equal to $\gamma.(d_{A\delta})$ if $\beta = \gamma\delta$ and zero otherwise.

Remark. The DG functor s_- is fully faithful.

We define DG functors $t_- : C\sec_- \to Chot_-$ and $s_+ : Chot_- \to C\sec_+$ by duality. Note that the composition $s_+t_- : C\sec_- \to C\sec_+$ coincides with the functor c_+ from 7.4.2; similarly, $s_-t_+ = c_-$. The functors

*) This construction goes back to the works of Toledo and Tong.

$t_-s_- : Chot_+ \to Chot_-$ and $t_+s_+ : Chot_- \to Chot_+$ are adjoint (just as the functors $c_\pm$, see 7.4.2).

We say that a morphism $f : A \to B$ in the homotopy category $Khot_\pm$ of $Chot_\pm$ is a *quasi-isomorphism* if all the morphisms $f_m := f_{\mathrm{id}_{\Delta_m}} : A_m \to B_m$ are quasi-isomorphisms. Quasi-isomorphisms form a localizing family. Denote the corresponding localized triangulated categories by $Dhot_\pm$.

The functors $s_\pm, t_\pm$ preserve quasi-isomorphisms, so they define functors between the derived categories. The adjunction morphisms for compositions of these functors are quasi-isomorphisms. So our derived categories $D\sec_\pm, Dhot_\pm$ are canonicaly identified.

Remarks. (i) A complex $A \in Dhot_+$ belongs to D_{tot} if and only if for any $\alpha : \Delta_m \to \Delta_{m+1}$ the α-component $d_{A\alpha} : \alpha.A_m \to A_{m+1}$ is a quasi-isomorphism of complexes (the differential on A_m is $d_{A\,\mathrm{id}_{\Delta_m}}$, same for A_{m+1}).

(ii) If the categories $\mathcal{M}_n$ have many injective objects then K^+hot_+ has many injective objects. Dually, if $\mathcal{M}_n$ have many projective objects then K^-hot_- has many projective objects (cf. 7.4.9).

7.4.11. Some of the above constructions make sense in the following slightly more general setting. Consider any family of DG categories $\mathcal{C}\cdot$ cofibered over (Δ). One has the DG categories $\mathcal{C}\sec_\pm = \sec_\pm(\mathcal{C}\cdot)$ (defined exactly as the categories $\sec_\pm(\mathcal{M}\cdot)$ in 7.4.2), and the corresponding homotopy categories. One defines the adjoint functors $c_\pm$ between the $\pm$ categories as in 7.4.2.

Assume in addition that we have $\mathcal{M}$ as in 7.4.1 and a family of cohomology functors $H : \mathcal{C}\cdot \to \mathcal{M}\cdot$ compatible with the fibered category structures. We get the corresponding cohomology functors $H_\pm : \mathcal{C}\sec_\pm \to \sec_\pm$. Localising our homotopy categories by H-quasi-isomorphisms we get the derived categories $\mathcal{D}\sec_\pm$. As in Lemma 7.4.4 the functors $c_\pm$ identify the categories $\mathcal{D}\sec_\pm$, so we may denote them simply $\mathcal{D}\sec$. One defines the categories $\mathcal{C}_{\mathrm{tot}\pm}$, etc., as in 7.4.5. Lemma 7.4.6 remains true, so we have the full triangulated subcategory $\mathcal{D}_{\mathrm{tot}} \subset \mathcal{D}\sec$ and the cohomology functor $H : \mathcal{D}_{\mathrm{tot}} \to \mathcal{M}_{\mathrm{tot}}$.

7.5. $\mathcal{D}$-module theory on smooth stacks II.

7.5.1. Let $\mathcal{Y}$ be an arbitrary smooth algebraic stack. Let $U\cdot$ be a hypercovering of $\mathcal{Y}$ such that each U_n is a disjoint union of (smooth) quasi-compact separated algebraic spaces (e.g., affine schemes). We call such $U\cdot$ an *admissible* hypercovering. Consider $U\cdot$ as a $(\Delta)^\circ$-algebraic space. The categories $\mathcal{M}(U\cdot)$ form a (Δ)-family of abelian categories as in 7.4.1; the corresponding category $\mathcal{M}_{\text{tot}}$ is $\mathcal{M}(\mathcal{Y})$. According to 7.4.7 we get the corresponding t-category $D_{\text{tot}} = D_{\text{tot}}(U\cdot, \mathcal{D})$ with core $\mathcal{M}(\mathcal{Y})$.

We may also consider DG categories $C(U\cdot, \Omega)$ together with the cohomology functors $H_{\mathcal{D}}\cdot : C(U\cdot, \Omega) \to \mathcal{M}(U\cdot)$, $H_{\mathcal{D}n}F_n = H_{\mathcal{D}}F_n[\dim U_n/\mathcal{Y}]$ for $F_n \in C(U_n, \Omega)$, and apply 7.4.11. We get a triangulated category $D_{\text{tot}}(U\cdot, \Omega)$ together with a cohomology functor $H_{\mathcal{D}} : D_{\text{tot}}(U\cdot, \Omega) \to \mathcal{M}(\mathcal{Y})$.

The categories $D_{\text{tot}}(U\cdot, \mathcal{D})$ and $D_{\text{tot}}(U\cdot, \Omega)$ are canonically identified. Namely, one has a functor $\Omega\cdot : C(U\cdot, \mathcal{D}) \to C(U\cdot, \Omega)$, $\Omega_n(M_n) := \Omega M_n[-\dim U_n/\mathcal{Y}]$. This functor is compatible with DG and fibered categories structures, and with the cohomology functors (i.e., $H = H_{\mathcal{D}}\cdot\Omega\cdot$). Therefore it yields an exact functor

$$\Omega : D_{\text{tot}}(U\cdot, \mathcal{D}) \to D_{\text{tot}}(U\cdot, \Omega) \tag{315}$$

This functor is an equivalence of categories. Indeed, though the functor $\mathcal{D}$ between $C(U\cdot, \Omega)$ and $C(U\cdot, \mathcal{D})$ is *not* compatible with the fibered category structures, it provides the functor $\mathcal{D} : C\sec_-(U\cdot, \Omega) \to C\sec_-(U\cdot, \mathcal{D})$, $(\mathcal{D}F)_n = \mathcal{D}F_n[\dim U_n/\mathcal{Y}]$ (use 7.2.8 to define α^*'s). This $\mathcal{D}$ is left adjoint to the corresponding Ω functor, and is compatible with the cohomology functors. The $\mathcal{D}$-Ω adjunction morphisms are quasi-isomorphisms (see 7.2.4, 7.2.5), so $\mathcal{D}$ yields the functor inverse to (315).

We denote the categories $D_{\text{tot}}(U\cdot, \mathcal{D})$ and $D_{\text{tot}}(U\cdot, \Omega)$ thus identified simply by $D_{\text{tot}}(U\cdot)$.

7.5.2. *Propositon.* There exists a canonical identification of t-categories $D_{\text{tot}}(U\cdot)$ for different admissible coverings of $\mathcal{Y}$.

For a proof see 7.5.5 below. We denote these categories thus identified by $D(\mathcal{Y})$; this is a t-category with core $\mathcal{M}(\mathcal{Y})$.

Before proving 7.5.2 let us show that if $\mathcal{Y}$ satisfies condition (310) then, indeed, we get the same category $D(\mathcal{Y})$ as in 7.3.2. By the way, this implies 7.3.4.

Choose a hypercovering $U\cdot$ of $\mathcal{Y}$ such that U_n are affine schemes. There is an obvious exact functor (restriction to $U\cdot$)

$$r: D(\mathcal{Y}, \Omega) \to D_{\mathrm{tot}}(U\cdot, \Omega) \tag{316}$$

7.5.3. *Lemma.* The functor r is an equivalence of categories.

Proof. Let us construct the inverse functor. For $F \in K_{\mathrm{tot}\,+}(\Omega)$ define the Ω-complex $\pi.F$ on $\mathcal{Y}$ as the total complex of Čech bicomplex with terms $\pi.F^{ab} := \pi_b(F^a)$, so $(\pi.F)^n = \bigoplus\limits_{a+b=n} F^{ab}$; here π_b are projections $U_b \to \mathcal{Y}$. Thus we have the exact functor $\pi. : K_{\mathrm{tot}\,+}(\Omega) \to K(\mathcal{Y}, \Omega)$. This functor preserves $\mathcal{D}$-quasi-isomorphisms (since, by (310), the projections π_b are affine), so it defines a functor $D_{\mathrm{tot}}(U\cdot, \Omega) \to D(\mathcal{Y}, \Omega)$.

We leave it to the reader to check that this functor is inverse to r (hint: for F as above the adjunction quasi-isomorphism $\pi^{\cdot}_{\Omega}\pi.F \to F$ comes from a canonical morphism $\pi^{\cdot}_{\Omega}\pi.F \to c_{-}F$ in $C\,\mathrm{sec}_{-}(U\cdot, \Omega)$).

□

7.5.4. *Remark.* The above lemma renders to the setting of 7.3.12 as follows. Let $\mathcal{Y}$ be any smooth stack such that the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is affine. Then the categories $D(\mathcal{Y})$ as defined in 7.3.12 and 7.5.1 are canonically equivalent. Indeed, let $\mathcal{Y}_i$ be a sequence of open substacks of $\mathcal{Y}$ as in 7.3.12, and $V_i \to \mathcal{Y}$ be a covering such that V_i are affine schemes. Then the V_i's form a covering of $\mathcal{Y}$. Let $U\cdot$ be the corresponding Čech hypercovering. Therefore U_a is disjoint union of components U_α labeled by sequences $\alpha = (\alpha_1, \alpha_2, ...)$, $\alpha_i \geq 0$, $\Sigma\alpha_i = a+1$, where U_α is fibered product over $\mathcal{Y}$ of α_1 copies of V_1, α_2 copies of V_2,... For $F \in C(\mathcal{Y}\cdot, \Omega)$ set $F_{U_\alpha} := F_{i_\alpha U_\alpha}$ where i_α is the minimal i such that α_i is non-zero (note that

$U_\alpha \in \mathcal{Y}_{sm}$. These F_{U_α} form an Ω-complex F on $U\cdot$ in the obvious manner which lies in $C_{\mathrm{tot}}(U\cdot,\Omega)$. The functor $C(\mathcal{Y}\cdot,\Omega) \to C_{\mathrm{tot}}(U\cdot,\Omega)$ commutes with the functor $H_{\mathcal{D}}$ so it defines a triangulated functor

$$r : D(\mathcal{Y}\cdot,\Omega) \to D_{\mathrm{tot}}(U\cdot,\Omega) \tag{317}$$

We leave it to the reader to check that this functor is an equivalence of categories, and that the corresponding identification of $D(\mathcal{Y})$'s in the sense of 7.3.12 and 7.5.2 does not depend on the auxiliary data of $\mathcal{Y}\cdot$ and $V\cdot$.

7.5.5. *Proof of 7.5.2.* We need to identify canonically the t-categories $D_{\mathrm{tot}}(U\cdot)$ for different $U\cdot$'s. Let $U'_\cdot$ be another admissible hypercovering. First we define a t-exact functor $\Phi = \Phi_V : D_{\mathrm{tot}}(U\cdot) \to D_{\mathrm{tot}}(U'_\cdot)$ in terms of some auxiliary data V. Then we show that Φ actually does not depend V, and it is an equivalence of categories.

Our V is a $(\Delta)^\circ \times (\Delta)^\circ$ -algebraic space $V\cdot\cdot$ over $\mathcal{Y}$ together with smooth morphisms $\pi : V_{mn} \to U_m$, $\pi' : V_{mn} \to U'_n$. We assume that π, π' are compatible with (Δ) projections in the obvious manner, $\pi'_{\cdot n} : V_{\cdot n} \to U'_n$ are hypercoverings, and $\pi'_{mn} : V_{mn} \to U'_n$ are affine morphisms. For $F \in K_{\mathrm{tot}\,+}(U\cdot,\Omega)$ we have Ω-complexes $F_{Vn} \in K_{\mathrm{tot}\,+}(V_{\cdot n},\Omega)$ - the pull-back of F to $V_{\cdot n}$. Set $\Phi_{Vn}F := \pi'_\cdot F_{Vn}$ (see the proof of 7.5.3 for the notation). This is an Ω-complex on U'_n. The Ω-complexes Φ_{Vn} form an Ω-complex $\Phi_V F \in K_{\mathrm{tot}\,+}(U'_\cdot,\Omega)$ in the obvious way such that $H_{\mathcal{D}}F = H_{\mathcal{D}}\Phi_V F$. Therefore we have a t-exact functor $\Phi_V : D_{\mathrm{tot}}(U\cdot,\Omega) \to D_{\mathrm{tot}}(U'_\cdot,\Omega)$ which induces the identity functor between the cores $\mathcal{M}(\mathcal{Y})$.

Assume that we have V_1 and V_2 as above. To identify the functors Φ_{V_i} choose another V as above, together with embeddings $V_1, V_2 \subset V$ compatible with all the projections which identify $(V_1)_{mn}$, $(V_2)_{mn}$ with a union of connected components of V_{mn}. The embeddings induce projections $\Phi_V F \to \Phi_{V_1}$, $\Phi_V F \to \Phi_{V_2} F$ which are obviously quasi-isomorphisms. Therefore we have identified the functors Φ_{V_i} between the derived categories.

We leave it to the reader to check that this identification does not depend on the auxiliary data of V.

Thus we have a canonical functor $\Phi = \Phi_{UU'} : D_{\mathrm{tot}}(U^{\cdot}, \Omega) \to D_{\mathrm{tot}}(U'^{\cdot}, \Omega)$. If $U''^{\cdot}$ is the third hypercovering then there is a canonical isomorphism of functors $\Phi_{UU''} = \Phi_{U'U''}\Phi_{UU'}$; we leave its definition to the reader, as well as verification of the usual compatibilities. Since Φ_{UU} is the identity functor we see that Φ's identify simultaneously all the categories $D_{\mathrm{tot}}(U^{\cdot})$. $\square$

7.5.6. Let $f : \mathcal{Y} \to \mathcal{Z}$ be a quasi-compact morphism of smooth stacks. Let us define the push-forward functor $f_* : D(\mathcal{Y})^+ \to D(\mathcal{Z})^+$. To do this consider any admissible hypercoverings $U^{\cdot}$ of $\mathcal{Y}$ and $W^{\cdot}$ of $\mathcal{Z}$. We get the $(\Delta)^\circ \times (\Delta)^\circ$-algebraic space $U^{\cdot} \underset{\mathcal{Z}}{\times} W^{\cdot}$. One may find a $(\Delta)^\circ \times (\Delta)^\circ$-algebraic space $V^{\cdot\cdot}$ together with morphism $\phi = (\phi_1, \phi_2) : V^{\cdot\cdot} \to U^{\cdot} \underset{\mathcal{Z}}{\times} W^{\cdot}$ such that the projections $V_{mn} \to U_m$ are smooth, $V_{mn} \to W_n$ are affine, and $V^{\cdot}_n \to \mathcal{Y} \underset{\mathcal{Z}}{\times} W_n$ are hypercoverings. Now for $F \in K^+_{\mathrm{tot}\,+}(U^{\cdot}, \Omega)$ let $F_{Vn} \in K^+_{\mathrm{tot}\,+}(V^{\cdot}_n, \Omega)$ be its pull-back to $V^{\cdot}_n$. Define the Ω -complex $f_{\cdot}F_n$ on W_n as the total complex of the Čech bicomplex with terms $\phi_{2\cdot}F_{Vn}$. These Ω-complexes form an object $f_{\cdot}F$ of $K^+_{\mathrm{tot}\,+}(W^{\cdot}, \Omega)$. The functor $f_{\cdot} : K^+_{\mathrm{tot}\,+}(U^{\cdot}, \Omega) \to K^+_{\mathrm{tot}}(W^{\cdot}, \Omega)$ preserves $\mathcal{D}$-quasi-isomorphisms hence it yields a functor $f_* : D(\mathcal{Y})^+ \to D(\mathcal{Z})^+$. We leave it to the reader to check that the construction of f_* does not depend on the auxiliary choices of U, W, V, and is compatible with composition of f's.

A smooth morphism of smooth stacks $f : \mathcal{Y} \to \mathcal{Z}$ yields a t-exact functor $f^\dagger = f^{\cdot}_\Omega : D(\mathcal{Z}) \to D(\mathcal{Y})$. Namely, choose admissible hypercoverings $U^{\cdot}$ of $\mathcal{Y}$, $W^{\cdot}$ of $\mathcal{Z}$ and a morphism $f_{\cdot} : U^{\cdot} \to W^{\cdot}$ compatible with f. The functor $f^{\cdot}_{\cdot\Omega} : K_{\mathrm{tot}\,\pm}(W^{\cdot}, \Omega) \to K_{\mathrm{tot}\,\pm}(U^{\cdot}, \Omega)$ preserves $\mathcal{D}$-quasi-isomorphisms, so it defines a functor $f^{\cdot}_\Omega$ between the derived categories. We leave it to the reader to check that this definition does not depend on the auxiliary choices, that our pull-back functor is compatible with composition of f's, and that in case when f is quasi-compact the functor $f^{\cdot}_\Omega$ is left adjoint to f_*.

7.5.7.

7.5.8. *Remarks.* (i) One may also try to define $D(\mathcal{Y})$ using appropriate non-quasi-coherent Ω-complexes in a way similar to the definition of derived category of $\mathcal{O}$-modules from [LMB93]6.3. Probably such a definition yields the same category $D^+(\mathcal{Y})$.

(ii) The "local" construction of derived categories is also convenient in the setting of $\mathcal{O}$-modules. For example, it helps to define the cotangent complex of an algebraic stack as a true object of the derived category (and not just the projective limit of its truncations as in [LMB93]9.2), and also to deal with Grothendieck-Serre duality.

(iii) Replacing $\mathcal{D}$-modules by perverse sheaves we get a convenient definition of the derived category of constructible sheaves on any algebraic stack locally of finite type.

7.6. **Equivariant setting.**

7.6.1. Let us explain parts 7.1.1 (a), (b) of the (finite dimensional) Hecke pattern. So let G be an algebraic group and $K \subset G$ an algebraic subgroup. Assume for simplicity that K is affine; then the stacks below satisfy condition (310) of 7.3.1. Set*) $\mathcal{H}^c := C(K\backslash G/K, \Omega)$, $\mathcal{H} := D(K\backslash G/K)$. We call these categories *pre Hecke* and *Hecke* category respectively. They carry canonical monoidal structures defined as follows.

Consider the morphisms of stacks

$$(K \setminus G/K) \times (K \setminus G/K) \xleftarrow{p} K \setminus G \underset{K}{\times} G/K \xrightarrow{\bar{m}} K \setminus G/K \tag{318}$$

Here $G \underset{K}{\times} G$ is the quotient of $G \times G$ modulo the K-action $k(g_1, g_2) = (g_1 k^{-1}, k g_2)$, p is the obvious projection, and $\bar{m}$ is the product map. For F_1, $F_2 \in \mathcal{H}^c$ set $F_1 \overset{c}{\circledast} F_2 := \bar{m}_{\cdot} p^{\cdot}_{\Omega}(F_1 \boxtimes F_2)$. The *convolution tensor product* $\overset{c}{\circledast}$ satisfies the obvious associativity constraint, so we have a monoidal structure on $\mathcal{H}^c$. We define the convolution tensor product $\circledast : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ as

*) Here the superscript "c" means that we deal with the true DG category of complexes, not the derived category.

the right derived functor of $\overset{c}{\circledast}$. One has $F_1 \circledast F_2 = \bar{m}_* p^{\cdot}_{\Omega}(F_1 \boxtimes F_2)$; if Ω-complexes F_1, F_2 are loose (see 7.3.7) then $F_1 \circledast F_2 = F_1 \overset{c}{\circledast} F_2$. Thus the associativity constraint for $\circledast$ follows from the one of $\overset{c}{\circledast}$, so $\mathcal{H}$ is a monoidal triangulated category. $\mathcal{H}^c$ and $\mathcal{H}$ have a unit object E: one has $E_G = i_{K\cdot}\Omega_K$ (here $i_K : K \hookrightarrow G$ is the embedding).

Let Y be a smooth variety with G-action. Consider the stack $\mathcal{B} := K \setminus Y$. The Hecke Action on $D(\mathcal{B})$ arises from the diagram

$$(K \setminus G/K) \times \mathcal{B} \xleftarrow{p_Y} K \setminus (G \underset{K}{\times} Y) \xrightarrow{\bar{m}_Y} \mathcal{B}. \tag{319}$$

Namely, for $F \in \mathcal{H}^c$, $T \in C(\mathcal{B}, \Omega)$ set $F \overset{c}{\circledast} T := \bar{m}_{Y\cdot} p^{\cdot}_{Y\Omega}(F \boxtimes T)$. As above $\overset{c}{\circledast}$ satisfies the obvious associativity constraint, so $C(\mathcal{B}, \Omega)$ is a unital $\mathcal{H}^c$-Module. Define $\circledast : \mathcal{H} \times D(\mathcal{B}) \longrightarrow D(\mathcal{B})$ as the right derived functor of $\overset{c}{\circledast}$. One has $F \circledast T = \bar{m}_{Y*} p^{\cdot}_{Y\Omega}(F \boxtimes T)$, and if F,T are loose (see 7.3.7) then $F \circledast T = F \overset{c}{\circledast} T$. Thus $D(\mathcal{B})$ is a $\mathcal{H}$-Module.

7.6.2. *Remarks.* (i) In the above definitions we were able to consider the unbounded derived categories since the projections $\bar{m}, \bar{m}_Y$ are representable (see7.3.11(ii)).

(ii) If $f : Z \to Y$ is a morphism of smooth varieties with G-action then $f_* : D(K \setminus Z) \to D(K \setminus Y)$ is a Morphism of $\mathcal{H}$-Modules.

7.6.3. Let Y be a smooth variety equipped with an action of an affine algebraic group K. Consider the stack $\mathcal{B} := K \setminus Y$. In the rest of 7.6 we are going to describe $D(\mathcal{B})$ in terms of appropriate equivariant complexes on Y. We will also introduce certain derived category $D(K \rotatebox{-20}{$\setminus$} Y)$ intermediate between $D(K \setminus Y)$ and $D(Y)$ that will be of use in 7.7.

Set $K_\Omega = (K, \Omega_K)$, $K^{\cdot}_\Omega = (K, \Omega^{\cdot}_K)$ (so $K^{\cdot}_\Omega$ is K_Ω with its de Rham differential skipped). These are group objects in the category of DG ringed spaced and graded ringed spaces respectively. Denote by $\mathfrak{k}$, $\mathfrak{k}_\Omega$, $\mathfrak{k}^{\cdot}_\Omega$ the Lie algebras of K, K_Ω, $K^{\cdot}_\Omega$ respectively. As a plain complex, $\mathfrak{k}_\Omega$ is equal to the cone of $id_{\mathfrak{k}}$ so $\mathfrak{k}^0_\Omega = \mathfrak{k} = \mathfrak{k}^{-1}_\Omega$. Since K is a subgroup of K_Ω and $K^{\cdot}_\Omega$ we have the corresponding Harish-Chandra pairs $(\mathfrak{k}_\Omega, K)$, $(\mathfrak{k}^\cdot_\Omega, K)$. Note that K_Ω modules are the same as DG $(\mathfrak{k}_\Omega, K)$-modules, and $K^\cdot_\Omega$-modules are the same as graded $(\mathfrak{k}^\cdot_\Omega, K)$-modules.

The K-action on Y yields the action of K_Ω on $Y_\Omega = (Y, \Omega)$ hence the action of $K^\cdot_\Omega$ on $Y^\cdot_\Omega = (Y, \Omega^\cdot)$. For a graded $\Omega^\cdot_Y$-module $F^\cdot_Y$ a $K^\cdot_\Omega$-action on $F^\cdot_Y$ is the same as a $(\mathfrak{k}^\cdot_\Omega, K)$-action. Explicitly, this is a K-action on $F^\cdot_Y$ together with a K-equivariant morphism $\mathfrak{k} \otimes F^\cdot_Y \to F^{\cdot -1}_Y$, $\xi \otimes f \mapsto i_\xi(f)$ (we assume that K acts on $\mathfrak{k}$ in the adjoint way) such that $i_\xi(\nu f) = <\xi, \nu> f + \nu i_\xi(f)$, $i_\xi^2 = 0$ for any $\xi \in \mathfrak{k}$ and $\nu \in \Omega^1_Y$.

7.6.4. Let F_Y be an Ω-complex on Y. A K-*action* on F_Y is a K-action on the graded $\mathcal{O}_Y$-module $F^\cdot_Y$ such that for any $k \in K$ the translation $k^* F^\cdot_Y \mathrel{\tilde{\to}} F^\cdot_Y$ is a morphism of Ω-complexes (i.e., it commutes with the differential). A K_Ω-*action* on F_Y is an action of K_Ω on F_Y considered as a DG module on Y_Ω. In other words, this is a $K^\cdot_\Omega$-action on the graded $\Omega^\cdot_Y$-module $F^\cdot_Y$ such that K acts on F_Y as on an Ω-complex and $\mathfrak{k}_\Omega$ acts on F_Y as a DG Lie algebra. The latter condition means that for any $\xi \in \mathfrak{k}$ one has $di_\xi + i_\xi d = Lie_\xi$ (here Lie is the $\mathfrak{k}$-action on $F^\cdot_Y$ that comes from the K-action). An Ω-complex equipped with a K-action is called a *weakly K-equivariant* Ω-complex, and that with K_Ω-action is called K_Ω-*equivariant* Ω-complex.

It is clear that for any Ω-complex F on the stack $\mathcal{B} := K \backslash Y$ the Ω-complex F_Y carries automatically a K_Ω-action.

7.6.5. *Lemma.* The functor $C(K \setminus Y, \Omega) \longrightarrow$ (K_Ω-equivariant Ω-complexes on Y) is an equivalence of DG categories. □

7.6.6. *Remark.* Assume we are in situation 7.6.1. Let $m : K \setminus G \times G/K \to K \setminus G/K$ be the product map. Set $F_1 \widetilde{\circledast} F_2 = m_\cdot(F_{1K\setminus G} \boxtimes F_{2G/K})$; this is an Ω-complex on $K \setminus G/K$. The K-action along the fibers of the projection $G \times G \to G \underset{K}{\times} G$ yields a K_Ω-action on $F_1 \widetilde{\circledast} F_2$ (with respect to the trivial K-action on $K \setminus G/K$). Its invariants coincide with $F_1 \overset{c}{\circledast} F_2$. Similarly, consider the map $m_Y : (K \setminus G) \times Y \to \mathcal{B}$; set $F \widetilde{\circledast} T := m_{Y\cdot}(F_{K\setminus G} \boxtimes T)$. The obvious K-action on $(K \setminus G) \times Y$ yields a K_Ω-action on this Ω-complex whose invariants coinside with $F \overset{c}{\circledast} T$.

7.6.7. We denote the category of weakly K-equivariant Ω-complexes on Y by $C(K \setminus Y, \Omega)$ and the corresponding homotopy and $\mathcal{D}$-derived categories by $K(K \setminus Y, \Omega)$, $D(K \setminus Y, \Omega)$ (a morphism of weakly equivariant Ω-complexes is called a $\mathcal{D}$-quasi-isomorphism if it is a $\mathcal{D}$-quasi-isomorphism of plain Ω-complexes).

7.6.8. *Remarks.* (i) The forgetful functor $C(\mathcal{B}, \Omega) \to C(K \setminus Y, \Omega)$ admits left and right adjoint functors $c^l, c^r : C(K \setminus Y, \Omega) \to C(\mathcal{B}, \Omega)$, $c^l(F_Y) = U(\mathfrak{k}_\Omega) \underset{U(\mathfrak{k})}{\otimes} F_Y$, $c^r(F_Y) = \mathrm{Hom}_{U(\mathfrak{k})}(U(\mathfrak{k}_\Omega), F_Y)$. These functors preserve quasi-isomorphisms, so they define adjoint functors between the derived categories.

(ii) The forgetful functor $C(K \setminus Y, \Omega) \to C(Y, \Omega)$ admits a right adjoint functor $\mathrm{Ind} : C(Y, \Omega) \to C(K \setminus Y, \Omega)$, $\mathrm{Ind}(T_Y)^\cdot = p_* m^*(T_Y^\cdot)$ where $m, p : K \times Y \rightrightarrows Y$ are the action and projection maps. These functors preserve quasi-isomorphisms so they yield the adjoint functors between the derived categories. The composition $c^r\,\mathrm{Ind}$ is the push-forward functor for the projection $Y \to \mathcal{B}$.

(iii) Remark 7.6.6 (ii) remains valid for weakly equivariant Ω-complexes.

(iv) Let $f : Z \to Y$ be a morphism of smooth varieties equipped with K-actions. The construction of the direct image functor from 7.3.6 passes to the weakly equivariant setting without changes, so we have the functor $f_* = Rf_\cdot : D(K \setminus Z, \Omega) \to D(K \setminus Y, \Omega)$. The functors f_* commute with the functors from (i), (ii) above. The same holds for the pull-back functors $f_\Omega^\cdot$ from 7.2.8, 7.3.6.

(v) Here is a weakly equivariant version of 7.6.1. Assume that Y from 7.6.1 carries in addition an action of an affine algebraic group G' that commutes with the G-action (we will write it as a right action). Consider the category $C(K \setminus Y /\, G', \Omega) = C(\mathcal{B} /\, G', \Omega)$ of Ω-complexes on Y equipped with commuting K_Ω- and G-actions. Then the corresponding derived category $D(\mathcal{B}{}_{/}{}^{/}\,G', \Omega)$ is an $\mathcal{H}$-Module. The $\mathcal{H}$-action is defined in the same way as in 7.6.1. Remark 7.6.6 remains valid.

7.6.9. Let us describe the $\mathcal{D}$-module counterpart of the above equivariant categories (see [BL] for details). For a $\mathcal{D}$-module M on Y a *weak K-action* on M is a K-action on M as on an $\mathcal{O}_Y$-module such that for any $k \in K$ the translation $k^*M \rightleftarrows M$ is a morphism of $\mathcal{D}$-mosules. A $\mathcal{D}$-module equipped with a weak K-action is called a *weakly K-equivariant* $\mathcal{D}$-module; the category of those is denoted by $\mathcal{M}(K{}^{\backslash}{}_{\backslash}\,Y)$ (as usual we write $\mathcal{M}^\ell$ or $\mathcal{M}^r$ to specify left and right $\mathcal{D}$-modules). The notations $C(K{}^{\backslash}{}_{\backslash}\,Y, \mathcal{D})$, $K(K{}^{\backslash}{}_{\backslash}\,Y, \mathcal{D})$, $D(K{}^{\backslash}{}_{\backslash}\,Y, \mathcal{D}) = D(K{}^{\backslash}{}_{\backslash}\,Y)$ are clear (cf. 7.2).

The functors $\mathcal{D}$ and Ω from 7.2.2 send weakly equivariant complexes to weakly equivariant ones, thus we have the adjoint DG functors

$$\mathcal{D} : C(K{}^{\backslash}{}_{\backslash}\,Y, \Omega) \to C(K{}^{\backslash}{}_{\backslash}\,Y, \mathcal{D}),\ \Omega : C(K{}^{\backslash}{}_{\backslash}\,Y, \mathcal{D}) \to C(K{}^{\backslash}{}_{\backslash}\,Y, \Omega) \tag{320}$$

and the mutually inverse equivalences of triangulated categories

$$D(K{}^{\backslash}{}_{\backslash}\,Y, \mathcal{D}) \Longleftrightarrow D(K{}^{\backslash}{}_{\backslash}\,Y, \Omega)\,. \tag{321}$$

As usual we denote these categories thus identified by $D(K{}^{\backslash}{}_{\backslash}\,Y)$.

7.6.10. *Remark.* For a weakly K-equivariant $\mathcal{D}$-module M the $\mathfrak{k}$-action on Y lifts to the $\mathcal{O}$-module M in two ways: either as the infinitesimal action defined by the K-action on M or via the $\mathfrak{k}$-action on Y $\sigma : \mathfrak{k} \to \Theta_Y$ and the $\mathcal{D}$-module structure on M. Denote these actions by $\xi, m \mapsto \mathrm{Lie}_\xi\, m$, $\sigma_\xi m$ respectively. Set $\xi^\natural m ::= \mathrm{Lie}_\xi\, m - \sigma_\xi m$. Then $\xi^\natural \in \mathrm{End}_{\mathcal{D}}\, M$ and $\natural : \mathfrak{k} \to \mathrm{End}_{\mathcal{D}}\, M$ is a $\mathfrak{k}$-action on M. Note that $\natural$ is trivial if and only if M is a K-equivariant $\mathcal{D}$-module, i.e., $M \in \mathcal{M}(\mathcal{B})$.

7.6.11. A *K-equivariant $\mathcal{D}$-complex on Y* is a complex N of weakly K-equivariant $\mathcal{D}$-modules together with morphisms $\mathfrak{k} \otimes N^\cdot \to N^{\cdot -1}$, $\xi \otimes n \mapsto i_\xi n$, such that for any $\xi \in \mathfrak{k}$ our has $i_\xi^2 = 0$, $di_\xi + i_\xi d = \xi^\natural$. By abuse of notation we denote the DG category of such complexes by $C(\mathcal{B},\mathcal{D})$. Note that any K-equivariant $\mathcal{D}$-module is a K-equivariant $\mathcal{D}$-complex in the obvious way, and for any K-equivariant $\mathcal{D}$-complex its cohomology sheaves are K-equivariant $\mathcal{D}$-modules. So we have the cohomology functor $H : C(\mathcal{B},\mathcal{D}) \to \mathcal{M}(\mathcal{B})$. Localizing the homotopy category of $C(\mathcal{B},\mathcal{D})$ by H-quasi-isomorphisms we get a triangulated category $D(\mathcal{B},\mathcal{D})$. It is easy to see that it is a t-category with core $\mathcal{M}(\mathcal{B})$.

For any $F \in C(\mathcal{B},\Omega)$ the $\mathcal{D}$-complex $\mathcal{D}F$ equipped with operators $i_\xi^{\mathcal{D}F} = i_\xi^F \otimes \mathrm{id}_{\mathcal{D}_Y}$ is K-equivariant. For any $N \in C(\mathcal{B},\mathcal{D})$ the Ω-complex ΩN equipped with the operators $i_\xi^{\Omega N}$ which act on $N^i \otimes \Lambda^{-j}\Theta_Y$ as $n \otimes \tau \mapsto i_\xi n \otimes \tau + (-1)^i n \otimes \sigma(\xi) \wedge \tau$ is a K_Ω-equivariant Ω-complex. Thus we have the adjoint functors $\mathcal{D}$, Ω

$$C(\mathcal{B},\Omega) \rightleftarrows C(\mathcal{B},\mathcal{D}) \tag{322}$$

and the mutually inverse equivalences of triangulated categories

$$D(\mathcal{B},\Omega) \rightleftarrows D(\mathcal{B},\mathcal{D})\,. \tag{323}$$

The latter equivalence identifies the above t-structure on $D(\mathcal{B},\mathcal{D})$ with that on $D(\mathcal{B},\Omega)$ defined in 7.3.2. This provides another proof of 7.3.4 in the particular case when our stack is a quotient of a smooth variety by a group action.

7.7. **Harish-Chandra modules and their derived category.**

7.7.1. Let G be an affine algebraic group, $K \subset G$ an algebraic subgroup, so we have the Harish-Chandra pair $(\mathfrak{g}, K)$. Consider the category $\mathcal{M}(K \setminus G\,{}_/{}^/\,G) = \mathcal{M}((K\setminus G)\,{}_/{}^/\,G)$ of $\mathcal{D}$-modules on G equipped with commuting K- and weak G-actions (where K and G act on G by left and right translations respectively). For $M \in \mathcal{M}(K \setminus G\,{}_/{}^/\,G)$ set $\gamma(M) = \gamma^r(M) := \Gamma(G, M_G)^G$; here we consider M_G as a right $\mathcal{D}$-module on G. This is a $(\mathfrak{g}, K)$-module: $\mathfrak{g}$ acts on $\gamma(M)$ by vector fields invariant by right G-translations (according to $\mathcal{D}$-module structure on M), and K acts by left K-translations.

7.7.2. *Lemma.* The functor $\gamma : \mathcal{M}(K \setminus G_{/}{}^{/} G) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ is an equivalence of categories.

Proof. Left to the reader (or see [Kas]). □

7.7.3. *Remarks.* (i) Set $\gamma^l(M) := \Gamma(G, M_G^l)^G$ where M_G^l is the left $\mathcal{D}$-module realization of M. This is a $(\mathfrak{g}, K)$-module by the same reason as above; one has the obvious identification $\gamma^l(M) = \gamma^r(M) \otimes \det \mathfrak{g}$.

(ii) There is a canonical isomorphism of vector spaces $\gamma^l(M) \xrightarrow{\sim} M_{G,1}^l = M_{K\setminus G,1}^l$ which assigns to a G-invariant section its value at $1 \in G$. The $(\mathfrak{g}, K)$-module structure on $M_{K\setminus G,1}^l$ may be described as follows. The K-action comes from the (weak) action of right K-translations on $K \setminus G$ (note that K is the stabilizer of $1 \in K \setminus G$), and the $\mathfrak{g}$-action comes from $\natural$-action of $\mathfrak{g}$ that corresponds to the weak G-action (see 7.6.10).

(iii) Let P be a K-module, and $\mathcal{P}$ the corresponding G-equivariant vector bundle on $K \setminus G$ with fiber $\mathcal{P}_1 = P$. We have $\mathcal{DP} = \mathcal{P} \otimes \mathcal{D}_{K\setminus G} \in \mathcal{M}((K \setminus G)_{/}{}^{/} G)$, and $\gamma(\mathcal{DP}) = U(\mathfrak{g}) \underset{U(\mathfrak{k})}{\otimes} (P \otimes \det \mathfrak{k}^*)$.

7.7.4. The above lemma provides, as was promised in 7.1.1(c), a canonical $\mathcal{H}$-Action on the derived category $D(\mathfrak{g}, K)$ of $(\mathfrak{g}, K)$-modules. Indeed, by 7.6.8(v) (and 7.6.9) we know that $D(K \setminus G_{/}{}^{/} G)$ is an $\mathcal{H}$-Module. And 7.7.2 identifies $D(\mathfrak{g}, K)$ with this category.

We give a different description of this Action in 7.8.2 below. Its equivalence with the present definition is established in 7.8.9, 7.8.10(i).

The rest of the Section (7.7.5-7.7.11) is a digression about $\mathcal{D}$-Ω equivalences in the Harish-Chandra setting; as a bonus we get in 7.7.12 a simple proof of Bernstein-Lunts theorem [BL]1.3. The reader may skip it and go directly to 7.8.

7.7.5. Here is a version of 7.7.2 for Ω-complexes.

Let $\Omega_{\mathfrak{g}}$ be the Chevalley DG-algebra of cochains of $\mathfrak{g}$, so $\Omega_{\mathfrak{g}}^{\cdot} = \Lambda^{\cdot}\mathfrak{g}^*$. It carries a canonical "adjoint" action of K_Ω (see 7.6.3 for notations). Namely,

K acts on $\Omega_{\mathfrak{g}}^{\cdot}$ in coadjoint way, and $\xi \in \mathfrak{k} = \mathfrak{k}_{\Omega}^{-1}$ acts as the derivation i_ξ of $\Omega_{\mathfrak{g}}^{\cdot}$ which sends $\nu \in \mathfrak{g}^* = \Omega_{\mathfrak{g}}^1$ to $\langle \nu, \xi \rangle$.

A $\Omega_{(\mathfrak{g},K)}$-*complex* is a DG $(\Omega_{\mathfrak{g}}, K_\Omega)$-module, i.e., it is a complex equipped with $\Omega_{\mathfrak{g}}$- and K_Ω-actions which are compatible with respect to the K_Ω-action on $\Omega_{\mathfrak{g}}$. For an $\Omega_{(\mathfrak{g},K)}$-complex T we denote the action of $\nu \in \mathfrak{g}^* = \Omega_{\mathfrak{g}}^1$, $\xi \in \mathfrak{k} = \mathfrak{k}_{\Omega}^{-1}$ on $T^{\cdot}$ by a_ν, i_ξ. Denote the DG category of $\Omega_{(\mathfrak{g},K)}$-complexes by $C\Omega_{(\mathfrak{g},K)}$ and its homotopy category by $K\Omega_{(\mathfrak{g},K)}$.

For $F \in C(K \setminus G \,_/{}^/\, G, \Omega)$ set $\gamma(F) := \Gamma(G, F_G)^G$. This is an $\Omega_{(\mathfrak{g},K)}$-complex. Indeed, $\Omega_{\mathfrak{g}}$ acts on it via the usual identification with DG algebra of differential forms on G that are invariant with respect to right G-translations, and K_Ω acts on $\gamma(F)$ since it acts on F_G (see 7.6.4, 7.6.5).

7.7.6. *Lemma.* The functor $\gamma : C(K \setminus G \,_/{}^/\, G, \Omega) \longrightarrow C\Omega_{(\mathfrak{g},K)}$ is an equivalences of DG categories.

Proof. Left to the reader. □

7.7.7. We identified $(\mathfrak{g}, K)$- and $\Omega_{(\mathfrak{g},K)}$-complexes with weakly G-equivariant complexes on $K \setminus G$. Let us write down the standard functors $\mathcal{D}$ and Ω in Harish-Chandra's setting. It is convenient to introduce a DG Harish-Chandra pair $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$ (the structure embedding $Lie K \hookrightarrow \mathfrak{k}_\Omega \times \mathfrak{g}$ is the diagonal map).

Let $DR_{\mathfrak{g}}$ be the Chevalley complex of cochains of $\mathfrak{g}$ with coefficients in $U\mathfrak{g}$ (considered as a left $U\mathfrak{g}$-module), so $DR_{\mathfrak{g}}^i = \Lambda^i \mathfrak{g}^* \otimes U\mathfrak{g}$. Now $DR_{\mathfrak{g}}$ is an $\Omega_{\mathfrak{g}}$-complex, and an $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-complex; those actions are compatible (here $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$ acts on $\Omega_{\mathfrak{g}}$ via the projection $(\mathfrak{k}_\Omega \times \mathfrak{g}, K) \to (\mathfrak{k}_\Omega, K)$, see 7.7.5). Namely, for $\nu \in \Omega_{\mathfrak{g}}$, $\epsilon = (\epsilon_l, \epsilon_r) \in \mathfrak{k} \times \mathfrak{g} = \mathfrak{k}_\Omega^0 \times \mathfrak{g}$, $\xi \in \mathfrak{k} = \mathfrak{k}_\Omega^{-1}$, $k \in K$, and $a = \alpha \otimes \upsilon \in DR_{\mathfrak{g}}$ one has $\nu a = \nu\alpha \otimes \upsilon$, $\epsilon a = \mathrm{Ad}_{\epsilon_l}(\alpha) \otimes \upsilon + \alpha \otimes (\epsilon_l \upsilon - \upsilon \epsilon_r)$, $\xi a = i_\xi(\alpha) \otimes \upsilon$, $ka = \mathrm{Ad}_k(\alpha) \otimes \mathrm{Ad}_k(\upsilon)$.

For a complex of $(\mathfrak{g},K)$-modules ($(\mathfrak{g},K)$-complex for short) V, set $\Omega V := \mathrm{Hom}_{\mathfrak{g}}(DR_{\mathfrak{g}},V)$; this is an $\Omega_{(\mathfrak{g},K)}$-complex in the obvious way. For an $\Omega_{(\mathfrak{g},K)}$-complex T set $\mathcal{D}T=\mathcal{D}_{(\mathfrak{g},K)}T := T \underset{\Omega_{\mathfrak{g}},\mathfrak{k}_\Omega}{\otimes} DR_{\mathfrak{g}} = (T \underset{\Omega_{\mathfrak{g}}}{\otimes} DR_{\mathfrak{g}})_{\mathfrak{k}_\Omega}$; this a $(\mathfrak{g},K)$-complex. Thus we have the adjoint DG functors

$$\mathcal{D}=\mathcal{D}_{(\mathfrak{g},K)} : C\Omega_{(\mathfrak{g},K)} \longrightarrow C(\mathfrak{g},K)\,, \qquad \Omega : C(\mathfrak{g},K) \longrightarrow C\Omega_{(\mathfrak{g},K)}\,. \tag{324}$$

Remark. For T as above let $\overline{T}^{\cdot} \subset T^{\cdot}$ be the kernel of all operators i_ξ, $\xi\in\mathfrak{k}$. This is a K- and $\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k})^*$-submodule of $T^{\cdot}$ (here $\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k})^* \subset \Lambda^{\cdot}\mathfrak{g}^* = \Omega^{\cdot}_{\mathfrak{g}}$), and the obvious morphisms

$$\Omega^{\cdot}_{\mathfrak{g}} \underset{\Lambda^{\cdot}(\mathfrak{g}/\mathfrak{k})^*}{\otimes} \overline{T}^{\cdot} \longrightarrow T^{\cdot}\,, \qquad \overline{T}^{\cdot} \underset{U\mathfrak{k}}{\otimes} U\mathfrak{g} \longrightarrow \mathcal{D}T^{\cdot} \tag{325}$$

are isomorphisms.

7.7.8. Let us return to the geometric situation. One has the obvious identification $\Gamma(G,DR_G)^G = DR_{\mathfrak{g}}$ (see 7.2.2 for notation; G acts on itself by right translations). For $M \in C((K\setminus G)\,{}_{/}{}^{/}\,G,\mathcal{D})$ there is a canonical isomorphism $\gamma(\Omega M) \mathrel{\widetilde{\to}} \Omega(\gamma M)$ of $\Omega_{(\mathfrak{g},K)}$-complexes defined as composition $\Gamma(G, Hom_{\mathcal{D}_G}(DR_G,M_G))^G = \mathrm{Hom}_{\mathcal{D}_G}(DR_G,M_G)^G = \mathrm{Hom}_{U\mathfrak{g}}(DR_{\mathfrak{g}},\gamma M)$. For $F \in C(K\setminus G\,{}_{/}{}^{/}\,G,\Omega)$ there is a similar canonical isomorphism $\gamma\mathcal{D}F \mathrel{\widetilde{\to}} \mathcal{D}\gamma F$ whose definition is left to the reader.

7.7.9. For an $\Omega_{(\mathfrak{g},K)}$-complex T set $H^{\cdot}_{\mathfrak{g}}T = H^{\cdot}\mathcal{D}T \in \mathcal{M}(\mathfrak{g},K)$. Then $H^{\cdot}_{\mathfrak{g}} : K\Omega_{(\mathfrak{g},K)} \to \mathcal{M}(\mathfrak{g},K)$ is a cohomological functor. Define a $\mathfrak{g}$-*quasi-isomorphism* as a morphism in $K\Omega_{(\mathfrak{g},K)}$ that induces isomorphism between $H^{\cdot}_{\mathfrak{g}}$'s. The $\mathfrak{g}$-quasi-isomorphisms form a localizing family; define $D\Omega_{(\mathfrak{g},K)}$ as the corresponding localization of $K\Omega_{(\mathfrak{g},K)}$. The functors $\mathcal{D}$, Ω yield mutually inverse equivalences of derived categories

$$D\Omega_{(\mathfrak{g},K)} \Longleftrightarrow D(\mathfrak{g},K) \tag{326}$$

where $D(\mathfrak{g},K) := D\mathcal{M}(\mathfrak{g},K)$. The equivalences γ yield equivalences of derived categories

$$D(K\setminus G\,{}_{/}{}^{/}\,G,\Omega) \mathrel{\widetilde{\to}} D\Omega_{(\mathfrak{g},K)}\,, \quad D((K\setminus G)\,{}_{/}{}^{/}\,G,\mathcal{D}) \mathrel{\widetilde{\to}} D(\mathfrak{g},K)\,. \tag{327}$$

7.7.10. *Remarks.* (i) Any $\mathfrak{g}$-quasi-isomorphism is a quasi-isomorphism; the converse might be not true.

(ii) Any $\Omega_{(\mathfrak{g},K)}$-complex T may be considered as an $\Omega_{\mathfrak{g}} = \Omega_{(\mathfrak{g},1)}$-complex (forget the K_Ω-action), so we have the corresponding complex of $\mathfrak{g}$-modules $\mathcal{D}_{\mathfrak{g}}T := T \otimes_{\Omega_{\mathfrak{g}}} DR_{\mathfrak{g}}$. The obvious projection $\mathcal{D}_{\mathfrak{g}}T \to \mathcal{D}_{(\mathfrak{g},K)}T$ is a quasi-isomorphism. This implies that a morphism of $\Omega_{(\mathfrak{g},K)}$-complexes is a $\mathfrak{g}$-quasi-isomorphism if and only if it is a $\mathfrak{g}$-quasi-isomorphism of $\Omega_{\mathfrak{g}}$-complexes.

7.7.11. The format of 7.7.7, 7.7.9 admits the folowing version. Recall that $DR_{\mathfrak{g}}$ is a $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-complex. Thus the above $\mathcal{D}_{\mathfrak{g}}T$ is a $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-complex, and for a $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-complex V the complex $\Omega V := Hom_{\mathfrak{g}}(DR_{\mathfrak{g}}, V)$ is a $\Omega_{(\mathfrak{g},K)}$-complex. The functors

$$\mathcal{D}_{\mathfrak{g}} : C\Omega_{(\mathfrak{g},K)} \longrightarrow C(\mathfrak{k}_\Omega \times \mathfrak{g}, K), \quad \Omega : C(\mathfrak{k}_\Omega \times \mathfrak{g}, K) \longrightarrow C\Omega_{(\mathfrak{g},K)} \tag{328}$$

are adjoint, as well as the corresponding functors between the homotopy categories. Passing to derived categories they become (use 7.7.10(ii)) mutually inverse equivalences

$$D\Omega_{(\mathfrak{g},K)} \rightleftarrows D(\mathfrak{k}_\Omega \times \mathfrak{g}, K). \tag{329}$$

The projection $(\mathfrak{k}_\Omega \times \mathfrak{g}, K) \to (\mathfrak{g}, K)$ yields a fully faithful embedding $C(\mathfrak{g}, K) \longrightarrow C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$ hence the exact functor

$$D(\mathfrak{g}, K) \longrightarrow D(\mathfrak{k}_\Omega \times \mathfrak{g}, K). \tag{330}$$

The following theorem is due to Bernstein and Lunts [BL] 1.3*):

7.7.12. *Theorem.* The functor (330) is equivalence of categories.

Proof. The functor Ω from (328) restricted to $C(\mathfrak{g}, K)$ coincides with Ω from (324). Now 7.7.12 follows from (326) and (329). The inverse functor $D(\mathfrak{k}_\Omega \times \mathfrak{g}, K) \longrightarrow D(\mathfrak{g}, K)$ sends V to $\mathcal{D}_{(\mathfrak{g},K)}\Omega V$. □

7.8. **The Hecke Action and localization functor.**

*) The authors of [BL] consider only bounded derived categories.

7.8.1. We are going to describe a canonical Hecke Action on the derived category of Harish-Chandra modules. We consider a twisted situation, i.e., representations of a central extension of $\mathfrak{g}$. Here is the list of characters.

Let G' be a central extension of G by $\mathbb{G}_m$ equipped with a splitting $K \to G'$. Therefore the preimage $K' \subset G'$ of K is identified with $K \times \mathbb{G}_m$. Set $\mathfrak{g}' := \operatorname{Lie} G'$, $\mathfrak{k}' := \operatorname{Lie} K' = \mathfrak{k} \times \mathbb{C}$. We have a Harish-Chandra pair $(\mathfrak{g}', K')$ and the companion DG pair $(\mathfrak{k}_\Omega \times \mathfrak{g}', K')$ (here the first component of the structure embedding $\mathfrak{k}' \hookrightarrow \mathfrak{k}_\Omega \times \mathfrak{g}'$ is the projection $\mathfrak{k}' \to \mathfrak{k}$).

Let $\mathcal{M}(\mathfrak{g}, K)'$ be the category of $(\mathfrak{g}', K')$-modules on which $\mathbb{G}_m \subset K'$ acts by the standard character; we call its objects $(\mathfrak{g}, K)'$-modules or, simply, Harish-Chandra modules. This is an abelian category. Similarly, let $C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$ be the category of those $(\mathfrak{k}_\Omega \times \mathfrak{g}', K')$-complexes on which $\mathbb{G}_m$ acts by the standard character; its objects are called $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$-complexes or, simply, *Harish-Chandra complexes*. This is a DG category which carries an obvious cohomology functor with values in $\mathcal{M}(\mathfrak{g}, K)'$. Denote the corresponding derived category by $D(\mathfrak{g}, K)'$; this is a t-category with core $\mathcal{M}(\mathfrak{g}, K)'$.

Remark. By a twisted version of the Bernstein-Lunts theorem $D(\mathfrak{g}, K)'$ is equivalent to the derived category of $\mathcal{M}(\mathfrak{g}, K)'$ *). We will not use this fact in the sequel since the Hecke Action is naturally defined in terms of $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$-complexes.

7.8.2. Now let us define a canonical $\mathcal{H}$-Action on $D(\mathfrak{g}, K)'$. First we define an Action of the pre Hecke monoidal DG category $\mathcal{H}^c := C(K \setminus G/K, \Omega)$ on $C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$; the Hecke Action comes after passing to derived categories.

Denote by $\mathcal{L}_G$ the line bundle over G that corresponds to the $\mathbb{G}_m$-torsor $G' \to G$. The left and right translation actions of G on itself lift canonically to G'-actions on $\mathcal{L}_G$. So a section of $\mathcal{L}_G$ is the same as a function ϕ on G' such that for $c \in \mathbb{G}_m$, $g' \in G'$ one has $\phi(cg') = c^{-1}\phi(g')$. Therefore the

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*) The twisted Bernstein-Lunts follows from the straight one (see 7.7.12) applied to the Harish-Chandra pair $(\mathfrak{g}', K')$.

right translation action of $\mathbb{G}_m \subset G'$ on sections of $\mathcal{L}_G$ is multiplication by the character inverse to the standard one.

Take a Harish-Chandra complex $V \in C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$. Set $\mathcal{V}_G := \mathcal{L}_G \otimes V$. Then $\mathcal{V}_G$ is a complex of left $\mathcal{D}$-modules on G. Indeed, the tensor product of the infinitesimal right translation action of $\mathfrak{g}'$ on $\mathcal{L}_G$ and the $\mathfrak{g}'$-action on V is a $\mathfrak{g}$-action on $\mathcal{V}_G$. The left $\mathcal{D}$-module structure on $\mathcal{V}_G$ is such that the left invariant vector fields act on $\mathcal{V}_G$ via the above $\mathfrak{g}$-action. The $\mathcal{D}$-complex $\mathcal{V}_G$ is weakly equivariant with respect to left G'-translations: they act as tensor product of the corresponding action on $\mathcal{L}_G$ and the trivial action on $\mathcal{V}$. Therefore, by 7.6.10, it carries a canonical $\mathfrak{g}'$-action $\natural$.

Remark. For $\theta \in \mathfrak{g}'$ consider a function $\theta^\natural : G \to \mathfrak{g}'$, $\theta^\natural(g) := Ad_g(\theta)$. Then for $v \in V$, $l \in \mathcal{L}_G$ one has $\theta^\natural(l \otimes v) = l \otimes \theta^\natural(v)$.

Take $F \in \mathcal{H}^c$. Then $F_G \otimes \mathcal{V}_G$ is an Ω-complex on G (see 7.2.3(ii)). It is K_Ω-equivariant with respect to the right K-translations. Namely, K acts as tensor product of the corresponding actions on F, $\mathcal{L}_G$, and the structure action on V; the operators i_ξ act as the sum of the corresponding operators for the right translation action on F and the structure ones for V. Denote by $(F \otimes \mathcal{V})_{G/K}$ the corresponding Ω-complex on G/K. The action of $\mathfrak{g}'$ on $F_G \otimes \mathcal{V}_G$ that comes from the action $\natural$ on $\mathcal{V}_G$ commutes with this K_Ω-action, so it defines $\mathfrak{g}'$-action on $(F \otimes \mathcal{V})_{G/K}$. We also denote it as $\natural$.

Remark. If V is a complex of $(\mathfrak{g}, K)'$-modules then $\mathcal{V}_G$ is a complex of left $\mathcal{D}_G$-modules strongly equivariant with respect to right K-translations. Let $\mathcal{V}_{G/K}$ be the corresponding complex of left $\mathcal{D}$-modules on G/K. One has $(F \otimes \mathcal{V})_{G/K} = F_{G/K} \otimes \mathcal{V}_{G/K}$.

Set $F \widetilde{\circledast} V := \Gamma(G, F_G \otimes \mathcal{V}_G)$ and

$$F \overset{c}{\circledast} V = \Gamma(G/K, (F \otimes \mathcal{V})_{G/K}) = (F \widetilde{\circledast} V)^{K_\Omega}. \tag{331}$$

These are $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$-complexes. Indeed, $\mathfrak{g}'$ acts according to $\natural$ action, K acts by tensor product of the left translation actions for F and $\mathcal{V}$, and the operators i_ξ are the corresponding operators for F. We leave it to the reader to check the Harish-Chandra compatibilities.

Now $\overset{c}{\circledast}$ defines an $\mathcal{H}^c$-Module structure on $C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$. Indeed, the associativity constraint $(F_1 \overset{c}{\circledast} F_2) \overset{c}{\circledast} V = F_1 \overset{c}{\circledast} (F_2 \overset{c}{\circledast} V)$ follows from the obvious identification

$$\Gamma(G, (F_1 \overset{c}{\circledast} F_2)^{\cdot} \otimes \mathcal{L}_G) = [\Gamma(G, F_1^{\cdot} \otimes \mathcal{L}_G) \otimes \Gamma(G, F_2^{\cdot} \otimes \mathcal{L}_G)]^{K_\Omega^{\cdot}}$$

where $K_\Omega^{\cdot}$ acts by tensor product of the right and left translation actions (see 7.6.5). We define the Hecke Action $\circledast : \mathcal{H} \times D(\mathfrak{g}, K)' \to D(\mathfrak{g}, K)'$ as the right derived functor of $\overset{c}{\circledast}$. If F is loose then $F \circledast V = F \overset{c}{\circledast} V$ so the associativity constraint for $\circledast$ follows from that of $\overset{c}{\circledast}$.

Remark. As folows from the previous Remark, for $M \in \mathcal{M}(K \setminus G/K) \subset \mathcal{H}$, $V \in \mathcal{M}(\mathfrak{g}, K)'$ one has

$$H^{\cdot} M \circledast V = H^{\cdot}_{DR}(G/K, M \otimes \mathcal{V}_{G/K}). \tag{332}$$

7.8.3. *Remark.* Assume that our twist is trivial, so $G' = G \times \mathbb{G}_m$. One has obvious equivalences $\mathcal{M}(\mathfrak{g}, K)' = \mathcal{M}(\mathfrak{g}, K)$ and $D(\mathfrak{g}, K) = D(\mathfrak{g}, K)'$ (see 7.7.11). So we defined a Hecke Action on $D(\mathfrak{g}, K)$. We will see in 7.8.9 that this Action indeed coincides with the one from 7.7.4.

Let us return to the general situation. Let U' be the twisted enveloping algebra of $\mathfrak{g}$; denote by $\mathfrak{Z}$ its subalgebra of $\mathrm{Ad}\, G$-invariant elements. The commutative algebra $\mathfrak{Z}$ acts on any Harish-Chandra complex in the obvious manner, so $C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$, hence $D(\mathfrak{g}, K)$, is a $\mathfrak{Z}$-category.

7.8.4. *Lemma.* The Hecke Actions on $C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$, $D(\mathfrak{g}, K)'$ are $\mathfrak{Z}$-linear.

Proof. Use the first Remark in 7.8.2. □

7.8.5. *Example.* (to be used in 5). Let $Vac' := U'/U' \cdot \mathfrak{k}$ be the twisted vacuum module. Let us compute $F \circledast Vac'$ explicitely. We use notation of 7.8.2. So, according to the second Remark in 7.8.2, we have the left $\mathcal{D}$-module $\mathcal{V}_{G/K}$ on G/K, weakly equivariant with respect to left G-translations, such that $\mathcal{V}_G = \mathcal{L}_G \otimes Vac'$. The embedding $\mathbb{C} \subset Vac'$ yields an embedding $\mathcal{L}_{G/K} \subset \mathcal{V}_{G/K}$. It is easy to see that the corresponding morphism of left $\mathcal{D}_{G/K}$-modules $\mathcal{D}_{G/K} \underset{\mathcal{O}_{G/K}}{\otimes} \mathcal{L}_{G/K} \to \mathcal{V}_{G/K}$ is an isomorphism of weakly G-equivariant $\mathcal{D}$-modules.

Remark. The $\mathfrak{g}'$-action on $\mathcal{D}_{G/K} \otimes \mathcal{L}_{G/K}$ that corresponds to $\natural$ is given by formula $\alpha'(\psi \otimes l) = \psi \otimes \alpha'(l) - \psi \cdot \alpha \otimes l$ where $\alpha' \in \mathfrak{g}'$, α is the corresponding left translation vector field on G/K, and $\alpha'(l)$ is the infinitesimal left translation of $l \in \mathcal{L}_{G/K}$.

So for $F \in \mathcal{H}^c$ one has $(F \otimes \mathcal{V})_{G/K} = F_{G/K} \otimes \mathcal{D}_{G/K} \otimes \mathcal{L}_{G/K} = \mathcal{D}(F_{G/K}) \underset{\mathcal{O}_{G/K}}{\otimes} \mathcal{L}_{G/K}$. Therefore

$$F \overset{c}{\circledast} Vac' = \Gamma(G/K, \mathcal{D}(F_{G/K}) \otimes \mathcal{L}_{G/K}). \tag{333}$$

Here the $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$-action on $\Gamma(G/K, \mathcal{D}(F_{G/K}) \otimes \mathcal{L}_{G/K})$ is defined as follows. The $\mathfrak{g}'$-action comes from the $\mathfrak{g}'$-action on $\mathcal{D}(F_{G/K}) \otimes \mathcal{L}_{G/K}$ described in the Remark above, the K-action is the action by left translations, and the operators i_ξ come from the corresponding operators on $F_{G/K}$.

Passing to the derived functors (which amounts to considering loose F in the above formula) we get

$$F \circledast Vac' = R\Gamma(G/K, \mathcal{D}(F_{G/K}) \otimes \mathcal{L}_{G/K}). \tag{334}$$

In particular, for $M \in \mathcal{M}(K \setminus G/K)$ one has

$$M \circledast Vac' = R\Gamma(G/K, M_{G/K} \otimes \mathcal{L}_{G/K}). \tag{335}$$

Here the $\mathfrak{g}'$-action on the r.h.s. comes from the $\mathfrak{g}'$-action on $M_{G/K} \otimes \mathcal{L}_{G/K}$ given by formula $\alpha'(m \otimes l) = m \otimes \alpha'(l) - m\alpha \otimes l$.

7.8.6. Let us explain part (d) of the "Hecke pattern" from 7.1.1. Let us first define the localization functor Δ. We use the notation of 7.8.1. Let Y be a smooth variety on which G acts, $\mathcal{L} = \mathcal{L}_Y$ a line bundle on Y. Assume that $\mathcal{L}$ carries a G'-action which lifts the G-action on Y in a way that $\mathbb{G}_m \subset G'$

acts by the character opposite to the standard one. The line bundle $\omega_Y \otimes \mathcal{L}$ carries the similar action.

We define a DG functor

$$\Delta_\Omega = \Delta_{\Omega\mathcal{L}} : C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)' \to C(K \setminus Y, \Omega) \tag{336}$$

as follows. Note that $(\mathfrak{g}', K')$, hence $(\mathfrak{k}_\Omega \times \mathfrak{g}', K')$, acts on $\omega_Y \otimes \mathcal{L}$ (since G' does). For a Harish-Chandra complex V consider the complex of $\mathcal{O}$-modules $\omega_Y \otimes \mathcal{L} \otimes V$. The tensor product of $(\mathfrak{k}_\Omega \times \mathfrak{g}', K')$-actions on $\omega_Y \otimes \mathcal{L}$ and V yields a $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-action on $\omega_Y \otimes \mathcal{L} \otimes V$. Set

$$\Delta_\Omega(V) := Hom_{\mathfrak{g}}(DR_{\mathfrak{g}}, \omega_Y \otimes \mathcal{L} \otimes V)[-\dim K]$$

(see 7.7.7 for notation). In other words $\Delta_\Omega(V)$ is the shifted Chevalley chain complex of $\mathfrak{g}$ with coefficients in $\omega_Y \otimes \mathcal{L} \otimes V$. This is an Ω-complex on Y. Since $DR_{\mathfrak{g}}$ and $\omega_Y \otimes \mathcal{L} \otimes V$ are $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-complexes our $\Delta_\Omega(V)$ is K_Ω-equivariant, i.e., $\Delta_\Omega(V) \in C(K \setminus Y, \Omega)$.

Note that $\Delta_\Omega(V)$ carries a canonical increasing finite filtration with successive quotients equal to $\Lambda^i \mathfrak{g} \otimes \omega_Y \otimes \mathcal{L} \otimes V[i - \dim K]$. Therefore Δ_Ω sends quasi-isomorphisms to $\mathcal{D}$-quasi-isomorphisms. So it yields a triangulated functor

$$L\Delta = L\Delta_{\mathcal{L}} : D(\mathfrak{g}, K)' \to D(K \setminus Y) \tag{337}$$

The above remark also shows that $L\Delta$ is a right t-exact functor. The corresponding right exact functor between the cores $\Delta_{\mathcal{L}} : \mathcal{M}(\mathfrak{g}, K)' \to \mathcal{M}^\ell(K \setminus Y)$ sends a $(\mathfrak{g}, K)'$-module V to a K-equivariant left $\mathcal{D}_Y$-module $(\mathcal{D}_Y \otimes \mathcal{L}) \underset{U(\mathfrak{g}')}{\otimes} V$. More generally, $H^i_{\mathcal{D}} L\Delta_{\mathcal{L}}(V) = H_{-i}(\mathfrak{g}, \mathcal{D}_Y \otimes \mathcal{L} \otimes V)$.

7.8.7. *Remarks.* (i) The above construcion used only the action of $(\mathfrak{g}', K')$ on $(Y, \mathcal{L})$ (we do not need the whole G'-action).

(ii) One may show that $L\Delta_{\mathcal{L}}$ is a left derived functor of $\Delta_{\mathcal{L}}$ (see Remark in 7.8.1).

(iii) Assume that $(\mathfrak{g}', K')$ is the trivial extension of $(\mathfrak{g}, K)$, so $(\mathfrak{g}, K)'$-modules are the same as $(\mathfrak{g}, K)$-modules, and $\mathcal{L}$ is $\mathcal{O}_Y$ with the obvious action of $(\mathfrak{g}', K')$. Then $\Delta_{\mathcal{L}}(V) = \mathcal{D}_Y \underset{U(\mathfrak{g})}{\otimes} V$, i.e., $\Delta_{\mathcal{L}}$ coincides with the functor Δ from 1.2.4.

7.8.8. *Proposition.* The functor $L\Delta_{\mathcal{L}} : D(\mathfrak{g}, K)' \to D(K \setminus Y)$ is a Morphism of $\mathcal{H}$-Modules.

Proof. It suffices to show that the functor $\Delta_{\Omega\mathcal{L}} : C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)' \to C(K\setminus Y, \Omega)$ is a Morphism of $\mathcal{H}^c$-Modules.

Take F, V as in 7.8.2. We have to define a canonical identification of Ω-complexes $\alpha : \Delta_\Omega(F \overset{c}{\circledast} V) \xrightarrow{\sim} F \overset{c}{\circledast} \Delta_\Omega(V)$ compatible with the associativity constraints. We will establish a canonical isomorphism $\widetilde{\alpha}$: $\Delta_\Omega(F \widetilde{\circledast} V) \xrightarrow{\sim} F \widetilde{\circledast} \Delta_\Omega(V)$ compatible with the K_Ω-actions (see 7.6.6, 7.8.2 for notation). One gets α by passing to K_Ω-invariants.

Let $m, p : G \times Y \to Y$ be the action and projection maps, $i : G \times Y \to G \times Y$ the symmetry $i(g, x) = (g, gx)$; one has $pi = m$. The G'-action on $\mathcal{L}_Y$ provides an i-isomorphism of line bundles $\widetilde{i} : \mathcal{O}_G \boxtimes \mathcal{L}_Y \xrightarrow{\sim} \mathcal{L}_G \boxtimes \mathcal{L}_Y$.

Below for a $\mathfrak{g}$-complex P we denote by $C(P)$ the Chevalley complex of Lie algebra chains with coefficients in P shifted by $\dim K$. So $C(P)^{\cdot} = C^{\cdot} \otimes P^{\cdot}$ where $C^a := \Lambda^{\dim K - a}\mathfrak{g}$. Consider the Ω-complexes $F_G \boxtimes \Delta_\Omega(V) = F_G \boxtimes C(\omega_Y \otimes \mathcal{L}_Y \otimes V)$ and $C((F_G \otimes \mathcal{V}_G) \boxtimes (\omega_Y \otimes \mathcal{L}_Y)) = C((F_G \otimes (\mathcal{L}_G \otimes V)) \boxtimes (\omega_Y \otimes \mathcal{L}_Y))$; here the $\mathfrak{g}$-action on $(F_G \otimes \mathcal{V}_G) \boxtimes (\omega_Y \otimes \mathcal{L}_Y)$ is the tensor product of the $\mathfrak{g}'$-action $\natural$ and the standard $\mathfrak{g}'$-action on $\omega_Y \otimes \mathcal{L}_Y$ (see 7.8.2).

There is a canonical i-isomorphism of Ω-complexes

$$\widetilde{\alpha}' : F_G \boxtimes \Delta_\Omega(V) \xrightarrow{\sim} C((F_G \otimes \mathcal{V}_G) \boxtimes (\omega_Y \otimes \mathcal{L}_Y))$$

defined as follows. For $f \in F_G$, $\lambda \in C^{\cdot}$, $l \in \omega_Y \otimes \mathcal{L}_Y$, $v \in V$ one has $\widetilde{\alpha}'(f \otimes \lambda \otimes l \otimes v) = a(\lambda) \otimes f \otimes \widetilde{i}(l) \otimes v$; here $a(\lambda) \in \mathcal{O}_{G \times Y} \otimes C^{\cdot}$ is a function $a(\lambda)(g, y) = Ad_g(\lambda)$. We leave it to the reader to check that α commutes with the differentials (use Remark in 7.8.2).

Now one has the obvious identifications $m_{\cdot}(F_G \boxtimes \Delta_\Omega(V)) = F \widetilde{\circledast} \Delta_\Omega(V)$ and $p_{\cdot} C((F_G \otimes \mathcal{V}_G) \boxtimes (\omega_Y \otimes \mathcal{L}_Y)) = \Delta_\Omega(F \widetilde{\circledast} V)$. Thus $\widetilde{\alpha}'$ defines the desired canonical isomorphism $\widetilde{\alpha}$. We leave it to the reader to check its compatibility with the K_Ω-actions and associativity constraints. $\square$

7.8.9. Consider the case when $Y = G$ with the left translation G-action, and $\mathcal{L} = \mathcal{L}_Y$ is the line bundle dual to $\mathcal{L}_G$ (see 7.8.2) equipped with the obvious G'-action by left translations. The right G'-translations act on our data. Therefore the Ω-complexes $\Delta_\Omega(V)$ are weakly G'-equivariant with respect to the right translation action of G'.

Let $C(K \setminus G\,{}_{/}{}^{/}\,G, \Omega)' \subset C(K \setminus G\,{}_{/}{}^{/}\,G', \Omega)$ be the subcategory of those weakly G'-equivariant Ω-complexes T that $\mathbb{G}_m \subset G'$ acts on T by the standard character. Let $D(K \setminus G\,{}_{/}{}^{/}\,G)'$ be the corresponding $\mathcal{D}$-derived category. The complexes $\Delta_\Omega(V)$ lie in this subcategory, so we have a triangulated functor $L\Delta : D(\mathfrak{g}, K)' \to D(K \setminus G\,{}_{/}{}^{/}\,G)'$. This categories are $\mathcal{H}$-Modules (for the latter one see 7.6.8(v), 7.6.9). By 7.8.8, $L\Delta$ is a Morphism of $\mathcal{H}$-modules. A variant of 7.7.6 and 7.7.11 shows that $L\Delta$ is an equivalence of t-categories.

7.8.10. *Remarks.* i) If G' is the trivial extension of G then $D(\mathfrak{g}, K)' = D(\mathfrak{g}, K)$ and $L\Delta$ coincides with the equivalence defined by the functor γ^{-1} from 7.7.2. This shows that the Hecke Actions from 7.7.4 and in 7.8.3 do coincide.

(ii) Assume that our extension is arbitrary. Then the pull-back functor $r : D(K \setminus G/K) \to D(K' \setminus G'/K')$ is a Morphism of monoidal categories, and the fully faithful embedding $D(\mathfrak{g}, K)' \hookrightarrow D(\mathfrak{g}', K')$ is r-Morphism of Hecke Modules. So the twisted picture is essentially equivalent to untwisted one for $(\mathfrak{g}', K')$. However in applications it is convenient to keep the twist (alias level, alias central charge) separately.

7.8.11. Let us explain the Γ part of the "Hecke pattern" (d) from 7.1.1. This subject is not needed for the main part of this paper, so the reader may skip the rest of the section. We treat a twisted version, so we are in situation 7.8.6. For $T \in C(K \setminus Y, \Omega)$ the $\mathcal{D}$-complex $\mathcal{D}T_Y$ on Y is K-equivariant (see

7.6.11). Let us consider $\mathcal{D}T_Y$ as an $\mathcal{O}$-complex equipped with a $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)$-action. Set $\Gamma_{\mathcal{L}}(T) := \Gamma(Y, \mathcal{D}T_Y \otimes (\omega_Y \otimes \mathcal{L}_Y)^*)$. This is a Harish-Chandra complex (recall that $(\mathfrak{g}', K)$ acts on $\omega_Y \otimes \mathcal{L}_Y$), so we have a DG functor $\Gamma_{\mathcal{L}} :\, C(K \setminus Y, \Omega) \to C(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$. Let

$$R\Gamma_{\mathcal{L}} :\, D(K \setminus Y) \to D(\mathfrak{g}, K)'$$

be its right derived functor. If T is loose then $\Gamma_{\mathcal{L}}(T) = R\Gamma_{\mathcal{L}}(T)$, so $R\Gamma_{\mathcal{L}}$ is correctly defined.

Note that $R\Gamma_{\mathcal{L}}$ is a left t-exact functor; let $\Gamma_{\mathcal{L}} : \mathcal{M}(K \setminus Y) \to \mathcal{M}(\mathfrak{g}, K)'$ be the corresponding left exact functor. One has $\Gamma_{\mathcal{L}}(M) = \Gamma(Y, M \otimes (\omega_Y \otimes \mathcal{L}_Y)^*)$. If we are in situation 7.8.7(iii) then this functor coincides, after the standard identification of right and left $\mathcal{D}$-modules, with the functor Γ from 1.2.4.

7.8.12. *Lemma.* The functor $R\Gamma_{\mathcal{L}}$ is a Morphism of $\mathcal{H}$-Modules.

Proof. It suffices to show that $\Gamma_{\mathcal{L}}$ is a Morphism of $\mathcal{H}^c$-Modules, i.e., to define for $F \in \mathcal{H}^c$, T as above a canonical isomorphism $\beta : \Gamma_{\mathcal{L}}(F \overset{c}{\circledast} T) \simeq F \overset{c}{\circledast} \Gamma_{\mathcal{L}}(T)$ compatible with the associativity constraints. Let us write down a canonical isomorphism $\widetilde{\beta} : \Gamma_{\mathcal{L}}(F \widetilde{\circledast} T) \simeq F \widetilde{\circledast} \Gamma_{\mathcal{L}}(T)$ compatible with the K_Ω-actions; one gets β by passing to K_Ω-invariants.

The G'-action on $\mathcal{L}$ yields an isomorphism $m_Y^*((\omega_Y \otimes \mathcal{L}_Y)^*) = \mathcal{L}_G \boxtimes (\omega_Y \otimes \mathcal{L}_Y)^*$, and the G-action on $\mathcal{D}_Y$ (as on a left $\mathcal{O}_Y$-module yields an isomorphism $m_Y^*(\mathcal{D}_Y) = \mathcal{O}_G \boxtimes \mathcal{D}_Y$. These isomorphisms identify $\Gamma_{\mathcal{L}}(F \widetilde{\circledast} T)^{\cdot}$ with $\Gamma(G \times Y, (F^{\cdot} \otimes \mathcal{L}_G) \boxtimes (\mathcal{D}T_Y^{\cdot} \otimes (\omega_Y \otimes \mathcal{L}_Y)^*))$. This vector space coincides with $\Gamma(G, F^{\cdot} \otimes \mathcal{L}_G) \otimes \Gamma(Y, \mathcal{D}T_Y^{\cdot} \otimes (\omega_Y \otimes \mathcal{L}_Y)^*)$ which is $(F \widetilde{\circledast} \Gamma_{\mathcal{L}}(T))^{\cdot}$. Our $\widetilde{\beta}$ is composition of these identifications. We leave it to the reader to check that this is an isomorphism of Harish-Chandra complexes compatible with the K_Ω-actions. $\square$

7.9. **Extra symmetries and parameters.**

7.9.1. In the main body of this paper (namely, in 5.4) we use an equivariant version of the Hecke pattern from 7.1.1. Namely, we are given an extra Harish-Chandra pair $(\mathfrak{l}, P)$ that acts on (G, K), and we are looking for an $(\mathfrak{l}, P)$-equivariant version of 7.1.1(a)-(d). Let us explain very briefly the setting; for all the details see the rest of this section. The Hecke category $\mathcal{H}$ is a derived version of the category of *weakly* $(\mathfrak{l}, P)$-equivariant $\mathcal{D}$-modules on $K\backslash G/K$. This is a monoidal triangulated category (which is the analog of 7.1.1(a) in the present setting). $\mathcal{H}$ acts on the appropriate derived category D_{HC} of $(\mathfrak{l} \ltimes \mathfrak{g}, P \ltimes K)$-modules; this is the Harish-Chandra counterpart similar to 7.1.1(c). The geometric counterpart looks as follows. Let X be a "parameter" space equipped with an $(\mathfrak{l}, P)$-structure $X^\wedge$ (see 2.6.4). We consider a family $Y^\wedge$ of smooth varieties with G-action parametrized by $X^\wedge$. We assume that the $(\mathfrak{l}, P)$-action on $X^\wedge$ is lifted to $Y^\wedge$ in a way compatible with the G-action. Then $\mathcal{H}$ acts on the $\mathcal{D}$-module derived category $D(\mathcal{B})$ of the X-stack $\mathcal{B} = (P \ltimes K) \setminus Y^\wedge$ (which is the version of 7.1.1(b)). We have an appropriate localization functor $L\Delta : D_{HC} \to D(\mathcal{B})$ which commutes with the Hecke Actions (this is 7.1.1(d)). For an algebra A with an $(\mathfrak{l}, P)$-action one has an A-linear version of the above constructions: one looks at Harish-Chandra modules with A-action and $\mathcal{D}$-modules with A_X-action (see 2.6.6 for the definition of A_X). The corresponding triangulated categories are denoted by $\mathcal{H}_A$, $D_{HC\,A}$, and $D(\mathcal{B}, A_X)$.

The constructions are essentially straightforward modifications of constructions from the previous sections; we write them down for the sake of direct reference in 5.4.

Remark. The equivariant Hecke pattern does not reduce to the plain one with G replaced by the group ind-scheme that corresponds to the Harish-Chandra pair $(\mathfrak{l} \ltimes \mathfrak{g}, P \ltimes G)$. Indeed, our $\mathcal{H}$ is much larger then the corresponding "plain" Hecke category: the latter is formed by *strongly* P-equivariant $\mathcal{D}$-modules on $K\backslash G/K$. In particular, $\mathcal{H}$ contains as a tensor subcategory the tensor category of $(\mathfrak{l}, P)$-modules. The above structure of fibration Y/X is needed to make the whole $\mathcal{H}$ act on $D(\mathcal{B})$.

7.9.2. So we consider a Harish-Chandra pair $(\mathfrak{l}, P)$ that acts on (G, K). Here P could be any affine group scheme (it need not be of finite type), but we assume that $\operatorname{Lie} P$ has finite codimension in $\mathfrak{l}$. Consider the DG category $\mathcal{H}^c$ of Ω-complexes F on $K \setminus G/K$ equipped with an $(\mathfrak{l}, P)$-action on F that lifts the $(\mathfrak{l}, P)$-action on G/K. Such F is the same as an $(\mathfrak{l}, P) \ltimes (K_\Omega \times K_\Omega)$-equivariant Ω-complex on G. We call $\mathcal{H}^c$ the $(\mathfrak{l}, P)$-*equivariant pre Hecke category.* The morphisms in the homotopy category of $\mathcal{H}^c$ which are $\mathcal{D}$-quasi-isomorphisms of plain Ω-complexes form a localizing family. The $(\mathfrak{l}, P)$-*equivariant Hecke category* $\mathcal{H}$ is the corresponding localization. So $\mathcal{H}$ is a t-category with core equal to the category of $\mathcal{D}$-modules on G/K equipped with a weak $(\mathfrak{l} \ltimes \mathfrak{k}, P \ltimes K)$-action (here K acts on G/K by left translations) such that the action of K is actually a strong one.

Now $\mathcal{H}^c$ is a DG monoidal category, and $\mathcal{H}$ is a monoidal triangulated category. Indeed, all the definitions from 7.6.1 work in the present situation.

Remark. Take a Harish-Chandra module $V \in \mathcal{M}(\mathfrak{l}, P)$. Assign to it the corresponding skyscraper sheaf at the distinguished point of G/K considered as an Ω-complex sitting in degree zero and equipped with the trivial K_Ω-action. This is an object of $\mathcal{H}^c$. The functors $\mathcal{M}(\mathfrak{l}, P) \to \mathcal{H}^c, \mathcal{H}$ are fully faithful monoidal functors. Note that $\mathcal{M}(\mathfrak{l}, P)$ belongs in a canonical way to the center of the (pre)Hecke monoidal category, i.e., for any V as above, $F \in \mathcal{H}$ there is a canonical isomorphism $V \circledast F \rightleftarrows F \circledast V$ compatible with tensor products of F's and V's. Indeed, both objects coincide with $V \otimes F$.

7.9.3. To define the Hecke Action on $\mathcal{D}$-modules we need to fix some preliminaries.

Let X be a smooth variety, Y be a $\mathcal{D}_X$-scheme. A $\mathcal{D}_X\Omega_{/X}$-*complex on* Y is a DG $\Omega_{Y/X}$-module equipped with a $\mathcal{D}_X$-structure (:= flat connection along the leaves of the structure connection on Y/X). Precisely, the $\mathcal{D}_X$-structure on Y defines on $\Omega_{Y/X}(\mathcal{D}_X) := \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \Omega_{Y/X}$ the structure of an associative DG algebra. Now a $\mathcal{D}_X\Omega_{/X}$-complex on Y is a left DG $\Omega_{Y/X}(\mathcal{D}_X)$-module which is quasi-coherent as an $\mathcal{O}_Y$-module.

The DG category $C(Y, \mathcal{D}_X\Omega_{/X})$ of $\mathcal{D}_X\Omega_{/X}$-complexes on Y is a tensor category (the tensor product is taken over $\Omega_{Y/X}$). The pull-back functor $C(\mathcal{M}^\ell(X)) \longrightarrow C(Y, \mathcal{D}_X\Omega_{/X})$, $M \to \Omega_{Y/X} \underset{\mathcal{O}_X}{\otimes} \Omega_{Y/X}$, is a tensor functor. In particular $C(Y, \mathcal{D}_X\Omega_{/X})$ is an $\mathcal{M}^\ell(X)$-Module (one has $M \circledast F = M \underset{\mathcal{O}_X}{\otimes} F$).

Note that for a $\mathcal{D}_X\Omega_{/X}$-complex F on Y we have an absolute Ω-complex $\Omega_X F$ defined as de Rham complex along X with coefficient in F*). So if Y is a smooth variety then we have a notion of $\mathcal{D}$-quasi-isomorphism of $\mathcal{D}_X\Omega_{/X}$-complexes. The corresponding localization of the homotopy category of $C(Y, \mathcal{D}_X\Omega_{/X})$ is denoted $D(Y, \mathcal{D}_X\Omega_{/X})$. The functor Ω_X : $D(Y, \mathcal{D}_X\Omega_{/X}) \longrightarrow D(Y, \Omega)$ is an equivalence of triangulated categories.

7.9.4. Now let X be a smooth variety equipped with a $(\mathfrak{l}, P)$-structure $X^\wedge$ (see 2.6.4). Let $Y^\wedge$ be a scheme equipped with an action of $(\mathfrak{l}, P) \ltimes G$ and a smooth morphism $p^\wedge : Y^\wedge \to X^\wedge$ compatible with the actions (so G acts along the fibers and $p^\wedge$ commutes with the actions of $(\mathfrak{l}, P)$). Set $Y := P \setminus Y^\wedge$. This is a smooth variety equipped with a smooth projection $p : Y \to X$. The $(\mathfrak{l}, P)$-action on $Y^\wedge$ defines a structure of $\mathcal{D}_X$-scheme on Y. The G-action on $Y^\wedge$ yields a horisontal G_X-action on Y (the group $\mathcal{D}_X$-scheme G_X was defined in 2.6.6).

Consider the stack $\mathcal{B} := K_X \setminus Y = (P \ltimes K) \setminus Y^\wedge$ fibered over X so we have the corresponding category of left $\mathcal{D}$-modules $\mathcal{M}^\ell(\mathcal{B})$ and the t-category $D(\mathcal{B})$ of Ω-complexes on $\mathcal{B}$. This t-category has a different realization in terms of $\mathcal{D}_X\Omega_{/X}$-complexes that we are going to describe.

Consider the DG group $\mathcal{D}_X$-schemes $G_{\Omega X} := (G_X, \Omega_{G_X/X})$, $K_{\Omega X}$. One defines a $K_{\Omega X}$-action on a $\mathcal{D}_X\Omega_{/X}$-complex on Y as in 7.6.4. Now we have the DG category $C(K_X \setminus Y, \mathcal{D}_X\Omega_{/X})$ of $K_{\Omega X}$-equivariant $\mathcal{D}_X\Omega_{/X}$-complexes

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*) As in 7.2 the functor Ω_X admits left adjoint functor $\mathcal{D}_X$.

on Y. Localizing its homotopy category by $\mathcal{D}$-quasi-isomorphisms we get the triangulated category $D(K_X \setminus Y, \mathcal{D}_X\Omega_{/X})$. The de Rham functor Ω_X identifies it with $D(\mathcal{B})$.

Now we can define the Hecke Action on $D(\mathcal{B})$. First let us construct the Action $\overset{c}{\circledast}$ of $\mathcal{H}^c$ on $C(K_X \setminus Y, \mathcal{D}_X\Omega_{/X})$. Indeed, for $F \in \mathcal{H}^c$ we have a $\mathcal{D}_X\Omega_{/X}$-complex F_X on G_X which is $K_{\Omega X}$-equivariant with respect to the left and right translations. So for $T \in C(K_X \setminus Y, \mathcal{D}_X\Omega_{/X})$ we have a $\mathcal{D}_X\Omega_{/X}$-complex $F \boxtimes T$ on the $\mathcal{D}_X$-scheme $G_X \times Y$ (the fiber product of G_X and Y over X). It is $K_{\Omega X}$-equivariant with respect to all the K_X-actions on $G_X \times Y$. So $F \boxtimes T$ descents to $G_X \underset{K_X}{\times} Y$. We define $F \overset{c}{\circledast} T \in C(K_X \setminus Y, \mathcal{D}_X\Omega_{/X})$ as the push-forward of the above complex by the action map $G_X \underset{K_X}{\times} Y \to Y$. The Hecke Action $\circledast : \mathcal{H} \times D(\mathcal{B}) \longrightarrow D(\mathcal{B})$ is the right derived functor of $\overset{c}{\circledast}$; as usually you may compute it using loose $\mathcal{D}_X\Omega_{/X}$-complexes.

Remark. For $W \in \mathcal{M}(\mathfrak{l}, P) \subset \mathcal{H}^c$ and T as above one has $W \overset{c}{\circledast} T = W \circledast T = W_X \otimes T$ (the $\mathcal{D}_X$-module W_X was defined in 2.6.6).

7.9.5. Let us define the Harish-Chandra categories. Let G' be as in 7.8.1 and assume that we are given a lifting of the $(\mathfrak{l}, P)$-action on G to that on G' which preserves $K \subset G'$ and fixes $\mathbb{G}_m \subset G'$. So we have the Harish-Chandra pair $(\mathfrak{l}, P) \ltimes (\mathfrak{g}', K')$. Let C_{HC} be the category of $(\mathfrak{l}, P) \ltimes (\mathfrak{k}_\Omega \ltimes \mathfrak{g}, K)'$-complexes, i.e., $(\mathfrak{k}_\Omega \times \mathfrak{g}, K)'$-complexes equipped with a compatible $(\mathfrak{l}, P)$-action (see 7.8.1 for notation). Let D_{HC} be the corresponding derived category. This is a t-category with core $\mathcal{M}_{HC} = \mathcal{M}(\mathfrak{l} \ltimes \mathfrak{g}, P \ltimes K)'$. Below we call the objects of C_{HC} and D_{HC} simply *Harish-Chandra complexes* and those of $\mathcal{M}_{HC}$ *Harish-Chandra modules.*

The pre Hecke category $\mathcal{H}^c$ acts on C_{HC}. Indeed, the constructions of 7.8.2 make perfect sense in our situation ($(\mathfrak{l}, P)$ acts on $F \overset{c}{\circledast} V$ by transport of structure). The $\mathcal{H}$-Action $\circledast$ on D_{HC} is the right derived functor of $\overset{c}{\circledast}$. The results of 7.8.4-7.8.5 render to the present setting without changes.

Remark. For $W \in \mathcal{M}(\mathfrak{l}, P) \subset \mathcal{H}^c$ and a Harich-Chandra complex V one has a canonical isomorphism of Harish-Chandra complexes $W \overset{c}{\circledast} V = W \circledast V = W \otimes V$.

7.9.6. Let us pass to the localization functor. The construction of 7.8.6 renders to our setting as follows. We start with $Y^\wedge$ as in 7.9.4. Assume that it carries a line bundle $\mathcal{L}_{Y^\wedge}$ and the $(\mathfrak{l}, P) \ltimes G$-action on $Y^\wedge$ is lifted to an action of $(\mathfrak{l}, P) \ltimes G'$ on $\mathcal{L}_{Y^\wedge}$ such that $\mathbb{G}_m \subset G'$ acts by the character opposite to the standard one. Let $\mathcal{L}_Y$ be the descent of $\mathcal{L}_{Y^\wedge}$ to Y defined by the action of P. This line bundle carries a canonical $\mathcal{D}_X$-structure that comes from the $\mathfrak{l}$-action on $\mathcal{L}_{Y^\wedge}$. It also carries a horisontal action of G'_X.

We have a DG functor

$$\Delta_\Omega = \Delta_{\Omega\mathcal{L}} : \ C_{HC} \longrightarrow C(K_X \setminus Y, \mathcal{D}_X \Omega_{/X}), \tag{338}$$

$\Delta_\Omega(V) = \mathrm{Hom}_{\mathfrak{g}_X}(DR_{\mathfrak{g}_X}, \omega_{Y/X} \times \mathcal{L}_Y \times V)[-\dim K]$ (cf. (336)). As in 7.8.6 this functor sends quasi-isomorphisms to $\mathcal{D}$-quasi-isomorphisms, so it yields a triangulated functor

$$L\Delta = L\Delta_{\mathcal{L}} : \ D_{HC} \longrightarrow D(\mathcal{B}) \tag{339}$$

which is right t-exact. The corresponding right exact functor between the cores $\Delta_{\mathcal{L}} : \mathcal{M}_{HC} \longrightarrow \mathcal{M}^\ell(\mathcal{B})$ sends V to the K_X-equivariant left $\mathcal{D}_Y$-module $(\mathcal{D}_{Y/X} \otimes \mathcal{L}_Y) \underset{U(\mathfrak{g}'_X)}{\otimes} V_X$.

The functors $\Delta_\Omega, L\Delta$ commute with the Hecke Action. Indeed, the proof of 7.8.8 renders to our setting word-by-word. In particular for any $W \in \mathcal{M}(\mathfrak{l}, P), V \in D_{HC}$ one has $L\Delta(W \otimes V) = W_X \otimes L\Delta(V)$.

7.9.7. *A-linear version.* Assume that in addition we are given a commutative algebra A equipped with an $(\mathfrak{l}, P)$-action. One attaches it to the above pattern as follows.

(i) Denote by $\mathcal{H}^c_A$ the DG category of objects $F \in \mathcal{H}^c$ equipped with an action of A such that the actions of A and $(\mathfrak{l}, P)$ are compatible and F is A-flat. Let $\mathcal{H}_A$ be the corresponding $\mathcal{D}$-derived category. One defines the convolution product as in 7.9.2 (the tensor product is taken over A) so $\mathcal{H}^c_A$ and $\mathcal{H}_A$ are monoidal categories. Let $\mathcal{M}(\mathfrak{l},P)^{fl}_A$ be the tensor category of flat A-modules equipped with an action of $(\mathfrak{l},P)$. As in the Remark in 7.9.2 one has canonical fully faithful monoidal functors $\mathcal{M}(\mathfrak{l},P)^{fl}_A \longrightarrow \mathcal{H}^c_A, \mathcal{H}_A$ which send $\mathcal{M}(\mathfrak{l},P)^{fl}_A$ to the center of Hecke categories.

(ii) Assume we are in situation 7.9.4. Consider the category $\mathcal{M}^\ell(\mathcal{B},A_X)$ of left $\mathcal{D}$-modules on $\mathcal{B}$ equipped with A_X-action (the $\mathcal{D}_X$-algebra A_X was defined in 2.6.6). Let $C(\mathcal{B},A_X\otimes\Omega)$ be the DG category of Ω-complexes on $\mathcal{B}$ equipped with an A_X-action and $D(\mathcal{B},A_X)$ be the localization of the corresponding homotopy category with respect to $\mathcal{D}$-quasi-isomorphisms. This is a t-category with core $\mathcal{M}^\ell(\mathcal{B},A_X)$. As in 7.9.4 one may also define this t-category in terms of $\mathcal{D}_X\Omega_{/X}$-complexes. Namely, let $C(K_X\setminus Y, A_X\mathcal{D}_X\Omega_{/X})$ be the DG category of objects of $C(K_X\setminus Y,\mathcal{D}_X\Omega_{/X})$ equipped with an A_X-action (commuting with the $K_{\Omega X}$-action). Localizing it by $\mathcal{D}$-quasi-isomorphisms we get the triangulated category $D(K_X\setminus Y, A_X\mathcal{D}_X\Omega_{/X})$. The de Rham functor Ω_X identifies it with $D(\mathcal{B},A_X)$.

The Hecke Action in the A-linear setting is defined exactly as in 7.9.4. The statement of the Remark in 7.9.4 remains true (you take the tensor product over A_X).

(iii) Assume we are in situation 7.9.5. One defines $C_{HC\,A}$ as the category of Harish-Chandra complexes equipped with a compatible A-action (so the actions of A and $(\mathfrak{k}_\Omega\times\mathfrak{g},K)'$ commute). Let $D_{HC\,A}$ be the corresponding derived category. This is a t-category with core $\mathcal{M}_{HC\,A}$ equal to the category of $(\mathfrak{l}\ltimes\mathfrak{g}, P\ltimes K)'$-modules equipped with a compatible A-action. All the constructions and results about the Hecke Action remain valid without changes. In the Remark in 7.9.5 you take $W\in\mathcal{M}(\mathfrak{l},P)^{fl}_A$; the tensor product $W\otimes V$ is taken over A. The A-linear setting for the localization functors requires no changes.

Remark. There are obvious functors (tensoring by A) which send the plain categories as above to those with A attached. These functors are compatible with all the structures we considered. The forgetting of the A-action functors $D(\mathcal{B}, A_X) \to D(\mathcal{B})$, $D_{HC\,A} \to D_{HC}$ are Morphisms of $\mathcal{H}$-Modules. They commute with the localization functors.

7.9.8. *Variant.* Assume that in addition to A we are given a morphism of commutative algebras $e : \mathfrak{Z} \to A$ compatible with the $(\mathfrak{l}, P)$-actions. Here $\mathfrak{Z} := U(\mathfrak{g})'^{\,\mathrm{Ad}\,G}$ (so if G is connected then $\mathfrak{Z}$ is the center of $U(\mathfrak{g})'$). Then $\mathfrak{Z}$ acts on any object of $\mathcal{M}_{HC\,A}$ or $C_{HC\,A}$ in two ways. Denote by $\mathcal{M}^e_{HC\,A}, C^e_{HC\,A}$ the categories of those objects on which the two actions of $\mathfrak{Z}$ coincide; let D^e_{HC} be the corresponding derived category. The Action of $\mathcal{H}^c_A$ on $C_{HC\,A}$ is $\mathfrak{Z}$-linear (see 7.8.4) so it preserves $C^e_{HC\,A}$. Thus we have an Action of $\mathcal{H}_A$ on $D^e_{HC\,A}$. The obvious functor $D^e_{HC\,A} \to D_{HC\,A}$ is a Morphism of $\mathcal{H}_A$-Modules.

Remark. If e is surjective then $\mathcal{M}^e_{HC\,A}$ is the full subcategory of $\mathcal{M}_{HC}$ that consists of Harish-Chandra modules killed by $\operatorname{Ker} e$. Same for $C^e_{HC\,A}$.

7.10. **$\mathcal{D}$-crystals.** Below we sketch a crystalline approach to $\mathcal{D}$-module theory. As opposed to the conventional formalism it makes no distinction between smooth and non-smooth schemes.

In this section "scheme" means "$\mathbb{C}$-scheme locally of finite type". Same for algebraic spaces and stacks. The formal schemes or algebraic spaces are assumed to be locally of ind-finite type*).

7.10.1. Let $f : Y \to X$ be a quasi-finite morphism of schemes. Then Grothendieck's functor $Rf^! : D^b(X, \mathcal{O}) \to D^b(Y, \mathcal{O})$ is left t-exact. Set $f^! := H^0 Rf^! : \mathcal{M}(X, \mathcal{O}) \to \mathcal{M}(Y, \mathcal{O})$; this is a left exact functor. Therefore the categories $\mathcal{M}(X, \mathcal{O})$ together with functors $f^!$ form a fibered category over the category of schemes and quasi-finite morphisms.

Here is an explicit description of $f^!$. According to Zariski's Main Theorem any quasi-finite morphism is composition of a finite morphism and an open embedding; let us describe $f^!$ in these two cases. If f is an open embedding

*):= any closed subscheme is of finite type.

(or, more generally, if f is étale) then $f^! = f^*$. If f is finite then $f^!$ is the functor right adjoint to the functor $f_* : \mathcal{M}(Y, \mathcal{O}) \to \mathcal{M}(X, \mathcal{O})$. Explicitely, $f_*\mathcal{O}_Y$ is a finite $\mathcal{O}_X$-algebra, and the functor f_* identifies $\mathcal{M}(Y, \mathcal{O})$ with the category of $f_*\mathcal{O}_Y$-modules which are quasi-coherent as $\mathcal{O}_X$-modules. Now for an $\mathcal{O}$-module M on X the corresponding $f_*\mathcal{O}_Y$-module $f_*f^!M$ is $\mathcal{H}om_{\mathcal{O}_X}(f_*\mathcal{O}_Y, M)$. In particular, if f is a closed embedding then $f^!M \subset M$ is the submodule of sections supported (scheme-theoretically) on Y.

The above picture extends to the setting of formal schemes (or algebraic spaces) as follows. For a formal scheme $\hat{X}$ we denote by $\mathcal{M}(\hat{X}, \mathcal{O})$ the category of discrete quasi-coherent $\mathcal{O}_{\hat{X}}$-modules$^{*)}$. For example, if $\hat{X}$ is the formal completion of a scheme V along its closed subscheme X then $\mathcal{M}(\hat{X}, \mathcal{O})$ coinsides with the category of $\mathcal{O}$-modules on V supported set-theoretically on X. If $\hat{X}$ is affine then for any $M \in \mathcal{M}(\hat{X}, \mathcal{O})$ one has $M = \bigcup M_{X'}$ where X' runs the (directed) set of closed subschemes of $\hat{X}$ and $M_{X'} \in \mathcal{M}(X', \mathcal{O})$ is the submodule of sections supported scheme-theoretically on X'. The pull-back functors $f^!$ extend in a unique manner$^{*)}$ to the setting of quasi-finite morphisms of formal algebraic spaces. Indeed, if $\hat{f} : \hat{Y} \to \hat{X}$ is such a morphism then to define $\hat{f}^! : \mathcal{M}(\hat{X}, \mathcal{O}) \to \mathcal{M}(\hat{Y}, \mathcal{O})$ we may assume that $\hat{X}$, $\hat{Y}$ are affine; now $\hat{f}^! M = \bigcup \hat{f}|^!_{Y'} M_{X'}$ where Y' is a closed subscheme of $\hat{Y}$ and $\hat{f}(Y') \subset X'$. We leave it to the reader to describe $\hat{f}^!$ explicitely if $\hat{f}$ is ind-finite$^{*)}$.

7.10.2. For a scheme or an algebraic space X denote by X_{cr} the category of diagrams $X \xleftarrow{j} S \xhookrightarrow{i} \hat{S}$ where j is a quasi-finite morphism and i a closed embedding of affine schemes such that the corresponding ideal $\mathcal{I} \subset \mathcal{O}_{\hat{S}}$ is nilpotent. We usually write this object of X_{cr} as $(S, \hat{S})$ or simply $\hat{S}$. A morphism $(S, \hat{S}) \to (S', \hat{S}')$ in X_{cr} is a morphism of schemes $\phi : \hat{S} \to \hat{S}'$ such that $\phi(S) \subset S'$ and $\phi|_S : S \to S'$ is a morphism of X-schemes.

$^{*)}$This category is abelian. For a more general setting see 7.11.4.

$^{*)}$We assume that they are compatible with composition of f's.

$^{*)}$$:= Y_{\mathrm{red}} \to X_{\mathrm{red}}$ is finite.

Note that for any ϕ as above the morphism $\phi : \hat{S} \to \hat{S}'$ is quasi-finite. Therefore the categories $\mathcal{M}(\hat{S}, \mathcal{O})$ together with the pull-back functors $\phi^!$ form a fibered category $\mathcal{M}^!(X_{cr}, \mathcal{O})$ over X_{cr}.

Sometimes it is convenient to consider a larger category $X_{\hat{cr}}$ which consists of similar diagrams as above but we permit $\hat{S}$ to be a formal scheme (so $\mathcal{I}$ is a pronilpotent ideal, i.e., $\hat{S}_{\mathrm{red}} = S_{\mathrm{red}}$). As above we have the fibered category $\mathcal{M}^!(X_{\hat{cr}}, \mathcal{O})$ over $X_{\hat{cr}}$.

7.10.3. *Definition.* A *$\mathcal{D}$-crystal on X* is a Cartesian section of $\mathcal{M}^!(X_{cr}, \mathcal{O})$. $\mathcal{D}$-crystals on X form a $\mathbb{C}$-category $\mathcal{M}_{\mathcal{D}}(X)$.

Explicitely, a $\mathcal{D}$-crystal M is a rule that assigns to any $(S, \hat{S}) \in X_{cr}$ an $\mathcal{O}$-module $M_{\hat{S}} = M_{(S,\hat{S})}$ on $\hat{S}$ and to a morphism $\phi : (S, \hat{S}) \to (S', \hat{S}')$ an identification $\alpha_\phi : M_{\hat{S}} \stackrel{\sim}{\rightarrow} \phi^! M_{\hat{S}'}$ compatible with composition of ϕ's.

In particular, if ϕ is a closed embedding defined by an ideal $\mathcal{I} \subset \mathcal{O}_{\hat{S}'}$ then $M_{\hat{S}}$ is the submodule of $M_{\hat{S}'}$ that consists of sections killed by $\mathcal{I}$.

In definition 7.10.3 one may replace X_{cr} by $X_{\hat{cr}}$: we get the same category of $\mathcal{D}$-crystals. Indeed, for $(S, \hat{S}) \in X_{\hat{cr}}$ one has $M_{\hat{S}} = \bigcup M_{(S,\hat{S}')}$ where $\hat{S}'$ runs the set of all subschemes $S \subset \hat{S}' \subset \hat{S}$.

7.10.4. *Variants.* Let $X_{cr}^{(i)}, .., X_{cr}^{(iv)}$ be the full subcategories of X_{cr} that consist of objects $(S, \hat{S})$ which satisfy, respectively, one of the following conditions (in (ii)-(iv) we assume that X is a scheme):

(i) $S \to X$ is étale.

(ii) $S \to X$ is an open embedding.

(iii) (assuming that X is affine) $S \stackrel{\sim}{\rightarrow} X$.

(iv) $S \to X$ is a locally closed embedding.

Denote by $\mathcal{M}_{\mathcal{D}}^{(i)}(X), .., \mathcal{M}_{\mathcal{D}}^{(iv)}(X)$ the categories of Cartesian sections of $\mathcal{M}^!(X_{cr}, \mathcal{O})$ over the corresponding subcategories $X_{cr}^{(a)}$. One has the obvious restriction functors $\mathcal{M}_{\mathcal{D}}(X) \to \mathcal{M}_{\mathcal{D}}^{(a)}(X)$. We leave it to the reader to check that these functors are equivalences of categories*).

*) It suffices to notice that 7.10.6, 7.10.7, 7.10.8 together with the proofs remain literally valid if we replace $\mathcal{M}_{\mathcal{D}}(X)$ by $\mathcal{M}_{\mathcal{D}}^{(a)}(X)$.

Remark. The category $X_{cr}^{(ii)}$ is (the underlying category of) Grothendieck's crystalline site of X, so $\mathcal{D}$-crystals are the same as crystals for the fibered category $\mathcal{M}^!(X_{cr}^{(ii)}, \mathcal{O})$ in Grothendieck's terminology. We consider X_{cr} as the basic set-up since it directly generalizes to the setting of ind-schemes (see 7.11.6).

7.10.5. Let $f : Y \to X$ be a quasi-finite morphism. It yields a faithful functor $Y_{cr} \to X_{cr}$ which sends $Y \xleftarrow{j} S \hookrightarrow \hat{S}$ to $Y \xleftarrow{fj} S \hookrightarrow \hat{S}$. We get the corresponding "restriction" functor $f^! : \mathcal{M}_{\mathcal{D}}(X) \to \mathcal{M}_{\mathcal{D}}(Y)$. It is compatible with composition of f's.

In particular, categories $\mathcal{M}_{\mathcal{D}}(U)$, where U is étale over X, form a fibered category over the small étale site $X_{ét}$ which we denote by $\mathcal{M}_{\mathcal{D}}(X_{ét})$.

7.10.6. *Lemma.* $\mathcal{D}$-crystals are local objects for the étale topology, i.e., $\mathcal{M}_{\mathcal{D}}(X_{ét})$ is a sheaf of categories. $\square$

7.10.7. Below we give a convenient "concrete" description of $\mathcal{D}$-crystals.

Assume we have a closed embedding $X \hookrightarrow V$ where V is a formally smooth*) formal algebraic space such that $X_{\mathrm{red}} = V_{\mathrm{red}}$*). Such thing always exists if X is affine: one may embed X into a smooth scheme W and take for V the formal completion of W along X.

For $n \geq 1$ let $V^{<n>}$ denotes the formal completion of V^n along the diagonal $V \subset V^n$ (or, equivalently, along $X \subset V^n$). The projections $p_1, p_2 : V^{<2>} \to V$, $p_{12}, p_{23}, p_{13} : V^{<3>} \to V^{<2>}$ are ind-finite, so we have the functors $p_i^! : \mathcal{M}(V, \mathcal{O}) \to \mathcal{M}(V^{<2>}, \mathcal{O})$, $p_{ij}^! : \mathcal{M}(V^{<2>}, \mathcal{O}) \to \mathcal{M}(V^{<3>}, \mathcal{O})$. Since V is formally smooth these functors are exact.

Denote by $\mathcal{M}_{\mathcal{D}V}(X)$ the category of pairs (M_V, τ) where $M_V \in \mathcal{M}(V, \mathcal{O})$ and $\tau : p_1^! M_V \xrightarrow{\sim} p_2^! M_V$ is an isomorphism such that

(340) $$p_{23}^!(\tau) p_{12}^!(\tau) = p_{13}^!(\tau).$$

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*)see 7.11.1.

*)i.e., the ideal of X in $\mathcal{O}_V$ is pronilpotent.

7.10.8. *Proposition.* The categories $\mathcal{M}_{\mathcal{D}}(X)$ and $\mathcal{M}_{\mathcal{D}V}(X)$ are canonically equivalent.

Proof. We deal with local objects, so we may assume that X is affine. For $M \in \mathcal{M}_{\mathcal{D}}(X)$ we have $M_V = M_{(X,V)} \in \mathcal{M}(V,\mathcal{O})$. Since $p_i^! M_V = M_{V^{<2>}}$ we have τ that obviously satisfies (340). Conversely, assume we have $(M_V,\tau) \in \mathcal{M}_{\mathcal{D}V}(X)$; let us define the corresponding $\mathcal{D}$-crystal M. For $(S,\hat{S}) \in X_{cr}$ choose $j' : \hat{S} \to V$ that extends the structure morphism $j : S \to X$ (such j' exists since V is formally smooth). Consider the $\mathcal{O}_{\hat{S}}$-module $j'^! M_V$. If $j'' : \hat{S} \to V$ is another extension of j then there is a canonical isomorphism $\nu_{j'j''} : j'^! M_V \xrightarrow{\sim} j''^! M_V$. Namely, (j',j'') maps $\hat{S}$ to $V^{<2>}$, hence $j'^! M_V = (j',j'')^! p_1^! M_V$; now use the similar description of $j''^! M_V$ and set $\nu_{j'j''} := (j',j'')^!(\tau)$. By (340) these identifications are transitive, so $j'^! M_V$ does not depend on the choice of j'. This is $M_{(S,\hat{S})}$. The definition of structure isomorphisms α_ϕ for M is clear. $\square$

7.10.9. *Corollary.* (i) For any X the category $\mathcal{M}_{\mathcal{D}}(X)$ is abelian.

(ii) For $\hat{S} \in X_{cr}$ the functor $\mathcal{M}_{\mathcal{D}}(X) \to \mathcal{M}(\hat{S},\mathcal{O})$, $M \mapsto M_{\hat{S}}$ is left exact.

(iii) For a quasi-finite $j : Y \to X$ the functor $j^! : \mathcal{M}_{\mathcal{D}}(X) \to \mathcal{M}_{\mathcal{D}}(Y)$ is left exact. If j is étale then $j^!$ is exact.

Proof. The statement (i) is true if X is affine. Indeed, choose $X \hookrightarrow V$ as in 7.10.7. The category $\mathcal{M}_{\mathcal{D}V}(X)$ is abelian since the functors $p_i^!$, $p_{ij}^!$ are exact, so we are done by 7.10.8.

If $j : U \to X$ is an étale morphism of affine schemes then the functor $j^! : \mathcal{M}_{\mathcal{D}}(X) \to \mathcal{M}_{\mathcal{D}}(U)$ is exact. Indeed, let $U \hookrightarrow V_U$ be the U-localization of $X \hookrightarrow V$ (so V_U is étale over V); then $j^!$ coincides with the étale localization functor $\mathcal{M}_{\mathcal{D}V}(X) \to \mathcal{M}_{\mathcal{D}V_U}(U)$ which is obviously exact.

Now (i) follows from 7.10.6. The rest is left to the reader. $\square$

7.10.10. *Lemma.* For an étale morphism $p : U \to X$ the functor $p^!$ admits a right adjoint functor $p_* : \mathcal{M}_{\mathcal{D}}(U) \to \mathcal{M}_{\mathcal{D}}(X)$. If p is an open embedding then $p^! p_*$ is identity functor.

Proof. Here is an explicit construction of p_*. For $(S,\hat S)\in X_{cr}$ set $S_U := S\underset{X}{\times}U$; let $\hat p_S : \hat S_U \to \hat S$ be the étale morphism whose pull-back to $S\hookrightarrow \hat S$ is the projection $S_U\to S$. So $(S_U,\hat S_U)\in U_{cr}$, and we have the functor $X_{cr}\to U_{cr}$, $(S,\hat S)\mapsto (S_U,\hat S_U)$.

Now for $N\in \mathcal{M}_{\mathcal{D}}(U)$ set $(p_*N)_{\hat S} := (\hat p_S)_\cdot N_{\hat S_U}$. The identifications α_ϕ come from the base change isomorphism $\phi^!\hat p_{S'\cdot} = \hat p_{S\cdot}\phi_U^!$. □

Now let $i : Y\hookrightarrow X$ be a closed embedding and $j : U := X\setminus Y\hookrightarrow X$ the complementary open embedding. Denote by $\mathcal{M}_{\mathcal{D}}(X)_Y$ the full subcategory of $\mathcal{M}_{\mathcal{D}}(X)$ that consists of those $\mathcal{D}$-crystals M that $j^!M=0$.

7.10.11. *Lemma.* (i) The functor $i^!$ admits a left adjoint functor $i_* : \mathcal{M}_{\mathcal{D}}(Y)\to\mathcal{M}_{\mathcal{D}}(X)$.

(ii) i_* sends $\mathcal{M}_{\mathcal{D}}(Y)$ to $\mathcal{M}_{\mathcal{D}}(X)_Y$ and

$$i_* : \mathcal{M}_{\mathcal{D}}(Y)\to\mathcal{M}_{\mathcal{D}}(X)_Y,\quad i^! : \mathcal{M}_{\mathcal{D}}(X)_Y\to\mathcal{M}_{\mathcal{D}}(Y)$$

are mutually inverse equivalences of categories.

(iii) Let $p : Z\to X$ be a quasi-finite morphism; set $Y_Z := Y\underset{Z}{\times}X$, so we have $i_Z : Y_Z\hookrightarrow Z$ and $p_Y : Y_Z\to Y$. Then one has a canonical identification of functors $p^!i_* = i_{Y*}p_Y^! : \mathcal{M}_{\mathcal{D}}(Y)\to\mathcal{M}_{\mathcal{D}}(Z)$.

Proof. Here is an explicit construction of i_*. Take a $\mathcal{D}$-crystal N on Y. For $(S,\hat S)\in X_{cr}$ set $S_Y := S\underset{X}{\times}Y$, so S_Y is a closed subscheme of S, hence of $\hat S$. The projection $S_Y\to Y$ is quasi-finite, so N yields a $\mathcal{D}$-crystal on S_Y. We define $(i_*N)_{(S,\hat S)}$ as the corresponding $\mathcal{O}$-module on $\hat S$ (see 7.10.3). The structure isomorphisms α_ϕ for i_*N come from the corresponding isomorphisms for N in the obvious manner.

The adjunction property of i_*, as well as properties (ii), (iii), are clear. □

7.10.12. *Proposition.* If X is smooth then $\mathcal{M}_{\mathcal{D}}(X)$ is canonically equivalent to the category $\mathcal{M}(X)$ of $\mathcal{D}$-modules on X.

Proof. We use description 7.10.7 of $\mathcal{M}_{\mathcal{D}}(X)$ for $V=X$. So a $\mathcal{D}$-crystal M amounts to a pair (M_X,τ) where $M_X\in\mathcal{M}(X,\mathcal{O})$ and $\tau : p_1^!M_X \mathrel{\widetilde{\rightarrow}} p_2^!M_X$ is an isomorphism of $\mathcal{O}$-modules on $X^{<2>}$ which satisfies (340). Let us show that such τ is the same as a right $\mathcal{D}$-module structure on M_X.

Consider $\mathcal{D}_X$ as an object of $\mathcal{M}(X^{<2>}.\mathcal{O})$ (via the $\mathcal{O}_X$-bimodule structure). There is a canonical isomorphism $\mathcal{D}_X \overset{\sim}{\rightarrow} p_1^! \mathcal{O}_X$ which identifies $\partial \in \mathcal{D}_X$ with the section $(f \otimes g \mapsto f\partial(g)) \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_{X^{<2>}}, \mathcal{O}_X) = p_1^! \mathcal{O}_X$. Therefore we have $M_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \overset{\sim}{\rightarrow} M_X \underset{\mathcal{O}_X}{\otimes} p_1^! \mathcal{O}_X \overset{\sim}{\rightarrow} p_1^! M_X$. Hence, by adjunction,

(341)
$$\operatorname{Hom}(p_1^! M_X, p_2^! M_X) = \operatorname{Hom}(p_{2\cdot} p_1^! M_X, M_X) = \operatorname{Hom}(M_X \otimes \mathcal{D}_X, M_X).$$

Here we consider $M_X \otimes \mathcal{D}_X$ as an $\mathcal{O}_X$-module via the right $\mathcal{O}$-module structure on $\mathcal{D}_X$. So $\tau : p_1^! M_X \rightarrow p_2^! M_X$ is the same as a morphism $M_X \otimes \mathcal{D}_X \rightarrow M_X$. One checks that the conditions on τ just mean that this arrow is a right unital action of $\mathcal{D}_X$ on M_X. See the next Remark for a comment and some details. $\square$

7.10.13. *Remark.* Let us discuss certain points of 7.10.12 in a more general setting. Since $\mathcal{O}_{X^{<2>}}$ is a completion of $\mathcal{O}_X \underset{\mathbb{C}}{\otimes} \mathcal{O}_X$ one may consider objects of $\mathcal{M}(X^{<2>}, \mathcal{O})$ as certain sheaves of $\mathcal{O}_X$-bimodules called Diff-*bimodules* on X*). If A, B are Diff-bimodules then such is $A \underset{\mathcal{O}_X}{\otimes} B$ (so $\mathcal{M}(X^{<2>}, \mathcal{O})$ is a monoidal category). Notice that $A \underset{\mathcal{O}_X}{\otimes} B$ is actually an object of $\mathcal{M}(X^{<3>}, \mathcal{O})$ in the obvious way. By adjunction, for any $C \in \mathcal{M}(X^{<2>}, \mathcal{O})$ a morphism of Diff-bimodules $A \underset{\mathcal{O}_X}{\otimes} B \rightarrow C$ is the same as a morphism $A \underset{\mathcal{O}_X}{\otimes} B \rightarrow p_{13}^! C$ in $\mathcal{M}(X^{<3>}, \mathcal{O})$. Thus for a Diff-algebra*) A its product amounts to a morphism $m : A \underset{\mathcal{O}_X}{\otimes} A \rightarrow p_{13}^! A$ in $\mathcal{M}(X^{<3>}, \mathcal{O})$ (we leave it to the reader to write associativity property in these terms). Similarly, for a (right) A-module M_X we may write the A-action as a morphism $a : M_X \underset{\mathcal{O}_X}{\otimes} A \rightarrow p_2^! M_X$ in $\mathcal{M}(X^{<2>}, \mathcal{O})$; the action (associativity) property

*) In [BB93] the term "differential bimodule" was used; we refer there for the details.

*) i.e., an algebra in the monoidal category of Diff-bimodules.

just says that the two morphisms $M_X \underset{\mathcal{O}_X}{\otimes} A \underset{\mathcal{O}_X}{\otimes} A \to p_3^! M$ in $\mathcal{M}(X^{<3>}, \mathcal{O})$ obtained from m and a coincide. Assume now that $A = \mathcal{D}_X$ or, more generally, A is a tdo. Then $m : A \underset{\mathcal{O}_X}{\otimes} A \to p_{13}^! A$ is an isomorphism*). If M_X is a (possibly, non-unital) A-module then $a : M_X \underset{\mathcal{O}_X}{\otimes} A \to p_2^! M_X$ is an isomorphism if and only if our module is unital.

7.10.14. We leave it to the reader to identify (in the smooth setting) the functors $f^!$, p_*, i_* from, respectively, 7.10.5, 7.10.10, and 7.10.11(i), with the standard $\mathcal{D}$-module functors.

Combining 7.10.12 and 7.10.11(ii) we see that if X is any algebraic space then $\mathcal{D}$-crystals on X are the same as $\mathcal{D}$-modules on X in the sense of [Sa91]*).

7.10.15. The rest of the section is a sketch of crystalline setting for tdo and twisted $\mathcal{D}$-modules. First we discuss crystalline $\mathcal{O}^*$-gerbes. In case of a smooth scheme such gerbe amounts to an étale localized version of the notion "tdo up to a twist by a line bundle". Then we define for a crystalline $\mathcal{O}^*$-gerbe $\mathcal{C}$ the corresponding abelian category of twisted $\mathcal{D}$-crystals $\mathcal{M}_{\mathcal{C}}(X)$.

7.10.16. As before, X is any algebraic space. The category X_{cr} carries a structure of site (étale crystalline topology): a covering is a family of morphisms $\{(S_i, \hat{S}_i) \to (S, \hat{S})\}$ such that $\{\hat{S}_i \to \hat{S}\}$ is an étale covering of $\hat{S}$. It carries a sheaf of rings $\mathcal{O}_{cr}$ where $\mathcal{O}_{cr}(S, \hat{S}) = \mathcal{O}(\hat{S})$. So we have the corresponding sheaf $\mathcal{O}_{cr}^*$ of invertible elements.

7.10.17. *Definition.* A *crystalline $\mathcal{O}^*$-gerbe* on X is an $\mathcal{O}_{cr}^*$-gerbe on X_{cr}*).

Explicitely, this means the following. Consider the sheaf of Picard groupoids $\mathcal{P}ic_{cr}$ on X_{cr} where $\mathcal{P}ic_{cr}(S, \hat{S}) := \mathcal{P}ic(\hat{S})$ (= the Picard groupoid of line bundles on $\hat{S}$). Now a crystalline $\mathcal{O}^*$-gerbe on X is a $\mathcal{P}ic_{cr}$-Torsor

*)Probably this property characterizes tdo's.

*)Saito prefers to deal with analytic setting, but his definitions have obvious algebraic version (and the above definitions have obvious analytic version).

*)i.e., a gerbe over X_{cr} with band $\mathcal{O}_{cr}^*$ in terminology of [De-Mi].

$\mathcal{C}$ over X_{cr} (i.e., $\mathcal{C}$ is a fibered category over X_{cr} equipped with an Action of $\mathcal{P}ic_{cr}$ which makes each fiber $\mathcal{C}(\hat{S}) = \mathcal{C}(S, \hat{S})$ a $\mathcal{P}ic(\hat{S})$-Torsor) such that locally on X_{cr} our $\mathcal{C}(S, \hat{S})$ is non-empty.

Crystalline $\mathcal{O}^*$-gerbes form a Picard 2-groupoid $\mathcal{G}_{cr}(X)$. The group of equivalence classes of gerbes is $H^2(X_{cr}, \mathcal{O}^*_{cr})$. For a pair of gerbes $\mathcal{C}$, $\mathcal{C}'$ Morphisms $\phi : \mathcal{C} \to \mathcal{C}'$ form a $\mathcal{P}ic(X_{cr})$-Torsor. Here $\mathcal{P}ic(X_{cr})$ is the Picard groupoid of $\mathcal{O}^*_{cr}$-torsors on X_{cr}*).

7.10.18. *Remarks.* (i) Let $X_{\acute{e}t\, cr}$ be the small étale crystalline site of X (as a category it equals $X^{(i)}_{cr}$ from 7.10.4, the topology is induced from X_{cr}). A crystalline $\mathcal{O}^*$-gerbe on X yields by restriction an $\mathcal{O}^*_{cr}$-gerbe on $X_{\acute{e}t\, cr}$. We leave it to the reader to check that we get an equivalence of the Picard 2-groupoids of gerbes*).

(ii) Our $\mathcal{G}_{cr}(X)$ is the Picard 2-groupoid associated to the complex $\tau_{\leq 2} R\Gamma(X_{cr}, \mathcal{O}^*_{cr}) = \tau_{\leq 2} R\Gamma(X_{\acute{e}t\, cr}, \mathcal{O}^*_{cr})$. To compute $R\Gamma$ look at the canonical ideal $\mathcal{I}_{cr} \subset \mathcal{O}_{cr}$ defined by $(\mathcal{O}_{cr}/\mathcal{I}_{cr})(S, \hat{S}) = \mathcal{O}(S)$. There is a canonical morphism of ringed topologies $i : X_{\acute{e}t} \to X_{\acute{e}t\, cr}$, $i^{-1}(S, \hat{S}) = S$, and $\mathcal{I}_{cr}$ fits into short exact sequence $0 \to \mathcal{I}_{cr} \to \mathcal{O}_{cr} \to i.\mathcal{O}_X \to 0$. Passing to sheaves of invertible elements we get the short exact sequence

$$0 \longrightarrow \mathcal{I}_{cr} \xrightarrow{\exp} \mathcal{O}^*_{cr} \longrightarrow i.\mathcal{O}^*_X \longrightarrow 0 \tag{342}$$

where exp is the exponential map (since each $\mathcal{I}_{cr}(S, \hat{S})$ is a nilpotent ideal our exp is correctly defined). Since $R\Gamma(X_{\acute{e}t\, cr}, i.\mathcal{O}^*_X) = R\Gamma(X_{\acute{e}t}, \mathcal{O}^*)$. one may use (343) to compute $R\Gamma(X_{cr}, \mathcal{O}^*_{cr})$. For example, since $H^0(X_{cr}, \mathcal{I}_{cr}) = 0$ the group $H^0(X_{cr}, \mathcal{O}^*_{cr})$ is the group $\mathcal{O}^*(X)_{con}$ of locally constant invertible functions on X.

(iii) Assume that X is smooth. Set $\Omega^{\geq 1}_X := (0 \to \Omega^1_X \to \Omega^2_X ..)$. According to Grothendieck, one has $R\Gamma(X_{cr}, \mathcal{O}_{cr}) = R\Gamma(X, \Omega_X)$ and $R\Gamma(X_{cr}, \mathcal{I}_{cr}) =$

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*)If X is smooth then such torsor is the same as a line bundle with flat connection on X.

*)We consider X_{cr} as the basic setting since it directly generalizes to the case of ind-schemes, see 7.11.6).

$R\Gamma(X_{cr}, \mathcal{C}one(\mathcal{O}_{cr} \to i_\cdot\mathcal{O}_X)[-1]) = R\Gamma(X, \Omega_X^{\geq 1})$. Thus (342) yields the long cohomology sequence

$$0 \longrightarrow \mathcal{O}^*(X)_{con} \longrightarrow \mathcal{O}^*(X) \xrightarrow{d\,log} \Omega^{1cl}(X) \longrightarrow H^1(X_{cr}, \mathcal{O}^*_{cr}) \longrightarrow$$

$$\longrightarrow Pic(X) \xrightarrow{c_1} H^2(X, \Omega_X^{\geq 1}) \longrightarrow H^2(X_{cr}, \mathcal{O}^*_{cr}) \longrightarrow Br(X) \longrightarrow 0.$$

Here $H^1(X_{cr}, \mathcal{O}^*_{cr})$ is the group of isomorphism classes of line bundles with flat connection on X. One has 0 at the right since $H^2(X_{\acute{e}t}, \mathcal{O}^*) = Br(X)$ is a torsion group and $H^3(X_{cr}, \mathcal{I}_{cr})$ is a $\mathbb{C}$-vector space.

(iv) If X is a scheme then one may consider a weaker topology $X_{Zar\,cr}$ (as a category it equals $X_{cr}^{(ii)}$ from 7.10.4). We get the corresponding Picard 2-groupoid $\mathcal{G}_{Zar\,cr}(X)$ of $\mathcal{O}^*_{cr}$-gerbes on $X_{Zar\,cr}$. By étale descent the pull-back functor $\mathcal{G}_{Zar\,cr}(X) \to \mathcal{G}_{cr}(X)$ is a fully faithful Morphism of Picard 2-groupoids, i.e., $\mathcal{G}_{Zar\,cr}(X)$ is the 2-groupoid of Zariski locally trivial crystalline $\mathcal{O}^*$-gerbes. It is easy to see*) that $\mathcal{C} \in \mathcal{G}_{cr}(X)$ belongs to $\mathcal{G}_{Zar\,cr}(X)$ if (and only if) the $\mathcal{O}^*$-gerbe $i^\cdot\mathcal{C}$ on $X_{\acute{e}t}$ is Zariski locally trivial. For example, if X is smooth then $H^2(X_{Zar}, \mathcal{O}^*) = 0$, so $\mathcal{G}_{cr}(X)/\mathcal{G}_{Zar\,cr}(X) = Br(X)$.

7.10.19. Below we give a convenient "concrete" description of (appropriately rigidified) crystalline $\mathcal{O}^*$-gerbes.

Assume we have $X \hookrightarrow V$ as in 7.10.7. For $\mathcal{C} \in \mathcal{G}_{cr}(X)$ and an infinitesimal neighbourhood $X' \subset V$ of X we have the $\mathcal{P}ic(X')$-Torsor $\mathcal{C}(X')$. Set $\mathcal{C}(V) := \varprojlim \mathcal{C}(X')$ (:= the groupoid of Cartesian sections of $\mathcal{C}$ over the directed set of X''s); this is a $\mathcal{P}ic(V)$-Torsor.

Consider pairs $(\mathcal{C}, \mathcal{E}_V)$ where $\mathcal{C} \in \mathcal{G}_{cr}(X)$ and $\mathcal{E}_V \in \mathcal{C}(V)$. Such objects form a Picard groupoid $\mathcal{G}^V_{cr}(X)$. Namely, a morphism $(\mathcal{C}, \mathcal{E}_V) \to (\mathcal{C}', \mathcal{E}'_V)$ is a pair (F, ν) where F is a Morphism $\mathcal{C} \to \mathcal{C}'$ and $\nu : F(\mathcal{E}_V) \simeq \mathcal{E}'_V$*). We are going to describe $\mathcal{G}^V_{cr}(X)$.

*)cf. 7.10.22.

*)Notice that such pairs have no symmetries, so $\mathcal{G}^V_{cr}(X)$ is a plain groupoid (while $\mathcal{G}_{cr}(X)$ is a 2-groupoid).

We use notation from 7.10.7. Let $\mathcal{R}$ be a line bundle on $V^{<2>}$ and $\beta : p_{12}^*\mathcal{R}\otimes p_{23}^*\mathcal{R} \widetilde{\rightarrow} p_{13}^*\mathcal{R}$ an isomorphism of line bundles on $V^{<3>}$ such that the following diagram of isomorphisms of line bundles on $V^{<4>}$ commutes (associativity condition):

$$\begin{array}{ccc} \mathcal{R}_{12}\otimes\mathcal{R}_{23}\otimes\mathcal{R}_{34} & \longrightarrow & \mathcal{R}_{13}\otimes\mathcal{R}_{34} \\ \downarrow & & \downarrow \\ \mathcal{R}_{12}\otimes\mathcal{R}_{24} & \longrightarrow & \mathcal{R}_{14} \end{array} \tag{343}$$

Here $\mathcal{R}_{ij}$ is the pull-back of $\mathcal{R}$ by projection $p_{ij} : V^{<4>} \to V^{<2>}$ and the arrows come from β.

Such pairs $(\mathcal{R},\beta)$ form a Picard groupoid $G(V)$ (with respect to tensor product).

7.10.20. *Proposition.* The Picard groupoids $\mathcal{G}_{cr}^V(X)$ and $G(V)$ are canonically equivalent.

Proof. For $(\mathcal{C},\mathcal{E}_V)\in\mathcal{G}_{cr}^V(X)$ set $\mathcal{R} := \mathcal{H}om(p_1^*\mathcal{E}_V, p_2^*\mathcal{E}_V)\in\mathcal{P}ic(V)$ and define β as the composition isomorphism; it is clear that $(\mathcal{R},\beta)\in G(V)$. So we have the Morphism of Picard groupoids $\mathcal{G}_{cr}^V(X)\to G(V)$.

The inverse Morphism assigns to $(\mathcal{R},\beta)$ the pair $(\mathcal{C},\mathcal{E}_V)$ glued from trivial gerbes by means of $(\mathcal{R},\beta)$. Namely, one defines $(\mathcal{C},\mathcal{E}_V)$ as follows. Since V is formally smooth the structure morphism $j : S\to X$ extends to $j' : \hat{S}\to V$. Now $\mathcal{C}(\hat{S})$ is a $\mathcal{P}ic(\hat{S})$-Torsor together with the following extra structure:

(i) For any j' as above we are given an object of $\mathcal{C}(\hat{S})$ denoted by $j'^*\mathcal{E}_V$.

(ii) If $j'' : \hat{S}\to V$ is another extension of j then we have an identification of line bundles $\theta_{j''j'} : \mathcal{H}om(j'^*\mathcal{E}_V, j''^*\mathcal{E}_V) \widetilde{\rightarrow} (j'',j')^*\mathcal{R}$.

We demand that (ii) identifies composition of $\mathcal{H}om$'s with the isomorphism defined by β. It is easy to see that such $\mathcal{C}(\hat{S})$ exists and unique (up to a unique equivalence). The fibers $\mathcal{C}(\hat{S})$ glue together to form a crystalline $\mathcal{O}^*$-gerbe in the obvious way. We have $\mathcal{E}_V\in\mathcal{C}(V)$ by construction. $\square$

7.10.21. *Remark.* Let $\mathcal{E}'_V$ be another object of $\mathcal{C}(V)$ and $(\mathcal{R}', \beta') \in G(V)$ the pair that corresponds to $(\mathcal{C}, \mathcal{E}'_V)$. Set $\mathcal{L} := \mathcal{H}om(\mathcal{E}_V, \mathcal{E}'_V) \in \mathcal{P}ic(V)$. Then $\mathcal{R}' = \operatorname{Ad}_{\mathcal{L}} \mathcal{R} := (p_2^* \mathcal{L}) \otimes \mathcal{R} \otimes (p_1^* \mathcal{L})^{\otimes -1}$ and $\beta' = \operatorname{Ad}_{\mathcal{L}} \beta$.

Now let $\mathcal{C}$ be any crystalline $\mathcal{O}^*$-gerbe on X, and assume that we have $X \hookrightarrow V$ as above. To use 7.10.20 for description of $\mathcal{C}$ one has to assure that $\mathcal{C}(V)$ is non-empty.

7.10.22. *Lemma.* Assume that X is affine and V is a union of countably many subschemes. Then $\mathcal{C}(V)$ is non-empty if*) $\mathcal{C}(X, X)$ is non-empty.

Proof. Let $X' \subset V$ be an infinitesimal neighbourhood of X. Then any $\mathcal{E}_X \in \mathcal{C}(X, X)$ admits an extension $\mathcal{E}_{X'} \in \mathcal{C}(X, X')$, and all such extensions are isomorphic. Now we have a sequence $X \subset X^{(1)} \subset X^{(2)} \ldots$ of infinitesimal neighbourhoods of X such that $V = \varinjlim X^{(n)}$. One defines by induction a sequence $\mathcal{E}_{X^{(n)}} \in \mathcal{C}(X, X^{(n)})$ together with identifications $\mathcal{E}_{X^{(n+1)}}|_{X^{(n)}} = \mathcal{E}_{X^{(n)}}$. This is $\mathcal{E}_V \in \mathcal{C}(V)$. $\square$

7.10.23. *Remarks.* (i) Consider the $\mathcal{O}^*$-gerbe $i^{\cdot}\mathcal{C}$ on $X_{\acute{e}t}$ (so $i^{\cdot}\mathcal{C}(U) = \mathcal{C}(U, U)$). Then $\mathcal{C}(X, X) \neq \emptyset$ if and only if $i^{\cdot}\mathcal{C}$ is a trivial gerbe, i.e., its class in $H^2(X_{\acute{e}t}, \mathcal{O}^*) = Br(X)$ vanishes*).

(ii) For any algebraic space X and $\mathcal{C} \in \mathcal{G}_{cr}(X)$ one may use 7.10.20 to describe $\mathcal{C}$ locally on $X_{\acute{e}t}$. Namely, there exists an étale covering U_i of X such that U_i are affine and $\mathcal{C}(U_i, U_i) \neq \emptyset$. Embed U_i into a smooth scheme and take for V_i the corresponding formal completion. Now, by 7.10.22, we may use 7.10.20, 7.10.21 to describe $\mathcal{C}_{U_i}$.

7.10.24. *Definition.* For $\mathcal{C} \in \mathcal{G}_{cr}(X)$ a *$\mathcal{C}$-twisted $\mathcal{D}$-crystal on X* is a Cartesian functor $M : \mathcal{C} \to \mathcal{M}^!(X_{cr}, \mathcal{O})$ such that for any $\mathcal{E} \in \mathcal{C}(\hat{S})$ and $f \in \mathcal{O}^*(\hat{S})$ one has $M(f_{\mathcal{E}}) = f \cdot \mathrm{id}_{M(\mathcal{E})}$.

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*)and, certainly, only if

*)This class is the image of the class of $\mathcal{C}$ by the map $H^2(X_{cr}, \mathcal{O}^*_{cr}) \to H^2(X_{cr}, i_{\cdot}\mathcal{O}^*) = Br(X)$.

The $\mathcal{C}$-twisted $\mathcal{D}$-crystals form a $\mathbb{C}$-category $\mathcal{M}_{\mathcal{C}}(X)$. It depends on $\mathcal{C}$ in a functorial way (to a Morphism $\mathcal{C} \to \mathcal{C}'$ there corresponds an equivalence of categories $\mathcal{M}_{\mathcal{C}}(X) \xrightarrow{\sim} \mathcal{M}_{\mathcal{C}'}(X)$, etc.).

The categories $\mathcal{M}_{\mathcal{C}}(U) = \mathcal{M}_{\mathcal{C}_U}(U)$, $U \in X_{\acute{e}t}$, form a sheaf of categories $\mathcal{M}_{\mathcal{C}}(X_{\acute{e}t})$ over $X_{\acute{e}t}$ in the obvious way.

Let $\mathcal{C}^{triv}$ be the trivialized gerbe, so $\mathcal{C}^{triv}(\hat{S}) = \mathcal{P}ic(\hat{S})$. The $\mathcal{C}^{triv}$-twisted $\mathcal{D}$-crystals are the same as plain $\mathcal{D}$-crystals. Namely, one identifies $M \in \mathcal{M}_{\mathcal{C}^{triv}}(X)$ with the $\mathcal{D}$-crystal $M_{\hat{S}} := M(\mathcal{O}_{\hat{S}})$.

Remark. In the above definition we may replace X_{cr} by $X_{\acute{e}t\, cr}$. If X is a scheme and $\mathcal{C} \in \mathcal{G}_{Zar\, cr}(X)$ then we may replace X_{cr} by $X_{Zar\, cr}$. One gets the same category $\mathcal{M}_{\mathcal{C}}(X)$.

7.10.25. Here is a twisted version of 7.10.7, 7.10.8. Assume we are in situation 7.10.19, so we have $(\mathcal{C}, \mathcal{E}_V) \in \mathcal{G}^V_{cr}(X)$ and the corresponding $(\mathcal{R}, \beta) \in G(V)$ (see 7.10.20). The category $\mathcal{M}_{\mathcal{C}}(X)$ may be described as follows. Let $\mathcal{M}_{\mathcal{R}}(X) = \mathcal{M}_{\mathcal{R}\beta}(X)$ be the category of pairs (M_V, τ) where $M_V \in \mathcal{M}(V, \mathcal{O})$ and $\tau : (p_1^! M_V) \otimes \mathcal{R} \xrightarrow{\sim} p_2^! M_V$ is an isomorphism in $\mathcal{M}(V^{<2>}, \mathcal{O})$ such that*)

$$p_{23}^!(\tau) p_{12}^!(\tau) = p_{13}^!(\tau). \tag{344}$$

7.10.26. *Lemma.* The categories $\mathcal{M}_{\mathcal{C}}(X)$ and $\mathcal{M}_{\mathcal{R}}(X)$ are canonically equivalent.

Proof. For $M \in \mathcal{M}_{\mathcal{C}}(X)$ set $M_V = M(\mathcal{E}_V) := \bigcup M(\mathcal{E}_{(X,X')})$, and define τ as composition of the isomorphisms $(p_1^! M_V) \otimes \mathcal{R} = M(p_1^* \mathcal{E}_V) \otimes \mathcal{R} = M((p_1^* \mathcal{E}_V) \otimes \mathcal{R}) = M(p_2^* \mathcal{E}_V) = p_2^! M_V$. The rest is an immediate modification of the proof of 7.10.8. □

7.10.27. *Lemma.* For any X and $\mathcal{C} \in \mathcal{G}_{cr}(X)$ the category $\mathcal{M}_{\mathcal{C}}(X)$ is abelian.

Proof. An obvious modification of the proof of 7.10.9. Use 7.10.23(ii), 7.10.22, 7.10.26. □

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*) We use β to identify the modules where the l.h.s. and r.h.s. of the equality lie.

7.10.28. From now on we assume that X is a smooth algebraic space. We want to compare the above picture with the usual setting of tdo and twisted $\mathcal{D}$-modules. First let us relate crystalline $\mathcal{O}^*$-gerbes and tdo[*)].

Look at 7.10.19 for $V = X$. Consider the Picard groupoid $\mathcal{G}_{cr}^{\sim}(X) := \mathcal{G}_{cr}^{V}(X)$ of pairs $(\mathcal{C}, \mathcal{E}_X)$ where $\mathcal{C}$ is a crystalline $\mathcal{O}^*$-gerbe on X and $\mathcal{E}_X \in \mathcal{C}(X)$.

Here is a convenient interpretation of $\mathcal{G}_{cr}^{\sim}(X)$. Consider $\mathcal{I}_{cr}$-gerbes on X (i.e., $\mathcal{I}_{cr}$-gerbes on X_{cr}). Since $H^0(X_{cr}, \mathcal{I}_{cr}) = 0$ these gerbes form a (shifted) Picard groupoid $\mathcal{G}\mathcal{I}_{cr}(X)$. The exponential morphism $\mathcal{I}_{cr} \hookrightarrow \mathcal{O}_{cr}^*$ yields the functor $\exp : \mathcal{G}\mathcal{I}_{cr}(X) \to \mathcal{G}_{cr}(X)$. Since $\mathcal{I}_{(X,X)} = 0$, for any $\mathcal{I}_{cr}$-gerbe $\mathcal{B}$ the groupoid $\mathcal{B}_X$ is trivial, so the groupoid $(\exp \mathcal{B})_X$ has a distinguished object $\mathcal{E}_{\mathcal{B}X}$ (defined up to a canonical isomorphism). Thus we defined a Morphism of Picard groupoids

$$\exp : \mathcal{G}\mathcal{I}_{cr}(X) \longrightarrow \mathcal{G}_{cr}^{\sim}(X), \tag{345}$$

$\mathcal{B} \mapsto (\exp \mathcal{B}, \mathcal{E}_{\mathcal{B}X})$. This is an equivalence of Picard groupoids (as follows from (342)).

Example. The "boundary map" for (342) yields the morphism of Picard groupoids $c : \mathcal{P}ic(X) \to \mathcal{G}\mathcal{I}_{cr}(X)$ (the crystalline Chern class). In terms of (345) it assigns to $\mathcal{L} \in \mathcal{P}ic(X)$ the pair $(\mathcal{C}^{triv}, \mathcal{L})$.

7.10.29. *Proposition.* $\mathcal{G}_{cr}^{\sim}(X)$ is canonically equivalent to the Picard groupoid $\mathcal{TDO}(X)$ of tdo's on X.

Proof. Let us identify, according to 7.10.20 for $V = X$, our $\mathcal{G}_{cr}^{\sim}(X)$ with $G(X)$. Now for $(\mathcal{R}, \beta) \in G(X)$ the corresponding tdo $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{(\mathcal{R},\beta)}$ is defined as follows. We use notation from 7.10.13. Consider $\mathcal{D}_X$ as a Diff-bimodule (an object of $\mathcal{M}(X^{<2>}, \mathcal{O})$). Set $\mathcal{D}_{\mathcal{R}} := \mathcal{D}_X \underset{\mathcal{O}_{X^{<2>}}}{\otimes} \mathcal{R}$. The multiplication morphism $m_{\mathcal{R}} : \mathcal{D}_{\mathcal{R}} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_{\mathcal{R}} \to p_{13}^{!} \mathcal{D}_{\mathcal{R}}$ is the tensor product of the corresponding morphism for $\mathcal{D}_X$ and β. One checks easily that $\mathcal{D}_{\mathcal{R}}$

[*)] see, e.g., [BB93] 2.1 for basic facts about tdo.

is a tdo and $G(X) \to \mathcal{TDO}(X)$, $(\mathcal{R},\beta) \mapsto \mathcal{D}_{\mathcal{R}}$ is a Morphism of Picard groupoids.

The inverse Morphism assigns to a tdo A on X the object $(\mathcal{R},\beta)$ where $\mathcal{R} := \mathcal{H}om_{\mathcal{O}_{X^{<2>}}}(\mathcal{D}_X, A)$ and β is defined by comparison of the multiplication morphisms m for $\mathcal{D}_X$ and A. We leave the details for the reader. □

7.10.30. *Remark.* Here is another (equivalent) way to spell out the above equivalence. By (345) $\mathcal{G}_{cr}^{\sim}(X)$ is equivalent to $\mathcal{GI}_{crys}(X)$, i.e., to the Picard groupoid associated with complex $\tau_{\leq 1}(R\Gamma(X_{crys}, \mathcal{I}_{X_{crys}})[1])$. According to [BB93] 2.1.6, 2.1.4, $\mathcal{TDO}(X)$ is the Picard groupoid associated with the complex $\tau_{\leq 1}(R\Gamma(X, \Omega_X^{\geq 1})[1])$. Now the above complexes are canonically quasi-isomorphic (see 7.10.18(iii)).

7.10.31. Here is a twisted version of 7.10.12. For $(\mathcal{C}, \mathcal{E}_X) \in \mathcal{G}_{cr}^{\sim}(X)$ consider the corresponding $(\mathcal{R},\beta) \in G(X)$ and the tdo $\mathcal{D}_{\mathcal{R}}$. Take $M \in \mathcal{M}_{\mathcal{C}}(X)$. According to 7.10.26 we may consider M as pair $(M_X, \tau) \in \mathcal{M}_{\mathcal{R}}(X)$. Since[*)] $p_1^! M_X = M_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ and $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_X \underset{\mathcal{O}_{X^{<2>}}}{\otimes} \mathcal{R}$ we may rewrite τ as an isomorphism

$$M_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_{\mathcal{R}} \stackrel{\sim}{\rightarrow} p_2^! M_X \tag{346}$$

in $\mathcal{M}(X^{<2>}, \mathcal{O})$. By adjunction, one may consider (346) as a morphism of $\mathcal{O}_X$-modules

$$M_X \otimes \mathcal{D}_{\mathcal{R}} \to M_X. \tag{347}$$

Denote by $\mathcal{M}^r(X, \mathcal{D}_{\mathcal{R}})$ the category of right $\mathcal{D}_{\mathcal{R}}$-modules on X.

7.10.32. *Proposition.* The morphism (347) is a right unital action of $\mathcal{D}_{\mathcal{R}}$ on M_X. The functor $\mathcal{M}_{\mathcal{C}}(X) \to \mathcal{M}^r(X, \mathcal{D}_{\mathcal{R}})$, $M \mapsto M_X$, is an equivalence of categories.

Proof. Left to the reader (see 7.10.12, 7.10.13). □

[*)] See the proofs of 7.10.12 and 7.10.29.

7.11. **$\mathcal{D}$-modules on ind-schemes.** In this section we discuss $\mathcal{D}$-module theory on formally smooth ind-schemes. Notice that the $\mathcal{D}$-crystal picture (see 7.10) makes immediate sense in the ind-scheme setting, and it is the conventional approach (differential operators, etc.) that takes some space to be written down.

7.11.1. An *ind-scheme* (in the strict sense) X is a "space" (i.e., a set valued functor on the category of commutative $\mathbb{C}$-algebras $A \mapsto X(A) = X(\operatorname{Spec} A)$) which may be represented as $\varinjlim X_\alpha$ where $\{X_\alpha\}$ is a directed family of quasi-compact schemes such that all the maps $i_{\alpha\beta} : X_\alpha \to X_\beta$, $\alpha \leq \beta$, are closed embeddings. If X can be represented as above so that the set of indices α is countable then X is said to be an $\aleph_0$-*ind-scheme*.*) If P is a property of schemes stable under passage to closed subschemes then we say that X satisfies the *ind-P* property if each X_α satisfies P.

Set $X_{\mathrm{red}} := \varinjlim X_{\alpha\,\mathrm{red}}$; an ind-scheme X is said to be *reduced* if $X_{\mathrm{red}} = X$.

A *formal scheme* is an ind-scheme X such that X_{red} is a scheme (see 7.12.17 for a discussion of the relation between this definition of formal scheme and the one from EGA). An $\aleph_0$-*formal scheme* is a formal scheme which is an $\aleph_0$-ind-scheme. The *completion* of an ind-scheme Z along a closed subscheme $Y \subset Z$ is the direct limit of closed subschemes $Y' \subset Z$ such that $Y'_{\mathrm{red}} = Y_{\mathrm{red}}$. In the case of formal schemes we write "affine" instead of "ind-affine". A formal scheme X is affine if and only if X_{red} is affine.

Following Grothendieck ([Gr64], [Gr67]), we say that X is *formally smooth* if for every A and every nilpotent ideal $I \subset A$ the map $X(A) \to X(A/I)$ is surjective. It is easy to see that for ind-schemes of ind-finite type formal smoothness is a local property (cf. the proof of Proposition 17.1.6 from [Gr67]).*) A morphism $X \to Y$ is said to be *formally smooth* if for any A,

*)Not all natural examples of ind-schemes are $\aleph_0$-ind-schemes; e.g., for every infinite-dimensional vector space V the functor $A \mapsto \operatorname{End}_A(V \otimes A)$ is an ind-scheme but not an $\aleph_0$-ind-scheme.

*)We do not know whether this is true for ind-schemes that are not of ind-finite type. For schemes the answer is "yes". This follows from Remark 9.5.8 in [Gr68a] and the

I as above the map from $X(A)$ to the fiber product of $Y(A)$ and $X(A/I)$ over $Y(A/I)$ is surjective.

Let X be an ind-scheme. A closed quasi-compact subscheme $Y \subset X$ is called *reasonable* if for any closed subscheme $Z \subset X$ such that $Y \subset Z$ the ideal of Y in $\mathcal{O}_Z$ is finitely generated. We say that X is *reasonable* if X is a union of its reasonable subschemes, i.e., it may be represented as $\varinjlim X_\alpha$ where all X_α are reasonable. A closed subspace of a reasonable ind-scheme is a reasonable ind-scheme; a product of two reasonable ind-schemes is reasonable.

Remark. Replacing the word "schemes" in the above definition by "algebraic spaces" we get the notion of an ind-algebraic space. All the discussion passes automatically to the setting of ind-algebraic spaces.

7.11.2. *Examples.* (i) An ind-affine ind-scheme X is the same as a pro-algebra, i.e., a pro-object R of the category of commutative algebras that can be represented as $\varprojlim R_\alpha$ so that the maps $R_\beta \to R_\alpha$, $\beta \geq \alpha$, are surjective. We write $X = \operatorname{Spf} R := \varinjlim \operatorname{Spec} R_\alpha$. A complete topological commutative algebra R whose topology is defined by open ideals $I_\alpha \subset R$ can be considered as a pro-algebra (set $R_\alpha := R/I_\alpha$). Not all pro-algebras are of this type because if the set of indices α is uncountable then the map from the set-theoretical projective limit of the R_α to R_{α_0} is not necessarily surjective*). Of course, an ind-affine $\aleph_0$-ind-scheme is the same as a complete topological algebra whose topology is defined by a countable or finite system of open ideals of R.

(ii) Let V be a Tate vector space (see 4.2.13). Then V (or, more precisely, the functor $A \mapsto V\widehat{\otimes}A$) is a reasonable ind-affine ind-scheme.

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following surprising result ([RG], p.82, 3.1.4): the property of being a projective module is local for the Zariski topology and even for the fpqc topology (without any finiteness assumptions!).

*)even if the maps $R_\beta \to R_\alpha$, $\beta \geq \alpha$, are surjective (as we assume).