Note Title 3/29/2007

Tom Bridgeland - Derived Categories in Algebraic Geometry

Theme - study algebraic varieties via categories of sheaves on them.

- like studying an algebra through its modules.

Restrict to X smooth projective varieties $/\mathbb{C}$

$X \rightsquigarrow \mathrm{Coh}\, X$ abelian category of coherent $\mathcal{O}_X$-modules

$\rightsquigarrow D(X) = D^b \mathrm{Coh}\, X$: beautiful symmetric object associated to X.
bounded derived category

How is geometry of X reflected in properties of these categories?

One motivation: string theory on X sees not X but $D(X)$.

<u>Lemma</u> • $\mathrm{Coh}\, X \cong \mathrm{Coh}\, Y \Longrightarrow X \cong Y$.

– ie $\mathrm{Coh}\, X$ <u>too strongly</u> tied to X.

• $\mathrm{Aut}\, \mathrm{Coh}\, X = \mathrm{Pic}\, X \rtimes \mathrm{Aut}\, X$

exact autoequivalences

This isn't true for $D(X)$: can find interesting pairs of varieties X, Y with $D(X) \simeq D(Y)$ (rare & special!) & interesting groups of symmetries $\mathrm{Aut}\, D(X)$.
(discrete groups)

<u>Recap on derived categories</u>

$\mathcal{A}$ abelian category e.g. $\mathrm{Mod}(R)$, $\mathrm{Coh}(X)$

$D(\mathcal{A}) = \mathrm{Complexes}(\mathcal{A})[(\text{Quasi isomorphisms})^{-1}]$

$$\begin{tikzcd} M^\bullet \arrow[d, "f^\bullet"] & \arrow[r] & M^{i-1} \arrow[r, "d"] \arrow[d, "f^{i-1}"] & M^i \arrow[r, "d"] \arrow[d, "f^i"] & M^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots & d^2=0 \\ N^\bullet & & N^{i-1} \arrow[r, "d"] & N^i \arrow[r, "d"] & N^{i+1} \arrow[r] & \end{tikzcd}$$

by hand invert all quasi isomorphisms

$f^\bullet : M^\bullet \to N^\bullet$ s.t. $H(f^\bullet): H^*(M^\bullet) \xrightarrow{\sim} H^*(N^\bullet)$ isomorphism.

$\Rightarrow$ identify a module with all of its resolutions.

Structure on $D(\mathcal{A})$: triangulated :

$\exists$ shift functor $(M^\bullet[1])^i = M^{i+1}$
shifting of labeling

& distinguished triangles : no longer abelian, can't talk about kernels & cokernels, triangles are substitutes :

$$\begin{tikzcd} M^\bullet \arrow[r, "f"] & N^\bullet \arrow[dl] \\ P^\bullet \arrow[u] & \end{tikzcd}$$

$P^\bullet$ = mapping cone.

- $\mathcal{A} \hookrightarrow D(\mathcal{A})$, $M \mapsto \cdots 0 \to 0 \to \overset{0}{M} \to 0 \to 0 \cdots$

- $\mathrm{Hom}^i(M^\bullet, N^\bullet) := \mathrm{Hom}(M^\bullet, N^\bullet[i])$
gives Ext groups $= \mathrm{Ext}^i(M,N)$ if $M,N \in \mathcal{A}$.

How can we get interesting maps

$$\begin{tikzcd}
\arrow[r] & 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \cdots \\
& 0 \arrow[r] & N \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 & \cdots
\end{tikzcd}$$

??

Well we've added new morphisms!

can replace by resolution

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

quasi isomorphism

& then can have interesting maps

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & M \arrow[r] \arrow[d] & 0 \\
I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & I^3 \arrow[r] & \cdot
\end{tikzcd}$$

: measured in this case by Ext^2.

Fourier-Mukai theory

Example (Mukai 1981)

$X = \mathbb{C}^n / \Lambda$ abelian variety (projective complex torus)

$\hat{X} = (\mathbb{C}^n)^* / \Lambda^* = \mathrm{Pic}^0 X$ dual abelian variety

... degree 0 line bundles on X

$\exists$ a universal line bundle (Poincaré bundle) $\mathcal{P}$ on $\hat{X} \times X$.

Mukai: $\Phi = R\pi_{X*}(\mathcal{P} \otimes \pi_{\hat{X}}^*(-))$ is an equivalence

$$D(\hat{X}) \longrightarrow D(X)$$

i.e.

$$\begin{tikzcd} & \hat{X} \times X \arrow[dl, "\pi_{\hat{X}}"'] \arrow[dr, "\pi_X"] & \\ \hat{X} & & X \end{tikzcd}$$

pull back a sheaf, tensor with $\mathcal{P}$, & push forward

Can be used to compute moduli spaces of vector bundles.

e.g. X elliptic curve, $\hat{X} = X$

E semistable vector bundle rank a degree b $(a, b > 0)$ $\Longleftrightarrow$ $\Phi(E)$ semistable v.b. rank b, deg $-a$

$\Rightarrow$ isomorphism of moduli spaces for different rank & degree!

in fact get action of $SL_2\mathbb{Z}$ on rank & degree:

$$\begin{array}{ccc} E & \longmapsto & E \otimes L \\ (a,b) & & (a, a+b) \end{array} \qquad b = \deg L$$

gives translation

Corollary: moduli spaces $\mathcal{M}_X(a,b)$ for any rank & degree are all isomorphic to X
(Atiyah 1957)

- Mukai's appearance of $SL_2\mathbb{Z}$: first hint of mirror symmetry!

[Hint to proof: all indecomposables are of this form]

<u>Theorem</u> (Bondal, Orlov). Y, X smooth projective / $\mathbb{C}$, n-folds

Any equivalence $\Phi : D(Y) \to D(X)$

is of the form $\Phi(-) = R\pi_{X*}(\mathcal{P} \otimes \pi_Y^*(-))$

for some object $\mathcal{P} \in D(Y \times X)$.

Bridgeland

- Such a functor is an equivalence

iff $\boxed{a.}$ $\mathrm{Hom}^i_{D(X)}(\Phi(\mathcal{O}_{y_1}), \Phi(\mathcal{O}_{y_2})) = 0$

unless $y_1 = y_2$ & $0 \le i \le n$

$\mathcal{O}_{y_i}$ = skyscraper sheaf at y_i

$\boxed{b.}$ $\mathrm{Hom}_{D(X)}(\Phi(\mathcal{O}_y), \Phi(\mathcal{O}_y)) = \mathbb{C} \quad \forall y \in Y$

$\boxed{c.}$ $\Phi(\mathcal{O}_y) \otimes \omega_X \simeq \Phi(\mathcal{O}_y)$

$\forall y \in Y \qquad \omega_X = \Lambda^n T^*_X$ canonical bundle

Fourier-Mukai case: $\hat{x} \in \hat{X}$

$\Phi(\mathcal{O}_{\hat{x}}) = \mathcal{P}_{\hat{x}}$ corresponding line bundle.

So a is easy, c is trivial since ω_X trivial

(c is vacuous for <u>Calabi-Yaus</u>, $\omega_X = 0$)

Bondal & Orlov also proved that if $\omega_X^{\pm 1}$ is ample $\Rightarrow$ $D(X) \simeq D(Y)$ implies $X \cong Y$ – nothing interesting happens.

Remarks ① These conditions are necessary: if Φ is an equivalence must preserve Homs & there are no Homs between different skyscrapers, & $\mathbb{C}$ many Homs from skyscraper to itself.
For c. $\mathcal{O}_y \otimes \omega_Y = \mathcal{O}_y$ for skyscrapers

& Φ must commute with $\otimes\omega$:
$\otimes\omega$ is a canonical categorical operation, given by Serre duality.

② "Tautology": in the statement of the Theorem Y is a fine moduli space for the objects $\{\Phi(\mathcal{O}_y) \quad y \in Y\} \subset D(X)$.

... ie an equivalence preserves families
– so to construct equivalences we should look for fine moduli spaces!

Threefolds flops

Theorem Suppose $W \dashrightarrow Y$ are birationally equivalent smooth projective threefolds with ω_W, ω_Y nef (e.g. Calabi-Yaus). Then $D(W) \cong D(Y)$.

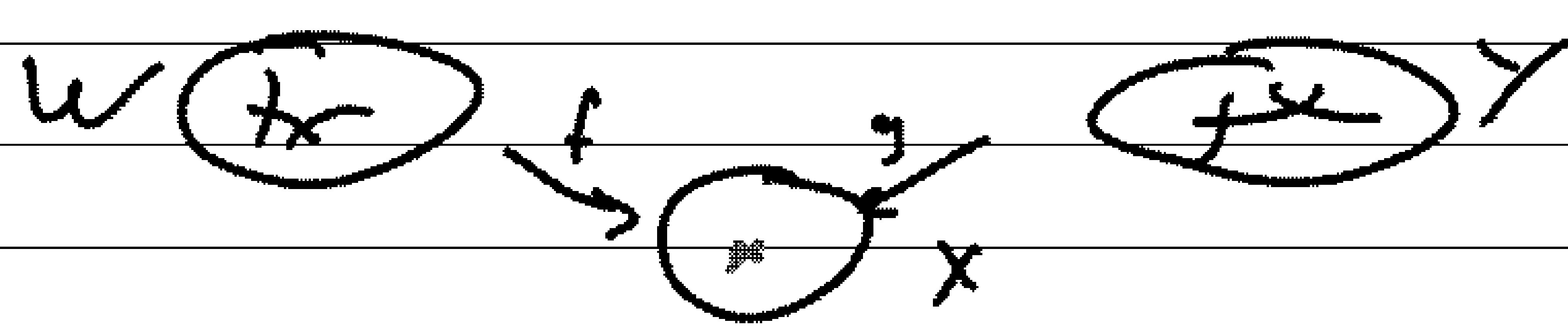
$\Rightarrow$ in particular W, Y have the same Hodge structures.

ω_Y nef means minimal model.

~ minimal models in dim >2 not unique.

Slogan Minimal model isn't unique, but its derived category is.

Proof First use result of Kollár: any birational transformation as above can be decomposed into basic steps where W, Y are related by a flop:



X has mild (Gorenstein rational singularities)

Maps collapse some $\mathbb{P}^1$'s

— so from W to Y is a <u>surgery</u> replacing some S^2's by others.

f, g contract finitely many rational curves
& $\omega_Y|_{C_i} = \mathcal{O}_{C_i}$.

The maps are isomorphisms in codimension one. Put extra condition to ensure W, Y not the same:

if divisor D is g-ample
$\iff -\varphi(D)$ is f-ample:

Switch sign of intersections with the curves C_i.

Have to prove $D(W) \xrightarrow{\sim} D(Y)$

$\Rightarrow$ How to construct W as a moduli space of objects in $D(Y)$?

... how is geometry of flop to be encoded in derived category?

Define a weird abelian subcategory
$\mathcal{A} \subset D(Y)$

$E \in \mathcal{A} \iff$
- $H^i(E) = 0$ unless $i = 0$ or -1
- $H^i Rf_* E = 0$ unless $i = 0$
- ....

Define $E \in \mathcal{A}$ to be a <u>perverse point sheaf</u>
if $\mathcal{O}_Y \twoheadrightarrow E$ in $\mathcal{A}$ quotient of $\mathcal{O}$
& if Chern character $ch(E) = ch(\mathcal{O}_y)$
same as for a skyscraper.

<u>Theorem</u> (Bridgeland) W is the fine moduli space of perverse point sheaves on Y & the universal object gives an equivalence
$D(Y) \to D(W)$.

... generalize to Mukai flops in 4 dim:

e.g. contract $\mathbb{P}^2 \subset T^*\mathbb{P}^2$, on other side get $T\mathbb{P}^2$ .... should be possible!

Michel van den Bergh: put a coherent sheaf of noncommutative algebras R over X
2×2 matrix algebra away from singular point —

$$\& \ D(\mathrm{Mod}(R)) \simeq D(W) \simeq D(Y)$$
$$\cup \qquad\qquad\qquad \cup$$
$$\mathrm{Mod}\, R \quad = \quad \mathcal{A}$$

— noncommutative resolution of singularities!

<u>Conjecture</u> $X \dashrightarrow Y$ birational Calabi-Yau smooth projective n-folds
$\Rightarrow D(X) \cong D(Y)$.

.... probably contains full complexity of minimal model program.

Known for complex symplectic 4-folds.

- would want to understand geography of t-structures on $D(X)$...
  nearby varieties should correspond to nearby t-structures.