Note Title

Gabi Farkas – M_{22} is of general type

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Fix genus g, write down explicitly the general curve of genus g.

eg $g = 5$ $\quad k_c$ canonical

$\deg k_c = 2g-2, \quad h^0(k_c) = g$

$\Rightarrow$ curve of degree 8 in $\mathbb{P}^4$

$C \hookrightarrow \mathbb{P}^4 \quad (a\ g^4_8)$

Now look at the quadrics containing this curve

.... ie study $\mathrm{Sym}^2 H^0(k_c) \longrightarrow H^0(k_c^{\,2})$

$$0 \to \underset{\substack{\parallel \\ \text{quadrics containing } C}}{I_2(C)} \to \underbrace{\mathrm{Sym}^2 H^0(\mathcal{O}_C(1))}_{15} \to \underbrace{H^0(\mathcal{O}_C(2))}_{3g-3=12} \to 0$$

short exact for non-hyperelliptic curves

$\Rightarrow$ have 3 quadrics in $\mathbb{P}^4$ containing C

$$\Rightarrow \quad C \subseteq \bigcap_{i=1}^{3} Q_i \quad \dots \deg 8$$

so must have equality: C is a type $(2,2,2)$ complete intersection in $\mathbb{P}^4$.

So look at Grassmannian of 3-planes in $|\mathcal{O}_{\mathbb{P}^4}(2)| \Rightarrow Gr(3, \mathbb{P}^{15})$

.... 36 dimensional space, which has a dominant rational map to $\mathcal{M}_5$ -

Can then find that $\exists!$ canonical curve in $\mathbb{P}^4$ passing through 12 general points.

<u>Def</u> X unirational $\Leftrightarrow \exists \mathbb{P}^N \dashrightarrow X$ dominant rational map

$\rightsquigarrow$ get $\mathcal{M}_5$ & in fact $\mathcal{M}_{5,12}$ are unirational

Theorem $\mathcal{M}_g$ is unirational for $g \le 14$.

($g \le 10$ easy, proved by Severi, Segre, Arbarello-Cornalba...)

$11 \le g \le 14$: Verra '05 ($g=14$)

Conjecture (Severi) $\mathcal{M}_g$ is unirational $\forall g$ (1915)

... believed into '70s, 80s, Mumford, Oort ...

Disproved by Harris-Mumford '80s.

Def. X smooth projective variety, $K_X = \Omega_X^{\dim X}$

$\dim H^0(X, K_X^{\otimes m}) \sim m^{\kappa(X)}$ $(m \gg 0)$

where $\kappa(X) :=$ Kodaira dimension of X.

$\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X\}$

X unirational $\Rightarrow \kappa(X) = -\infty$ (i.e. all plurigenera vanish)

If $\kappa(X) = \dim X \Rightarrow X$ is of <u>general type</u>.

eg surfaces in $\mathbb{P}^3$

$$\underbrace{0 \quad 1 \quad 2 \quad 3}_{\text{uniruled}} \qquad \underbrace{4}_{K3} \qquad \underbrace{\geq 5}_{\text{general type}}$$

$$\kappa(X) = \quad -\infty \qquad\qquad 0 \qquad\qquad 2$$

<u>Theorem</u> (Harris-Mumford-Eisenbud)

$\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$.

<u>Corollary</u> $[C] \in \mathcal{M}_g$ general curve of genus $g \geq 24$

whenever $C \subseteq S$ surface s.t. C moves on S ($\dim |C| \geq 1$)

$\iff h^0(S, \mathcal{O}_S(C)) \geq 2$

$\Rightarrow$ S must be birational to $\mathbb{P}^1 \times C$

..... eg get $C \subseteq S$ moving by finding C as a hyperplane section of a surface $S \subseteq \mathbb{P}^N$.

So generic curve of large genus is <u>not</u> the hyperplane section of a surface

Pf Suppose $C \in \{C_\lambda\}_{\lambda \in \mathbb{P}^1} \Rightarrow$

$$m : \mathbb{P}^1 \longrightarrow \overline{M}_g \qquad m(\lambda) = [C_\lambda]$$

If m is non-constant $\Rightarrow \exists$ rational curve through generic point of $\overline{M}_g$

i.e. $\overline{M}_g$ is uniruled $\Rightarrow \kappa(\overline{M}_g) = -\infty$

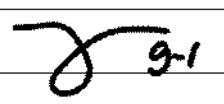
... if $0 \neq \omega \in H^0(K_{\overline{M}_g}^{\otimes m})$

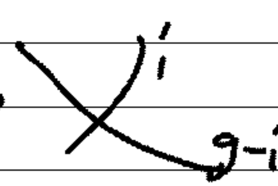
$$\Rightarrow \quad 0 \neq m^*(\omega) \in H^0(K_{\mathbb{P}^1}^{\otimes m}) = 0. \blacksquare$$

About proof of Harris-Mumford-Eisenbud:

$$\overline{M}_g \setminus M_g = \Delta_0 \cup \dots \cup \Delta_{[g/2]}$$

irreducible divisors

$\Delta_0 \longleftrightarrow$ 

$\Delta_i \longleftrightarrow$ 

$$\delta_i = [\Delta_i] \in \mathrm{Pic}\,\overline{M}_g$$

Hodge class : $\lambda = c_1(\mathbb{E} = H^0(\omega_C))$

Thm $g \geq 3$ $\mathrm{Pic}(\overline{M}_g)$ is freely generated by λ & δ's

Mumford: $k_{\overline{M}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots$
$- 2\delta_{[g/2]}$

Can this be effective?
λ is big & nef:
$\exists\ \overline{M}_g \xrightarrow[\varphi]{} \mathbb{P}^N$ s.t. $\lambda = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$
So λ has lots of sections
So are there modular forms of wt 13
vanishing to order 2 on the boundary?
i.e. is $k_{\overline{M}_g}$ effective?

If we can construct an effective divisor D on $\overline{M}_g$
$D = a\lambda - b_0\delta_0 \cdots - b_{[g/2]}\delta_{[g/2]}$, $a, b \geq 0$
s.t. slope $s(D) = \dfrac{a}{\min(b_i)} < s(k_{\overline{M}_g}) = \dfrac{13}{2}$
then $k_{\overline{M}_g} = \alpha\lambda + \beta \cdot D + (\geq 0 \text{ combo of } \delta_i)$
with $\alpha, \beta > 0$.
$\lambda >> 0$, $D >> 0$ $\rightsquigarrow$ k will have lots of sections

(this holds for D's coming from interior: has to do with normal bundle of Δ_i being negative)

Brill-Noether theory:

C genus g $\quad \mathrm{Pic}^d(C) \supset W^r_d(C) =$

$\{L \in \mathrm{Pic}^d : h^0(L) \geq r+1\}$

... determinantal variety $\Rightarrow$ i.e. $C \xrightarrow[|L|]{g^r_d} \mathbb{P}^r$

Virtual dimension of $W^r_d(C)$

$= \rho(g,r,d) = g-(r+1)(g-d+r)$

B-N theorem: For C general $\dim W^r_d(C) = \rho(g,r,d)$

Example $r=1$ $\quad W^1_d(C) \quad \rho = 2d-g-2 \geq 0 \iff$

$d \geq \frac{g+2}{2}$

$\iff$ covers of $\mathbb{P}^1$ of degree d

Hurwitz divisor : $g = 2d-1$, $\rho = -1$

$\Rightarrow$ divisor $\mathcal{M}^1_{g,d}$ = curves of $g = 2d-1$ which can be written as $C \xrightarrow{d:1} \mathbb{P}^1$

$\rightsquigarrow$ extend to boundary using theory of <u>admissible coverings</u>

Find $\overline{\mathcal{M}}^1_{g,d} = c\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i\geq 1} i(g-i)\delta_i\right)$

Slope $s(\overline{\mathcal{M}}^1_{g,d}) = 6 + \frac{12}{g+1} < \frac{13}{2} \Leftrightarrow g \geq 25$ & odd

Even $g = 2d-2$: $\dim W_d(C) = 0 \Rightarrow$
every C of genus $2d-2$ has $< \infty$ g^1_d's.

Gieseker-Petri divisor: $\{ C \in \mathcal{M}_{2d-2}$ s.t. $\exists L \in W_d(C)$ s.t. $h^0(L^{\otimes 2}) \geq 4 \}$

$s(GP_d) < \frac{13}{2} \Leftrightarrow g \geq 24$

Harris' Slope Conjecture $\quad D \subseteq \overline{M}_g$

$$\Rightarrow \quad s(D) \geq s(\overline{M}^1_{g,k}) = 6 + \frac{12}{g+1}$$

.. would imply $g \leq 22 \quad \kappa(\overline{M}_g) = -\infty$

$g = 23$ we know $\kappa(\overline{M}_{23}) \geq 2$.

Theorem $\overline{M}_{22}$ is of general type

$g = 15, 16$: known $\kappa(\overline{M}_g) = -\infty$

$17 \leq g \leq 21$ unknown

Proof Construct an effective divisor of slope < 6.5.

$$\mathcal{W}^6_{25} := \{ [C, L],\ C \in M_{22},\ L \in W^6_{25}(C) \}$$

$$\begin{array}{c} \mathcal{W}^6_{25} \\ \downarrow \pi \\ M_{22} \end{array}$$

$\pi^{-1}(C) = W^6_{25}(C)$, dim generically $\rho(22, 6, 25) = 1$

so fibers are curves,
genus ~ a million & a half

Construct two tautological vector bundles on W^6_{25}

$$\mathcal{E}(C,L) = H^0(C,L), \quad rk(\mathcal{E}) = 7$$

$$\mathcal{F}(C,L) = H^0(C,L^{\otimes 2}), \quad rk(\mathcal{F}) = 29$$

$$\underset{28}{Sym^2 \mathcal{E}} \xrightarrow{\varphi} \underset{29}{\mathcal{F}}$$

Z = first degeneracy locus of φ

$$= \{ [C,L] \in W^6_{25} \text{ s.t. } rk\, \varphi(L) \leq 27 \}$$

virtual codimension of Z is 2.

$\Rightarrow$ expect $\pi_* Z = D_{22}$ to be a (virtual) divisor on $\mathcal{M}_{22}$

<u>Theorem</u> D_{22} is an honest divisor on $\mathcal{M}_{22}$ & its class is

$$\overline{D}_{22} = 132\,22768 \cdot \left(\frac{17121}{2636}\lambda - \delta_0 - \frac{14511}{2636}\delta_1 - \sum_2^{11} b_i \delta_i \right)$$

So slope $s(\overline{D}_{22}) = \frac{17121}{2636} = 6.49506\ldots < 6.5$

& $\overline{M}_{22}$ is of general type.

[easier construction gives slope $6.5003\ldots$]

<u>Theorem</u> $D \subset \overline{M}_g$ effective divisor with $s(D) < 6 + \frac{12}{g+1} \Rightarrow D$ must contain the locus of curves on K3 surfaces.

Proofs: must compactify, study limit linear series, & then use Porteous type formulas.

$g = 23$: expected construction:

$$D_{23} \quad \leftarrow \quad \begin{array}{c} W^6_{26} \\ \downarrow \\ M_{23} \end{array} \quad \text{fibers are surfaces}$$

$s(D_{23}) < 6.5 \qquad S^2 H^0(L) \longrightarrow H^0(L^2) \ldots$