# Gabi Farkas - Koszul divisors on moduli

Note Title 10/6/2006

General method to produce "the best" divisors on moduli spaces

Thm ('80s: Harris-Mumford-Eisenbud)
$\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$
... disproving an ancient conjecture of Severi that $\kappa(\overline{\mathcal{M}}_g) = -\infty$

Conjecture $\kappa(\overline{\mathcal{M}}_g) = -\infty$ $g \leq 22$

Theorem! $\overline{\mathcal{M}}_{22}$ is of general type

Let $R_g$ = moduli of Pryms of dim $g-1$ $= \{ (C, \eta) : C \in \mathcal{M}_g, \eta \in \mathrm{Jac}(C)[2] \}$

$$\begin{array}{ccc} & R_g & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle Pr} \\ \mathcal{M}_g & & A_{g-1} \end{array}$$

$Pr$ = Prym map for $\tilde{C} \xrightarrow{2:1} C$

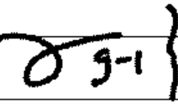
Thm 2 $\overline{\mathcal{R}}_g$ is of general type $g \geq 12$

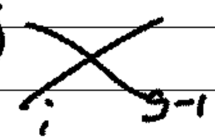
... method of proof the same & much more general.

Slogan: look at syzygies of object you're parametrizing.

Recap on Harris-Mumford-Eisenbud :

$\overline{\mathcal{M}}_g \setminus \mathcal{M}_g = \Delta_0 \cup \dots \cup \Delta_{[g/2]}$ boundary divisors

$\Delta_0 = \{$ 

$\}$ $\Delta_i = \{$ 

$\}$

$\delta_i = [\Delta_i]$ in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ Picard of stack

Hodge class $\lambda = c_1(\mathbb{E})$ Hodge bundle

$\mathbb{E}(C) = H^0(\omega_C)$

$\mathrm{Pic}\, \overline{\mathcal{M}}_g$ freely generated by $\lambda, \delta_i$ $(g \geq 3)$.

For any effective divisor $D$ on $\overline{\mathcal{M}}_g$ which doesn't come from boundary $\Longrightarrow$
$D = a\lambda - \Sigma\, b_i \delta_i$ with $a, b_i \geq 0$.

$$K_{\overline{M}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - \dots - 2\delta_{[g/2]}$$

Is $K_{\overline{M}_g}$ effective?

(general method)

If $\exists D \subseteq \overline{M}_g$ effective divisor with

$D = a\lambda - \sum b_i \delta_i$ s.t. the <u>slope</u>

$s(D) = a/\min(b_i) < s(K_{\overline{M}_g}) = \frac{13}{2}$

$\Rightarrow$ can write $K = \alpha\lambda + \beta D + (\geq 0 \text{ combo of } \delta_i)$

with $\alpha, \beta > 0$.

- $\lambda$ is big & nef $\Longrightarrow$ entire class is big

$\Rightarrow \overline{M}_g$ of general type

Examples of effective divisors on $\overline{M}_g$:

1) Hurwitz divisors : $g = 2k-1$

$$M^1_{g,k} = \{[C] \in M_g : \exists\, C \xrightarrow{k:1} \mathbb{P}^1\}$$

irreducible divisor.

$$[\overline{M}^1_{g,k}] = c\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i\geq 1} i(g-i)\delta_i\right)$$

$\therefore$ need theory of admissible covers to compactify.

$s(\overline{\mathcal{M}}^1_{g,k}) = 6 + \frac{12}{g+1} < \frac{13}{2}$ iff $g \geq 25$ odd

2. Gieseker - Petri divisor

$g = 2k-2$ $\exists$ fin many pencils $g^1_k$

(Catalan # of such)

$$\mathcal{GP}_k = \{ [C] \in \overline{\mathcal{M}}_g : \exists\ g^1_k\ A \in \mathrm{Pic}^k,\ h^0(A)=2$$
$$\text{s.t. } h^0(A^{\otimes 2}) \geq 4 \}$$

$\rightsquigarrow$ general type for even $g \geq 24$.

3. K3 divisor (Farkas - Popa)

$g = 10$ $K_{10} = \{ C \in \mathcal{M}_{10} : C \text{ sits on a K3 surface} \}$

$$= \{ C \in \mathcal{M}_{10} : \exists\ E \in SU_2(K_C) \text{ s.t. } h^0(E) \geq 7 \}$$

$\rightsquigarrow$ a rank 2 Brill-Noether divisor on $\mathcal{M}_{10}$.

expected (& actual) dimension is $-1$, find such not for generic curve but for a divisor of curves

$$\overline{\mathcal{K}}_{10} \equiv 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - 15\delta_5$$

$s(\overline{K}_{10}) = 7 <$ slope of Brill-Noether divisor.

— all of these come from uniform construction!

(will cover all known divisors on $\overline{\mathcal{M}}_g$)

<u>General construction</u>

Fix $g$ & $r, d \geq 0$ s.t.

$$\rho(g,r,d) = g - (r+1)\underbrace{(g-d+r)}_{s} = 0$$

Brill-Noether #

$g = rs + s$

$d = rs + r$

$$\mathcal{G}^r_d = \text{stack } \{ [C,L] : [C] \in \mathcal{M}_g,\ L \in \mathrm{Pic}^d(C),\ h^0(L) \geq r+1 \}$$

$\downarrow \sigma$

$\mathcal{M}_g$

fibers are $W^r_d(C)$

Virtual dim of fiber is $\rho(g,r,d) = 0$

Brill-Noether theory: $\exists !$ component of $\mathcal{G}^r_d$ mapping onto $\mathcal{M}_g$ & no components map onto codim one loci.

So get Koszul complex for $(C,L) \in \mathcal{G}^r_d$:

$$\Lambda^{i+1} H^0(L) \otimes H^0(L^{j-1}) \xrightarrow{d_{i+1,j-1}} \Lambda^i H^0(L) \otimes H^0(L^j) \xrightarrow{d_{i,j}} \Lambda^{i-1} H^0(L) \otimes H^0(L^{j+1}) \longrightarrow$$

$$K_{ij}(C,L) = \operatorname{Ker} d_{ij} / \operatorname{Im} d_{i+1,j-1}$$

$$U_i := \{ (C,L) \in \mathcal{G}^r_d : K_{i,2}(C,L) \neq 0 \}$$

..... $K_{ij}(C,L) = 0 \quad j \geqslant 3$

& $\dim K_{i,2} - \dim K_{i+1,1} = $ fixed

So above is only stratification we get for curves.

Set $Z_i = \sigma_*(U_i) \subset \mathcal{M}_g$

Can compactify

$$\begin{tikzcd} \mathcal{G}^r_d \arrow[r] \arrow[d] & \overline{\mathcal{G}}^r_d \arrow[d, "\sigma"] \\ \mathcal{M}_g \arrow[r, hook] & \overline{\mathcal{M}}_g \end{tikzcd}$$

"limit $g^r_d$'s"

... not a full compactification over $\overline{\mathcal{M}}_g$, throw away some subvarieties of high codimension.

e.g. $K_{0,2}$ : $\mathrm{Sym}^2 H^0(L) \to H^0(L^{\otimes 2})$

$K_{0,2}(C,L) \neq 0 \iff$ this map is non surjective

$K_{1,2}(C,L) \neq 0 \iff I_2(C,L) \otimes H^0(L) \to I_3(C,L)$ not surjective

not cut out by quadrics..

equivalent to Green-Lazarsfeld $N_p$ property.

Theorem $r = 2s+si+i$, $g = rs+s$ $d = rs+r$

$\exists$ 2 vector bundles over stack $\overline{\mathcal{G}}^r_d$, $A$ & $B$, of same rank & map $\phi: A \to B$

s.t. $\overline{U_i}$ = degeneration locus of $\phi$

& moreover $\overline{Z}_i = \sigma(\overline{U}_i)$,
with virtual class $[\overline{Z}_i] = [\sigma_* c_1(B-A)]$
$= a\lambda - b_0\delta_0 - \ldots - b_{[g/2]}\delta_{[g/2]}$
w/ explicitly given coefficients

$$\text{Slope} = \frac{6 f(s,i)}{s(i+2)\, g(s,i)} \quad \text{for complicated explicit } f, g$$

$$\leq 6 + \frac{12}{g+1} \text{ with equality} \iff s=1.$$

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$\boxed{s=1 \text{ case}}$ : $g = 2i+3$ $\quad d = 2g-2$, $r = g-1$

$\mathcal{G}^r_d = G^{g-1}_{2g-2}$ : ie only have canonical linear system,
$= \mathcal{M}_g$

$$\Rightarrow s(Z_{2i+3,i}) = 6 + \frac{12}{g+1}$$

get in fact Harris-Mumford divisor — USING Green's conjecture!

$Z_{2i+3,i} = \{[C] \in \mathcal{M}_{2i+3} : K_{i,2}(C, K_C) \neq 0\}$

$i=0$ : not projectively normal canonical curves
$\Longleftrightarrow$ hyperelliptic

$i=1$ : trigonal curves . .

$Z_{2i+3,i} = \{ C \in \mathcal{M}_{2i+3} \quad C \xrightarrow{i+2:1} \mathbb{P}^1 \}$

Green's conjecture on syzygies.
– so recover Harris-Mumford formula modulo Green's conjecture.

$\boxed{s=2}$ $g = 6i+10 \quad d = 9i+12 \quad s = 3i+4$

$\{ C \xrightarrow{g^{3i+4}_{9i+12}} \mathbb{P}^{3i+4} \text{ s.t. } K_{i,2} \neq 0 \}$

$i=0$ case $\quad s(\overline{Z}_{10,0}) = 7$ : this is the K3 divisor.

$i=2$ : $g=22 \quad Z_{22,2} =$ C of genus 22 $\hookrightarrow \mathbb{P}^{10}$
with $K_{2,2} \neq 0$

... it's cut out by quadrics, but look for a linear relation among quadrics

$s(\overline{Z}_{22,2}) = 6.500392\ldots$

Can apply this technique to $\mathcal{M}_{g,n}$, Pryms, moduli of K3s, etc...

Theorem $\mathcal{M}_{22}$ is of general type

Pf Construct effective divisor $D_{22}$ on $\overline{\mathcal{M}}_{22}$ with $s(D_{22}) < 6.5$.

Look at $\mathcal{G}^6_{25}$ over $\mathcal{M}_{22}$

(restrict to locus where $\mathcal{G}^6_{25} = W^6_{25}$)

... construct tautological bundles $\mathcal{E}, \mathcal{F}$,

$\mathcal{E}(C,L) = H^0(C,L)$, $\quad rk = 7$

$\mathcal{F}(C,L) = H^0(C,L^{\otimes 2})$ $\quad rk = 29$

$$\underset{rk\ 28}{\mathrm{Sym}^2 \mathcal{E}} \longrightarrow \underset{rk\ 29}{\mathcal{F}}$$

$\sigma^{-1}(c) = W^6_{25}(C)$ curve of large genus $> 10^6 \ldots$

$U$ = first degeneration locus : where $\mathrm{rk}\,\phi \le 27$, virtual codim $= 2$.

$D_{22} = \sigma_*(U)$ virtual divisor

<u>Theorem</u> $D_{22}$ is an honest divisor & $\overline{D_{22}}$

$$= \mathrm{const} \cdot \left( \frac{17121}{2636} \lambda - \delta_0 - \frac{14511}{2636} \delta_1 - \sum b_j \delta_j \right)$$

$b_j > 1$ $j \ge 2$

$$s(\overline{D_{22}}) = \frac{17121}{2636} = 6.49506\ldots < 6.5 \qquad \blacksquare$$