D. Auroux – Relative Fukaya, Relative Mirror

Note Title 3/12/2008

(after ideas of Kontsevich, Seidel, Hori-Vafa, ..)
part joint w/ Katzarkov & Orlov

Mirror symmetry for Calabi-Yau

$$(X, J, \Omega, \omega) \longleftrightarrow (X^\vee, J^\vee, \Omega^\vee, \omega^\vee)$$

complex, hol volume form, Kähler form

Relation: SYZ (after Kontsevich-Soibelman & Gross-Siebert):
dual torus fibrations with singularities over common base.

Recipe: 1 look for SLAG torus fibration (with singularities)

$$\begin{array}{ccc} T^n & \hookrightarrow & X \\ & & \downarrow f \\ & & B \end{array}$$

B = some moduli of SLAG tori in X

Structure: $T_b B = H^1(f^{-1}(b), \mathbb{R})$
$= H^{n-1}(f^{-1}(b), \mathbb{R})$
canonical identification

$\mathbb{Z}$ lattices $\Rightarrow$ natural integer affine structures on B:
two of them, & mirror symmetry switches them

(2) tentative mirror

$$\begin{array}{ccl} M & = & \{ (L, \nabla) \mid L \text{ Slag } T^n \subset X \\ & & \quad \& \ \nabla \text{ flat } U(1) \text{ connection/gauge} \} \\ \downarrow & & \quad \downarrow \\ B & \ni & L \end{array}$$

(3) Instanton corrections: modify above to make sense of the geometry near the singularities.

Homological Mirror Symmetry (Kontsevich)

$$D^b(\mathrm{Coh}\, X) \simeq D^\pi F(X^\vee)$$

(Karoubi completed derived) Fukaya category,

b $D^{\pi}\mathcal{F}(X) \simeq D^b Coh(X^\vee)$

Motivation for SYZ from this:

$\mathcal{O}_p \in D^b Coh\, X^\vee$ point objects

$\longleftrightarrow$ hope it corresponds to (L, ∇) on X. Look at moduli for this

so points of mirror naively should parametrize pairs L, ∇ on X.

Fukaya (A_∞) category $\mathcal{F}(X, \omega)$

X compact symplectic [roughly! cf. Seidel, FOOO]

Objects = pairs (L, ∇) L compact Lagrangian $\subset X$

∇ = flat connection on trivial unitary line bundle/L

(+ B field: should be projectively flat not flat; should have nontrivial bundles; should have coisotropic objects also...)

$$\mathrm{Hom}((L,\nabla),(L',\nabla')) \simeq CF^*(L,L')$$ Floer complex

$$= \mathbb{C}^{|L \cap L'|} \quad \text{when } L \pitchfork L'$$

(grading from Maslov index...)

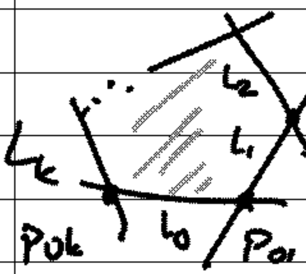
have differential, composition, ... k-fold compositions:

$$m_k : \mathrm{Hom}(L_0,L_1) \otimes \dots \otimes \mathrm{Hom}(L_{k-1},L_k) \longrightarrow \mathrm{Hom}(L_0,L_k)\ [2-k]$$

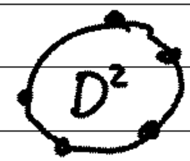
[$m_1 = \partial$, m_2 = product, ... m_k = Massey products]

$$m_k(p_{01}, p_{12}, \dots p_{k-1\,k})$$ intersection points

$$= \sum_{p_{0k} \in L_0 \cap L_k} \left(\sum_{\beta \in \pi_2(X, \cup L_i)} \# \mathcal{M}((p_{i\,i+1}),\beta)\, e^{-\int_\beta \omega}\, \mathrm{hol}_{\partial\beta}(\nabla) \right) p_{0k}$$



Choose J almost-complex structure



$k+1$ marked points

$\mathcal{M}((p_{i,i+1}), \beta)$ = moduli of pseudo-hol. discs of this type with class β

Today: (X, ω, J) smooth compact Kähler

$D \subset X$ anticanonical divisor $D \in |-K_X|$
with normal crossing singularities
(later will be smooth)

[X Fano works great, unknown for X of general type]

$X \setminus D$ open Calabi-Yau
Ω hol. volume form on $X \setminus D$ with poles on D
(weak CY: Ω does not have constant form)

Look for SLag T^n fibration, all fibers compact

$$\begin{array}{ccc} T^n & \longrightarrow & X \setminus D \\ & & \downarrow \\ & & B \end{array}$$

B affine base w/ singularities & boundary

$\rightsquigarrow$ build the mirror M = open CY
(+ instanton correction)

Points of $M \longleftrightarrow (L, \nabla)$, $L \subset X \setminus D$.

How to build the mirror to $\underline{\underline{X}}$ itself from this?

$\Rightarrow$ equip mirror M with a superpotential (LG model): $W : M \longrightarrow \mathbb{C}$
holomorphic function.

Meaning of W: what happens to tori L when we add the divisor?
their Floer homology gets messed up:
L will bound holomorphic discs passing through D

1, ... in fact Fukaya category has an m_0: <u>curved</u> A_∞ category, has an obstruction term:

For each Lagrangian have

$m_{0_{(L,\nabla)}} \in \mathrm{End}((L,\nabla))$ (ends of identity)

& $m_1^2 = m_2(m_{0_{L_1}}, -) - m_2(\cdot, m_{0_{L_0}})$

on $\mathrm{Hom}(L_0, L_1)$

instead of $m_1^2 = 0$ ...

In our case m_0 is a scalar multiple of the identity

$$m_0 = W(L,\nabla) \cdot Id_{(L,\nabla)}$$

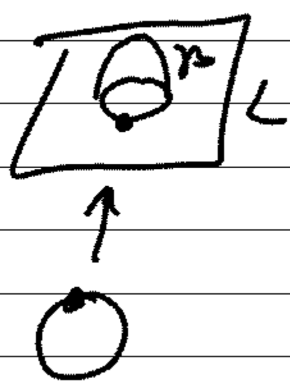
If we look only at SLAGs in CYs

Maslov index $= 0$ & don't have discs with nonzero degree ending on L,

but otherwise do have to face m_0.

Definition $W(L,\nabla) = \sum_{\substack{\beta \in \pi_2(X,L) \\ \mu(\beta)=2}} \deg\left(ev_* \mathcal{M}(\beta)\right) \cdot e^{-\int_\beta \omega} \cdot \mathrm{hol}_{\partial\beta}(\nabla)$

($\mu(\beta)=2 \Longleftrightarrow \beta \cdot D = 1$
when L is SLAG)



degree of n-chain on L
which is number of discs
through a given point of L

(moduli space has dim $n-3+$ Maslov index ..)

This is ill defined holomorphic function on $M = \{(L,\nabla)\}$ - have instanton corrections
= modifications of mirror as analytic space
which makes open GW invariants well defined.

[W depends on choice of point p & holomorphic structure — need to change M to get well defined function]

2. Even if $HF(L,L) = H^*(\mathrm{Hom}(L,L), \partial)$ well defined, it's usually $= 0$:

$HF(L,L) = 0$ unless (L,∇) is a critical point of W (ie category is localized on singularities of W: LG model ...]

$m_1^2 \neq 0$ <------> matrix factorizations.

HMS in this context (Kontsevich)

- $D^b Coh\, X \simeq D^\pi F(M, W)$

$= D^\pi F(M, F)$ relative Fukaya wrt fiber F of W

- $D^\pi F(X) \simeq D\mathcal{B}(M, W)$

Orlov category of singularities

From now on assume D is smooth.

$D \subset X$ is itself a CY hypersurface holomorphic volume form of D = residue of that on $X - D$. $\Omega_D = \mathrm{Res}_D \, \Omega$

So SYZ $\rightsquigarrow$ D carries its own SLag fibration (by T^{n-1})

e.g. X toric, $D = \bigcup$ toric strata [not smooth!]

$X \setminus D = (\mathbb{C}^*)^n$, $\Omega = \prod \frac{dz_i}{z_i}$

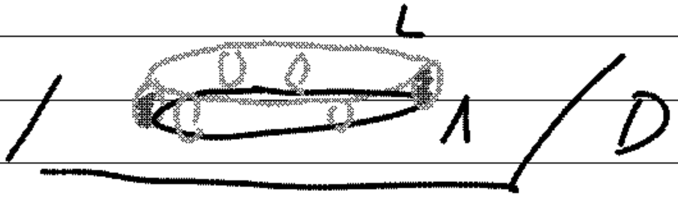
T^n orbits are SLag.

As we go to walls (ignore corners!) they collapse to T^{n-1}, orbits on boundary divisors.

<u>Conj</u> Near its boundary, the SLag T^n fibration on $X - D$ consists of tori:

S^1-fibered over SLag $T^{n-1} \subset D$

(ie $\exists$ SLag torus fibration with such properties...)



: L bounds hol discs along the fibers over Λ, & these are the minimal ones.

δ = class of meridian disc

$z_\delta = e^{-\int_\delta \omega} \mathrm{hol}_{\partial\delta}(\nabla)$ holo function on M near boundary

Near ∂, $W = z_\delta + O(1)$

∴ bounded roughly by 1 near ∂M.

$\partial M = \{ |z_\delta| = 1 \} \supset F = \{ z_\delta = 1 \}$
"fiber of W"

L is collapsed on D & ∇ is pulled back from $\Lambda \subset D$

So F = SYZ mirror to D !

Fact $\exists$ restriction functor from relative Fukaya

$$\mathcal{F}(M, F) \longrightarrow \mathcal{F}(F)$$

(on objects will be taking the boundary)

<u>Conjecture</u>

$$\begin{tikzcd} D^b \mathrm{Coh}\, X \arrow[r, "\simeq"] \arrow[d, "\text{restrict}"] & D^\pi \mathcal{F}(M, F) \arrow[d, "\text{restrict}"] \\ D^b \mathrm{Coh}\, D \arrow[r, "\simeq"] & D^\pi \mathcal{F}(F) \end{tikzcd}$$

<u>Theorem</u> (Auroux-Katzarkov-Orlov)
True for $X = \mathbb{CP}^2$ blown up at k points on cubic, D = proper transform of cubic
(on cubic this is Polishchuk-Zaslow)

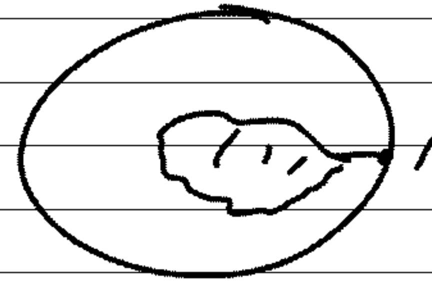
<u>Def</u> (Kontsevich, Seidel after Hori-Iqbal-Vafa)
M symplectic manifold with convex boundary, carries near ∂M a symplectic fibration

$$\begin{tikzcd} F \arrow[r] \arrow[d] & M \arrow[d, "f"] \\ 1 & \in D^2 \end{tikzcd}$$

(outside a compact subset)

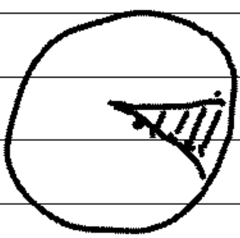
$L \subset M$ properly embedded Lag. submanifold is <u>admissible</u> if $\partial L \subset F$ (eg empty)

L near ∂, $f|_L \in \mathbb{R}_+$



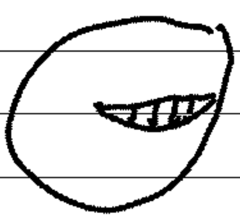
2. Hamiltonian perturbations:
$H: M \to \mathbb{R}$ supported near ∂M
$H = H(|f|)$ depends only on $|f|$

Φ^t = hamiltonian flow



Φ^t

<u>Def</u> Objects of $\mathcal{F}(M,f)$ are admissible (L,D)



$\tilde{\Phi}^t$

$\mathrm{Hom}_{\mathcal{F}(M,f)}(L,L') =$

$CF_*(L, \tilde{\Phi}^t(L'))$

$t>0$ small.

- no intersections near boundary

For <u>restriction</u>, need to fix the boundary pointwise $\Rightarrow$ use $\tilde{\Phi}$ to rotate

Restriction functor $(L, \nabla) \longmapsto (\partial L \subset F, \nabla|_{\partial L})$

On morphisms: perturb by $\widetilde{I}$ & count discs....

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mirror

$$\begin{tikzcd}
D^b \mathrm{Coh}\, X \arrow[d] \arrow[dr] & & & D^\pi \mathcal{F}(M,F) \arrow[d] \arrow[dr, "\mathrm{res}"] & \\
D^b \mathrm{Coh}\, D & D^b \mathrm{Coh}(X \setminus D) & & D^\pi \mathcal{F}(F) & D^\pi \mathcal{W}(M,w)
\end{tikzcd}$$

Abouzaid-Seidel

$$\mathrm{Hom}_{\mathcal{W}(M,F)}(L, L') = \lim_{t \to \infty} CF_*(L, \phi^t(L'))$$

rotate more & more:

mirror to $\otimes\, \mathcal{O}(D)$: sections over $X \setminus D$ are sections over $X \otimes \mathcal{O}(nD)$...