K. Costello - Calculations of categorical Gromov-Witten invariants

Note Title 3/10/2008

Def. A Calabi-Yau category of dim d is an A_∞ category $\mathcal{C}$ together with a nondegenerate symmetric pairing

$$\mathrm{Hom}^i(C, C') \otimes H^{d-i}(C', C) \longrightarrow \mathbb{k}$$

ground field

[implements Serre duality with trivial canonical bundle ....]

s.t. if

$$\begin{tikzcd} C_0 \arrow[r, "f_0"] & C_1 \arrow[r] & \cdots \arrow[r, "f_{n-1}"] & C_n \arrow[lll, bend left, "f_n"] \end{tikzcd}$$

$\langle f_0, m_n(f_1, \dots, f_n) \rangle$ is cyclically symmetric for n even, anti-sym n odd

$\rightsquigarrow$ e.g. $\langle f_0, m_2(f_1, f_2) \rangle = \langle f_2, m_2(f_0, f_1) \rangle$

Frobenius condition — looks like Frobenius algebra with many objects.

Examples 1. $\mathcal{C} = \mathbb{C}$ category with 1 object

2. X smooth projective CY variety,
$\mathcal{C} = D^{\infty}_{b}(X)$ A_∞-version of bounded derived category of coherent sheaves

3. X compact symplectic, $\mathcal{C} = \mathrm{Fuk}(X)$

4. $\mathcal{M}$ a 4-manifold w/ conformal class of metrics
$\mathcal{C}$ = category of Yang-Mills bundles on $\mathcal{M}$

To such a $\mathcal{C}$ can associate a 2d TFT (Kontsevich, Costello):

Let $\mathcal{M}(n,m) = \{ \Sigma$ a Riemann surface with $m+n$ analytically param. boundary components & each conn. component has ≥ 1 incoming $\}$

TFT give operations on Hochschild chains of $\mathcal{C}$ indexed by chains on $\mathcal{M}(n,m)$:

$$C_*(\mathcal{M}(n,m)) \otimes CH_*(\mathcal{C})^{\otimes n} \longrightarrow CH_*(\mathcal{C})^{\otimes m}$$

(or really something quasi-iso to $CH(\mathcal{C})^{\otimes n}$).

TFT $\rightsquigarrow$ a partition function $Z(\mathcal{C})$ which is roughly a generating function for $\int_{\mathcal{M}_{g,n}}$ of cochains coming from the TFT

Examples: $\mathcal{C} = \mathbb{C} \Rightarrow Z(\mathbb{C})$ is the generating function for integrals of ψ classes on Deligne-Mumford space — GW theory of a point. (Kontsevich - Witten).

$\mathcal{C} = \mathrm{Fuk}(X)$ X compact symplectic
Suppose the map $HH_*(\mathrm{Fuk}(X)) \longrightarrow H^*(X)$ is an isomorphism. [Kontsevich, Seidel]

<u>Conjecture</u> $Z(\mathcal{C})$ is the generating function for the GW invariants of X.
(Obstruction: fundamental chains on moduli spaces)

$\mathcal{C} = D^\infty_b(X)$, X a CY

$Z(\mathcal{C})$ is conjecturally the generating function for the B-model analog of GW.

Today: a program to compute $Z(\mathcal{C})$ from $\mathcal{C}$ explicitly.

What kind of an object is $Z(\mathcal{C})$?

Let $V = CH_*(\mathcal{C})$ (for simplicity take $d=0$)

This is a mixed complex: $D: V \to V$ Connes B-operator, $D^2 = [D, d] = 0$

- geometrically D is a circle action: it gives V an action of $H_*(S^1)$:

$(S^1)^{n+m} \circlearrowright \mathcal{M}(n,m)$ rotating the parametrizations, inducing an action of

$H_*(S^1)^{n+m}$ on $C_*(\mathcal{M}(n,m))$

& TFT operations intertwine the circle actions.

V^{S^1} homotopy invariants

$$:= R\mathrm{Hom}_{H_* S^1}(\mathbb{k}, V)$$

$$= (V[[t]], d_V + t \cdot D)$$

V_{S^1} homotopy coinvariants $= V \overset{L}{\underset{H_* S^1}{\otimes}} \mathbb{k} = (V[t^{-1}], d_V + tD)$

& $V_{Tate} = (V((t)), d_V + tD)$:

this is a symplectic vector space!

$\langle\,,\rangle : V^{\otimes 2} \longrightarrow \mathbb{k}$ symmetric pairing, induces a symplectic form

$$\Omega(v f(t), w g(t)) = \langle v, w\rangle \operatorname{Res} f(t) g(-t) dt$$

[Givental pairing]. This form is preserved by

$$d_{V_{Tate}} = d_V + tD.$$

Weyl algebra: $W =$ $\mathbb{k}((\hbar))$-algebra gen. by V_{Tate} & relation $[\alpha, \beta] = \hbar\, \Omega(\alpha, \beta)$

The Fock space $\mathcal{F} = W / W V^{S'}$:
$V^{S'}$ is an isotropic subspace of V_{Tate}

Rough answer: the partition function is roughly an element $Z \in \mathcal{F}[[\lambda]]$ closed under $d_{\mathcal{F}}$,

$Z = \exp\left(\sum \hbar^{g-1} \lambda^{2g-2+n} S_{g,n} \right)$

$S_{g,n}$ corresponds to $\mathcal{M}_{g,n}$

[$\mathcal{F} \sim \mathrm{Sym}\, t^{-1} V_{S'}$, half of $V_{Tate} = V^{S'} \oplus t^{-1} V_{S'}$]

<u>More precisely</u> The partition function is a formal deformation of the pair
$(\mathcal{F}, |\phi\rangle)$ Fock space & vacuum vector:
$\mathcal{F}$ is a W-module
$|0\rangle : W \longrightarrow \mathcal{F}$. , ie an

$W[[\lambda]]$-module $\mathcal{M}$ flat over $\mathbb{k}[[\lambda]]$
$\mathcal{M}/\lambda \mathcal{M} \simeq \mathcal{F}$, & an element $Z \in \mathcal{M}$,
$Z \bmod \lambda = |0\rangle$

e.g. $\mathcal{C} = \mathbb{C}$, $V = \mathbb{C}$, $V_{Tate} = \mathbb{C}((t))$

$F = \mathrm{Sym}^2\, t^{-1}\mathbb{C}[t^{-1}]$, $\mathcal{M} = F[[\lambda]]$.

In this case $Z = \exp\Big(\sum \lambda^{2g-2+n} \hbar^{g-1} \cdot$

$$\int_{\overline{\mathcal{M}}_{g,n}} (\psi_1^{k_1} \cdots \psi_n^{k_n})\, t_1^{-k_1-1} \cdots t^{-k_n-1} \Big)$$

<u>Theorem</u> There is a unique functorial way to associate to a CY category $\mathcal{C}$ a pair $(\mathcal{M}, Z)$ s.t.

1. This is functorial in a strong sense in $\mathcal{C}$
2. Grading $|\lambda| = -2$, $|\hbar| = +2$.
3. If we rescale the inner product on $\mathcal{C}$ by μ & λ by μ^{-1}, everything is unchanged
4. A certain normalization in semi classical limit mod λ^2 (ie of $S_{0,3}$)

Explanation of 1: functorial for CY categories enriched over dg symmetric monoidal categories

In practice: $(\mathcal{M}, Z)$ come from $\mathcal{M}_n, \langle , \rangle$ by some formula..

How to find the partition function?

1. Guess a construction ... maybe a matrix integral?

2. Compute: the universal partition function is an element of the dg Lie algebra constructed from graphs, compute term by term.

   e.g. $\int_{\overline{\mathcal{M}}_{1,1}} \psi = \frac{1}{24}$.

[Calculated in transparencies ...]