J. Kamnitzer - Categorification of Reshetikhin -

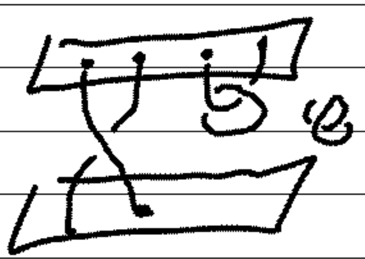
Note Title 3/13/2008

Turaev invariants of tangles + knots using derived categories of coherent sheaves on varieties arising from geometric Satake

Let $\mathfrak{g}/\mathbb{C}$ be a semisimple Lie algebra.
$U_q\mathfrak{g}$ quantum group

Λ_+ = dominant weights

Tangle: embedding of $S^1 \cup S^1 \cup \ldots \cup I \cup \ldots \cup I \to \mathbb{R}^2 \times [0,1]$



labelled by Λ_+ & oriented

To a labelled tangle T with $(\lambda_1, \ldots \lambda_n)$, $(\mu_1, \ldots \mu_m)$ labels at top & bottom $\Longrightarrow$

R-T invariant $\psi(T): V(\lambda_1) \otimes \ldots \otimes V(\lambda_n)$
$$\overset{\shortparallel}{V}(\underline{\lambda})$$

$$\longrightarrow V(\mu_1) \otimes \ldots \otimes V(\mu_m) = V(\underline{\mu})$$

map of $U_q\mathfrak{g}$ modules

... project T to the plane

$$\psi\left({}^{\lambda}\cup^{\lambda^\vee}\right) : V(\lambda)\otimes V(\lambda^\vee) \longrightarrow \mathbb{C}[q,q^{-1}]$$

$$\psi\left(\cap\right) : \mathbb{C}[q,q^{-1}] \longrightarrow V(\lambda)\otimes V(\lambda^\vee)$$

$$\psi\left({}^{\lambda}\times^{\mu}\right) : V(\lambda)\otimes V(\mu) \longrightarrow V(\mu)\otimes V(\lambda)$$

use R-matrix ....
check these satisfy all the Reidemeister moves

K link : $\psi(K) \in \mathbb{C}[q,q^{-1}]$ invariant
$\mathfrak{g} = \mathfrak{sl}_2 \Rightarrow$ get Jones polynomial

<u>Categorification</u> (Crane-Frenkel ; Khovanov)

Instead of $V(\underline{\lambda})$ associate a (graded) triangulated category $D(\underline{\lambda})$
& to a tangle T an (isom class of) functors $\psi(T) : D(\underline{\lambda}) \longrightarrow D(\underline{\mu})$

with $K_{\mathbb{C}}(D(\underline{\lambda})) \simeq V(\underline{\lambda})$ isom of vector spaces

so that $[\psi(T)]$ is the old $\psi(T)$.

For K a link $\rightsquigarrow$ an object of D(graded vect)

$\leftarrow\cdots\rightarrow$ bigraded vector space, with Euler characteristic $\in \mathbb{C}[q,q^{-1}]$ the old invariant.

- constructed by Khovanov for $\mathfrak{g} = \mathfrak{sl}_2$, λ = standard

  . . .

<u>Geometric Satake</u> $G^\vee$ Langlands dual group (adjoint)

$$Gr = G^\vee(\mathbb{C}((t))) / G^\vee(\mathbb{C}[[t]])$$

$= G^\vee$ bundles on $\mathbb{P}^1$ + triv. outside a point.

$$Gr \tilde{\times} \cdots \tilde{\times} Gr = \{ P_0, \dots P_n, \varphi_1, \dots \varphi_n : P_0 \xrightarrow[\varphi_1]{} P_1 \xrightarrow[\varphi_2]{} \cdots \xrightarrow[\varphi_n]{} P_n \}$$

isoms away from 0

$G^\vee(\mathbb{C}[[t]]) \curvearrowright Gr$, orbits labelled by $\lambda \in \Lambda_+ \; (Gr^\lambda)$

$\Rightarrow$ convolution map $Gr^{\lambda_1} \tilde{\times} \cdots \tilde{\times} Gr^{\lambda_n} \longrightarrow Gr$

$$Gr^{\lambda_1} \tilde{\times} \cdots \tilde{\times} Gr^{\lambda_n} =: Gr(\underline{\lambda})$$

Theorem (Ginzburg; Mirković–Vilonen)

There is an equivalence

$$\mathrm{Perv}_{G^\vee(\mathbb{C}[[t]])} Gr \xrightarrow{\sim} \mathrm{Rep}\, G$$

with $IC(\overline{Gr^\lambda}) \longleftrightarrow V(\lambda)$

$p_* IC_{\overline{Gr(\underline{\lambda})}} \longleftrightarrow V(\underline{\lambda})$

We'll assume λ_i minuscule (eg fund reps of $\mathfrak{sl}_n$)

$\Rightarrow \overline{Gr^{\lambda_i}} = Gr^{\lambda_i}$, smooth

get isomorphism $H_*(Gr(\underline{\lambda})) \simeq V(\underline{\lambda})$

e.g. $\mathfrak{g} = \mathfrak{sl}_2$, λ_i standard

$Gr^{\lambda_i} = \mathbb{P}^1$, $Gr(\underline{\lambda})$ iterated $\mathbb{P}^1$ bundle

Observe: $H_*(Gr(\underline{\lambda})) = K(DCoh(Gr(\underline{\lambda})))$

So choose $D(\underline{\lambda}) = DCoh(Gr\, \underline{\lambda})$

$$\mathrm{Hom}_{g\text{-rep}}(V(\underline{\lambda}), V(\underline{\mu})) = \mathrm{Hom}_{Perv}(P_* \mathbb{C}_{Gr_{\underline{\lambda}}}, P_* \mathbb{C}_{Gr_{\underline{\mu}}})$$

$$= H_{top}(Z(\underline{\lambda}, \underline{\mu}))$$

where $Z(\underline{\lambda}, \underline{\mu}) = Gr(\underline{\lambda}) \underset{Gr}{\times} Gr(\underline{\mu})$

We want to define $\psi(\mid\mid\mid\mid \overset{\lambda_i \; \lambda_{i+1}}{\times} \mid\mid\mid)$ $\quad \underline{\lambda} = (\lambda_1, \dots \lambda_n)$

$\underline{\lambda}' = (\lambda_1, \dots \lambda_{i+1}, \lambda_i, \dots \lambda_n)$

$$\rightsquigarrow Z_i(\underline{\lambda}) = \{ P_0, P_0' : P_j = P_j' \quad i \neq j \}$$

$$\subset Gr\,\underline{\lambda} \times Gr\,\underline{\lambda}'$$

$$= \left\{ \begin{tikzcd} P_0 \arrow[r, "\varphi_1"] & P_1 \arrow[r] & \cdots \arrow[r] & P_{i-1} \arrow[r] \arrow[dr] & P_i \arrow[r] & P_{i+1} \arrow[r] & \cdots \arrow[r] & P_n \\ & & & & P_i' \arrow[ur] & & & \end{tikzcd} \right\}$$

$$\Psi(\mid\mid\mid\mid \cup \mid\mid\mid) : \quad X_i(\underline{\lambda}) = \{ P_\cdot : P_{i-1} \to P_i \to P_{i+1} \text{ is an isom} \}$$

$$\downarrow$$

$$Gr(\lambda_1, \dots \lambda_{i-1}, \lambda_{i+2}, \dots \lambda_n)$$

$$\begin{tikzcd} X_1(\underline{\lambda}) \arrow[r, hook, "j"] \arrow[d, "q"] & Gr(\underline{\lambda}) \\ Gr(\underline{\lambda}') & \end{tikzcd}$$

$\Psi(||||\cup||) : DCoh(Gr\ \underline{\lambda})$
$\longrightarrow DCoh(Gr(\underline{\lambda}'))$

$\Psi = q_* j^*$.

$\Psi(|||||\cap|||) =$ adjoint.

<u>Theorem</u> (S. Cautis - K.) For $G = SL_m$ & $\lambda_i \in \{\omega_1, \omega_{m-1}\}$ this extends to

$$\{(\underline{\lambda}, \underline{\mu})\text{-tangles}\} \longrightarrow \left\{ \begin{array}{l} \text{isom classes of functors } DCoh^{\mathbb{C}^*}(Gr\ \underline{\lambda}) \\ \qquad \longrightarrow DCoh^{\mathbb{C}^*}(Gr\ \underline{\mu}) \end{array} \right\}$$

- When $G = SL_2$, $\Psi(K)$ agrees with Khovanov homology
- There is an isom $K\, DCoh(Gr(\underbrace{\omega_1 \ldots \omega_1}_{n})) \xrightarrow{\sim} V^{\otimes n}$ as $\mathbb{C}[q,q^{-1}]$-modules agreeing with Reshetikhin-Turaev
- For SL_m, K link $[\Psi(K)]$ is the R-T invariant.

[Note in particular $\Psi(\underline{\lambda}')$ is an equivalence of categories — uses results of Hörja, Arno-Bezrukavnikov, Seidel-Thomas, Kawamata-Namikawa]

"Bad news" $\mathfrak{g} = sl_4$ $\underline{\lambda} = (\omega_2, \omega_2)$
$\Rightarrow T_1(\underline{\lambda})$ is not an equivalence of categories (Namikawa) [correct using Chuang-Rouquier ?]
(w/ Cautis & T. Licata)

Symplectic geometry approach

Assume $\sum \lambda_i \in Q$ root lattice ($\Leftrightarrow$ rep has invariants)

${}_0Gr(\underline{\lambda}) = \{(P_\cdot) : P_n \text{ is trivial}\} \subset Gr(\underline{\lambda})$ open

$\mathfrak{g} = \underline{sl}_2$: these varieties are resolutions of Slodowy slices

${}_0Gr_{x_1,\dots x_n}(\underline{\lambda}) = \{(P_\cdot, \varphi_\cdot)$ sequences with $\varphi_i : P_{i-1} \to P_i$ isom off x_i Hecke type λ, P_n triv$\}$

${}_0Gr_{(\mathbb{P}^1)^n}(\underline{\lambda}) \longrightarrow (\mathbb{P}^1)^n$ locally trivial family off diagonals

$\mathfrak{sl}_2$: this is an open in a product of $\mathbb{P}^1$'s
$\longleftrightarrow$ reg ss matrices in a Slodowy slice

Theyre affine varieties, with middle homology
$V(\underline{\lambda})^G$

- reasonable to consider $DF({}_0Gr_{(x_1 \ldots x_n)}(\underline{\lambda}))$
derived Fukaya

<u>Theorem</u> (Seidel-Smith, Manolescu):
When $G = SL_m$ & $\lambda_i \in \{1, m-1\}$ there
is a map $\{(\underline{\lambda}, \underline{\mu})$-tangles$\} \longrightarrow$ functors $DF(\underline{\lambda}) \to DF(\underline{\mu})\}$