V. Lafforgue - Geometric Langlands on $\mathbb{P}^1$ & Arthur's SL_2

Note Title 3/14/2008

(starts from discussions w/ A. Genestier)

1. Ginzburg, Bezrukavnikov-Finkelberg on $D_{G(O)}(Gr)$
2. GL for $\mathbb{P}^1$
3. Constant sheaf on Bun_G
4. Action of $H^\bullet(Bun_G)$ on D-mod(Bun_G)

1. Geometric Satake: $Perv_{G(O)}(Gr) \simeq Rep({}^LG)$

Ginzburg, B.-F.: $D_{G(O)}(Gr) \simeq$

$$D^{perf}(\mathrm{Spec}\ \underbrace{\mathrm{Sym}^\bullet\, {}^L\mathfrak{g}[-2]}_{\text{dg algebra with differential } 0} / {}^LG)$$

$$IC_V \longmapsto \text{free object } V \otimes \mathrm{Sym}\, {}^L\mathfrak{g}[-2]$$

Equivalence of monoidal triangulated categories
On left convolution, on right tensor product

$$R\mathrm{Hom}_{D_{G(O)}(Gr)}(I(V), I(W)) = \left(V^* \otimes W \otimes \mathrm{Sym}\, {}^L\mathfrak{g}[-2] \right)^{{}^LG}$$

$$\begin{tikzcd} D_{G(O)}(Gr) \arrow[rr, "\sim"] \arrow[dr, "H^\bullet_{G(O)}"'] & & D^{perf}(\mathrm{Spec}\,\mathrm{Sym}\,{}^L\mathfrak{g}[-2]/{}^LG) \arrow[dl, "\times"] \\ & D^{perf}(H^\bullet_{G(O)}(Gr)) & \end{tikzcd}$$

use Kostant section

Ginzburg: $H^\bullet_{G(O)}(Gr) = U(\mathrm{Lie}\, J)$ as

- $\mathbb{Z}$-graded algebra;

J is the group scheme of regular centralizers, a flat commutative group scheme over ${}^L\mathfrak{g}^* /\!/ {}^LG = {}^L\mathfrak{t}^* /\!/ W$ Chevalley space

$$\begin{array}{c} I \\ \downarrow \\ {}^L\mathfrak{g}^* \end{array} \quad I_x = \{ g \in {}^LG,\ \mathrm{Ad}^* g\,(x) = x \}$$

nonflat noncommutative

but have $\chi : {}^L\mathfrak{g}^* \longrightarrow {}^L\mathfrak{g}^* /\!/ {}^LG$ ((Chevalley))

with $\chi^*(J) = I$ on ${}^L\mathfrak{g}^{*,reg}$

& $\chi^*(J) \longrightarrow I$ everywhere.

Kostant section $\sigma : {}^L\mathfrak{g}^* /\!/ {}^LG \longrightarrow {}^L\mathfrak{g}^{*,reg}$
gives a morphism of stacks,

$$\left[\left({}^L\mathfrak{g}^* /\!/ {}^LG \right) / J \right] \xrightarrow{[\sigma]} \left[{}^L\mathfrak{g}^* / {}^LG \right]$$

K above is global sections $\left({}^L\mathfrak{g}^* /\!/ {}^LG , \sigma^* M \right)$

& $U(\mathrm{Lie}\, J)$ acts by derivations of the action

$\mathbb{G}_m$ - equivariant morphism

$${}^L\mathfrak{g}^* /\!/ {}^LG \times {}^LG \longrightarrow {}^L\mathfrak{g}^*$$

$$(x, g) \longmapsto \mathrm{Ad}^*(g)(x)$$

where $\mathbb{G}_m$ acts on ${}^L\mathfrak{g}^*$ by $x \mapsto x^2$

& on Chevalley space in induced way,

& on ${}^LG$ by torus of a principal SL_2.

$$\mathrm{Sym}\,{}^L\mathfrak{g}[-2] \longrightarrow \left(\mathrm{Sym}\,{}^L\mathfrak{g}[-2]\right)^{{}^LG} \times \widetilde{\mathrm{Reg}\,{}^LG}$$

$$K(M) = \Big(M \otimes_{\mathrm{Sym}\,{}^L\mathfrak{g}[-2]} \big(\quad " \quad\big)\Big)^{{}^LG}$$

Koszul duality:

$$D^{\mathrm{perf}}\left(\mathrm{Spec}\ \mathrm{Sym}\,{}^L\mathfrak{g}[-2]/{}^LG\right)$$

$$= D^{b,\,\mathrm{fin\ type}}\Big(\underbrace{\mathrm{Spec}\ \mathrm{Sym}\,{}^L\mathfrak{g}^*[1]/{}^LG}_{\text{moduli space of local systems on } S^2.}\Big)$$

$$M \longmapsto M \otimes_{\mathrm{Sym}\,{}^L\mathfrak{g}[-2]} \mathbb{C}$$

<u>Remark</u> $H^\bullet_{G(O)}(Gr)$ acts on the category $D_{G(O)}(Gr)$

(H^k gives map $\mathrm{Id} \longrightarrow \mathrm{Id}[k]$)

$\Rightarrow$ $H^\bullet_{G(O)}(Gr) = U\,\mathrm{Lie}\,J$ acts on $D^{\mathrm{perf}}\left(\mathrm{Sym}\,{}^L\mathfrak{g}[-2]/{}^LG\right)$

by derivations along the gerbe:
for H a commutative group
$\mathfrak{h} = \mathrm{Lie}\, H$ acts on $\mathcal{O}$-modules on $\mathcal{B}_H$
(note $\chi^*(T) \longrightarrow I$ maps to the center of I)

$x \in X$ smooth projective curve $/\mathbb{C}$

p. 56 of Beilinson–Drinfeld on Hitchin system:
nice proof that $\mathrm{Locsys}_{{}^LG}$ is a local complete intersection:

$\mathrm{Locsys}_{{}^LG} \subset \mathrm{Locsys}_{{}^LG, x}$ (local systems with simple pole at x)

... in fact lives in smooth substack of $\mathrm{Locsys}_{{}^LG, x}$.

This embedding is cut out by the equation

$\mathrm{Res}_x = 0$: $\quad \mathrm{Locsys}_{{}^LG, x} \xrightarrow[\mathrm{Res}_x]{} {}^L\mathfrak{g}_x^c$

$X = \mathbb{P}^1$: $\mathrm{Locsys}_{{}^LG, X}^{\text{trivialized at } x} = pt$

$\Rightarrow \mathrm{Locsys}_{{}^LG}^{\text{triv at } x} = pt \times_{{}^L\mathfrak{g}} pt = \mathrm{Spec}\ \mathrm{Sym}\ {}^L\mathfrak{g}^*[1]$

$\Rightarrow \mathrm{Locsys}_{{}^LG} = pt \times_{{}^L\mathfrak{g}} pt \,/\, {}^LG$: isomorphism depends on choice of $x \in X$

Let's switch to the ℓ-adic setting:

$D_c^{b,0}(\mathrm{Bun}_G)$ $\quad X = \mathbb{P}^1$

$D_{G(O)}(G)$

we consider only extensions by 0 of sheaves on finite unions of strata to Bun_G.

"free of rank 1" over spherical Hecke algebra

Generator $j_! \mathbb{Q}_\ell$ $\quad j : BG \hookrightarrow \mathrm{Bun}_G$ trivial bundle

<u>Prop</u> $D_{G(O)}(G) \xrightarrow{\sim} D_c^{b,0}(\mathrm{Bun}_G)$

$$M \longmapsto M \otimes j_! \mathbb{Q}_\ell$$

$j_! \mathbb{Q}_\ell$ corresponds to the trivial module
over $A = \mathrm{Sym}\, {}^L\mathfrak{g}^*[1]$
(first Fourier coefficient)

For ${}^LG = SL_2$, $IC_{\mathrm{Sym}^a(\text{standard})}$
corresponds to extension by 0 of
$\mathbb{Q}_\ell$ on $\{\mathcal{O} \oplus \mathcal{O}(a), \mathcal{O}(1) \oplus \mathcal{O}(a-1) \ldots\}$ locus

$\mathbb{Q}_\ell$ on $\mathrm{Bun}_G \longleftrightarrow (\mathrm{Sym}\, {}^L\mathfrak{g}[-2])^{{}^LG} \otimes \widetilde{\mathrm{Reg}\, {}^LG}$
as object of $D^{\mathrm{perf}}(\mathrm{Sym}\, {}^L\mathfrak{g}[-2]/{}^LG)$
but is <u>acyclic</u> in $D^{\mathrm{perf}}(\mathrm{Sym}\, {}^L\mathfrak{g}^*[1]/{}^LG)$
– harder to see

X relative curve $\Rightarrow$ can consider
$\downarrow$ Langlands in families:
S $D(\mathcal{D}\text{-mod}(\mathrm{Bun}_G)) \longleftrightarrow$
$D((\mathcal{O},\mathcal{D})\text{-mod}(\mathrm{LocSys}_{{}^LG}))$

— $\mathcal{D}$-modules on total space of the family
$\longleftrightarrow$ $\mathcal{O}$-mod along fibers with $\mathcal{D}$-module
structure over base.

We'll use this for $\mathbb{P}^1/\mathbb{G}_m \longrightarrow B\mathbb{G}_m$.

. . .

<u>Question</u> $X/\mathbb{C}$ arbitrary,

$H^\bullet(Bun_G) \circlearrowright D(\mathcal{D}\text{-mod}(Bun_G))$

$\underset{?}{\longleftrightarrow} D(\mathcal{O}(Locsys_{{}^LG}))$

So want to guess an action
on other side — useful to see where
duality takes constant $\mathcal{D}$-module $\mathcal{O}$.

$Bun_G \times X \xrightarrow{ev} BG$

P G-invariant polynomial on $\mathfrak{g}$, degree k
$\Rightarrow c_p \in H^{2k}(BG)$

Believe all cohomology classes are Künneth
components of the inverse image
under ev of c_p.

$ev^*(c_p)$ acts on $D(D\text{-mod}(Bun_G \times X))$
as map $Id \longrightarrow Id[2k]$

Now consider $D((\mathcal{O},D)\text{-mod}(Locsys_{G^\vee} \times X))$

eg GL_n: $\begin{array}{c} E \\ \downarrow \\ Y \end{array}$ vector bundle

$$H^\bullet(\mathbb{P}(E)) = \frac{H^\bullet(Y)[y]}{y^n - c_1(E)y^{n-1} + \dots + (-1)^n c_n(E)}$$

$y = c_1(\mathcal{O}(1))$

$\Rightarrow$ Hecke correspondence

$$\begin{tikzcd} & \mathbb{P}(E) \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ Bun_G \times X & & Bun_G \times X \end{tikzcd}$$

$\Rightarrow H^\bullet(\mathbb{P}(E))$ acts on the Hecke functor
$H_{st} \longrightarrow H_{st}[2]$ of standard rep

$\Longleftrightarrow Id \longrightarrow H_{st^*} \circ_{st} [2]$
$= H_{adjoint}[2]$

Problem: Find $\mathrm{Id} \longrightarrow H_{{}^L\mathfrak{g}^*}[2]$ morphism from Id to coadjoint Hecke functor $\longrightarrow$ recovers all the components of the Chern classes this way.

Recall $R\mathrm{Hom}_{D_{G(\mathcal{O})}(Gr)}(IC_V, IC_W)$

$$= \left(V^* \otimes W \otimes \mathrm{Sym}^\bullet {}^L\mathfrak{g}[-2]\right)^{{}^LG}$$

For $V = 1$, $W = {}^L\mathfrak{g}^*$ get element of degree 1.

<u>Guess</u> for the corresponding map

$\mathrm{Id} \longrightarrow (\text{—} \otimes {}^L\mathfrak{g}^*, \varepsilon)$ Hecke functor on Galois side

on $D((\mathcal{O},D)\text{-mod}(\mathrm{Loc}S^\vee/S_{{}^LG\times k}))$:

$Locsys_{\check{G}} \subset Locsys_{\check{G},x}$

$\Rightarrow D(\mathcal{O}(Locsys_{\check{G}}))$

$= D(\mathcal{O}(\mathrm{Spec}\ \mathrm{Sym}_{Locsys_{\check{G},x}}\ \check{\mathfrak{g}}^{*,\varepsilon}_{x}[1]))$

dga $\qquad d\xi = \langle \xi, R \rangle$

Now say same in families: have horizontal connection on this sheaf of dg algebras:

$E \to Y$ vector bundle, s section

$Z = \{s=0\}$

$D(\mathcal{O}(Z)) = D(E, \mathrm{Spec}\ \mathrm{Sym}_{\mathcal{O}_Y} \underbrace{E^*[1]}_{\xi})$

$N \oplus N[1] \underset{\mathcal{O}_Y}{\otimes} E^*$

$\begin{pmatrix} \xi & 0 \\ s\xi & \xi \end{pmatrix}$

Case of $\mathbb{P}^1$: $\quad Locsys_{\check{G}} \times \mathbb{P}^1 = \mathrm{Spec}\ \mathrm{Sym}\ \check{\mathfrak{g}}[1]/\check{G} \times \mathbb{P}^1$

ison depends on points ....

$\mathcal{O} \in D(\mathcal{D}\text{-mod}(Bun_G))$

$\Longleftrightarrow \mathcal{N} \in D(\mathcal{O}(LocSys_{{}^LG}))$???

Dually consider the functor of global sections $D(\mathcal{D}\text{-mod}(Bun_G)) \xrightarrow[R\Gamma]{} D(Vect)$

(generalize restriction to Kostant section)

- use Koszul duality to understand formal neighborhood of trivial local system

$\simeq D(\mathcal{O}(Spec\, B/{}^LG))$

$B = \langle H^1(X) \otimes {}^L\mathfrak{g},\ H^2(X) \otimes {}^L\mathfrak{g} \rangle$

$[a \otimes x, b \otimes y] = ab \otimes [x,y]$

$G = GL_n$: $H_{St}(\mathcal{O}) = H^\bullet(\mathbb{P}(E))$

$= \mathcal{O} \otimes H^\bullet(\mathbb{P}^{n-1})$, action of y

is given by $\begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{pmatrix}$

$\Rightarrow$ N as above, expect

$$N \otimes St_{\mathcal{E}} = N \otimes H^\bullet(\mathbb{P}^{n-1})$$

standard

... take $N = \mathcal{S}_{H^\bullet(\mathbb{P}^{n-1}), \gamma}$

local system with fiber $H^\bullet(\mathbb{P}^{n-1})$ with nontrivial γ action

– in general $H^\bullet(\ast, \mathrm{Isom}(\mathcal{E}, -))$

$$\circlearrowleft \atop \gamma$$