A. Okounkov - Monodromy of the Quantum

Note Title 3/11/2008

Differential Equations for the Hilbert Scheme of Points on $\mathbb{C}^2$

(w/ R. Bezrukavnikov)

X variety, D divisor

(for us $X_n = \mathrm{Hilb}(\mathbb{C}^2, n)$, $\mathcal{O}(1) = c_1(\mathcal{O}/I)$)

Quantum differential equation

$\psi(q)$ function valued in $H^*(X)$

$$q\frac{d}{dq}\psi(q) = D * \psi(q)$$

$*$ = quantum multiplication:

q-deformation of product on $H^*(X)$

defined by $(\alpha * \beta, \gamma) = \sum_{\beta \in H_2(X)} q^{\beta D} \cdot \#\left\{\begin{matrix}\text{rat curves} \\ \text{of deg } \beta \\ \text{meeting} \\ \alpha^\vee, \beta^\vee, \gamma^\vee\end{matrix}\right\}$

$\#\{\text{curves}\}$: really $\int_{[M_{0,3}(X,\beta)]} (\text{---})$

We'll have $\psi \in \bigoplus H^*_{T^2}(\mathrm{Hilb}(\mathbb{C}^2, n))_{\text{localized}}$

$$\underset{\text{Nakajima}}{\cong} \mathbb{C}(t_1, t_2) \cdot \Lambda$$

where Λ = symmetric functions = $\mathbb{C}[p_1, p_2, \ldots]$

$p_k = \sum x_i^k$ power sum.

$n \longleftrightarrow$ degree of symmetric polynomial.

$\mathrm{Lie}\, T^2 = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right\}$

(we localize for convenience – in the end the answer is integral)

Operators α_{-k} = mult. by p_k $\qquad k > 0$

$$\alpha_k = k \frac{\partial}{\partial p_k}$$

Let M_D denote $D * \_\_$, quantum mult,

<u>Theorem</u> (Okounkov-Pandharipande)

$$M_D = (t_1+t_2) \sum_{k>0} \left[\frac{k}{2} \frac{(-q)^k+1}{(-q)^k-1} - \frac{1}{2} \frac{(-q)+1}{(-q)-1} \right] \alpha_{-k} \alpha_k$$

$$+ \frac{1}{2} \sum_{k,l>0} \left[t_1 t_2 \alpha_{k+l} \alpha_{-k} \alpha_{-l} - \alpha_{-k-l} \alpha_k \alpha_l \right]$$

ie operator is diagonal in the basis

$$P_\mu = \prod_i P_{\mu_i}$$

part studied by Lehn

So the diff eq has regular singularities, can study its monodromy

This interpolates between "all symmetric functions known to humans"

This monodromy controls all generating functions in equivariant GW / DT theory

Residues at $q = 0, \infty$

$\longrightarrow$ (second quantized) quantum Calogero-Moser hamiltonian ... near $0, \infty$ get just Schrödinger equation for quantum CM written in momenta coordinates

+ time dependent perturbation:

nonautonomous integrable system.

Residues at minus roots of unity:

$$\mathrm{Res}_q = (t_1 + t_2) \sum_{\substack{k \text{ s.t.} \\ (-q)^k = 1}} \alpha_{-k} \alpha_k$$

so exponents all belong to $(t_1 + t_2)\mathbb{Z}_{\geq 0}$

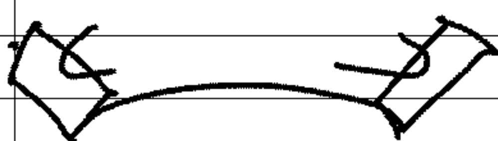
<u>Theorem</u> [OP] The monodromies for parameters (t_1, t_2) & $(t_1 + l, t_2 + m)$ for $(l, m) \in \mathbb{Z}^2$ are isomorphic

(for generic t_1, t_2 with known singularities)

- intertwining operator is the generating function for GW/DT invariants of $\mathcal{O}(l)\oplus\mathcal{O}(m)\longrightarrow\mathbb{P}^1$

relative to the fibers over $0,\infty$

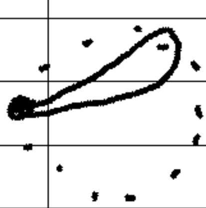
- ie count curves with specified intersections



$q=-1$ is actually a regular point, which is where we expand to get DT invariants so if we base monodromy at -1

& conjugate everything by

$$\Gamma P_\mu = \prod_i (2\pi i)\,\frac{\mu_i^{(1,-t_i)\mu_i}}{\Gamma(t_1\mu_i)\,\Gamma(t_2\mu_2)}\,P_\mu$$

-1  then the monodromy is a Laurent polynomial in $T_i = e^{2\pi i t_i}$

ie actually get <u>same</u> monodromy as we shift by integers

M_D is self adjoint (property of quantum product!) so monodromy is unitary wrt hermitian form

$$\langle P_\mu, P_\lambda \rangle = \delta_{\lambda\mu}\ \underset{\substack{\text{standard} \\ \text{factor}}}{z(\mu)} \prod_i \left(T_1^{\mu_i/2} - T_1^{-\mu_i/2}\right)\left(T_2^{\mu_i/2} - T_2^{-\mu_i/2}\right)$$

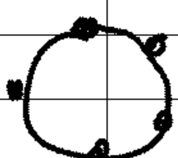
& $\overline{T_1} = T_1^{-1}$

..... this is the Macdonald inner product
= natural inner product on T-equivariant K-theory of the Hilbert schemes (Serre pairing)

Bezrukavnikov: monodromy should act by equivalences on $D^b_T(\mathrm{Hilb})$
& for every point of $\widetilde{\mathbb{C} \setminus \mathrm{sing}}$
should have a Bridgeland stability condition on $D^b_T(\mathrm{Hilb})$ & equivalences are deck transformations

--- cones (in progress) from quantization in characteristic p & rep theory of Cherednik algebras
[should be Picard-Fuchs on equivariant mirror]
- currently prove by calculating both sides.

We already know monodromy at $0, \infty$, so concentrate on roots of unity



look at universal cover $\widetilde{S^1}$ minus preimages of singular points.

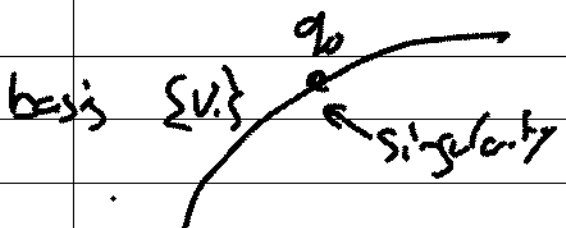
To every interval here we associate a basis of our vector space $K_{S^\bullet}$ so as to get basis of Schur functors at -1.

$$\dim_q : K \longrightarrow \mathbb{C}[\log q]$$

"central charge" $\quad f \longmapsto (f, (-q)^D)$

(D) is a nilpotent operator, $(-q)^D$ is a polynomial operator in $\log q$, can pair with $( )$

At $q=-1$ this is $S_\lambda \longleftrightarrow \dim(\lambda)$
dim of corresponding irrep

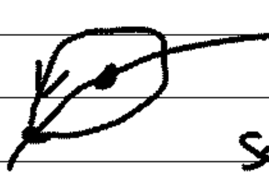


need (given basis $\{v_i\}$) to construct the monodromy R associated to the given singularity + the basis in the next interval.

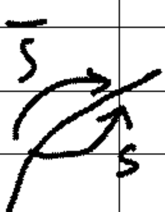
Assume the basis is ordered wrt order of vanishing of $\dim_q(v_i)$ as $q \to q_0$

R is then the unique unitary operator with increasing powers of T_1T_2 on diagonal

$$R = \begin{pmatrix} \ddots & & 0 \\ * & (T_1T_2)^{l_i} & \\ & & \ddots \end{pmatrix}$$ which is unitary.



This is monodromy – so we need something like a square root to carry our basis across the singularity



- want to factor $R = S\bar{S}^{-1}$
  - this will only be true <u>asymptotically</u>
- take Gauss-Bruhat-Birkhoff-Grothendieck-Riemann-Hilbert ....

factorization:

use variables $T = T_1 T_2$ & $\bar{T} = T_1/T_2$

$$R = \underbrace{\begin{pmatrix} 1 & & \\ & 1 & \\ T^{\geq 0} & & \ddots \end{pmatrix}}_{\bar{N}^{-1}} \underbrace{\begin{pmatrix} \ddots & & 0 \\ & T^{l_i} & \\ 0 & & \ddots \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ T^{\leq 0} & & 1 \end{pmatrix}}_{N}$$

New basis: apply $S = \sqrt{D}\, N$

This is basis of irreducibles in the t-structure associated to the corresponding stability condition.

$R \longleftrightarrow$ Serre functor

Powers of T tell us which Ext groups come in