D. Orlov – D-branes of type B in LG models

Note Title 3/12/2008

Landau-Ginzburg models: generalization of σ-model

σ-model: (X, ω, B) X smooth alg. variety / complex mfld
ω : Kähler form on X $(1,1)$
B : real closed 2-form

LG model : (X, ω, B, w)
add superpotential w: regular function on X
(so σ-model is case $w=0$)

(X,ω,B) with X not Calabi-Yau
$\Rightarrow$ its mirror will be an LG model not σ-model: w measures non Calabi-Yauness of X.

In σ-model have two categories:

- D-branes of type A = $D\mathrm{Fuk}(X,\omega,B)$ Fukaya
- D-branes of type B = $D^b\,\mathrm{Coh}\, X$

(or version with compact support)

D-branes of type B in LG model:
[type A: Fukaya-Seidel category...]
two definitions!

$\boxed{\text{Def. 1.}}$ Matrix factorizations (M. Kontsevich)

Y alg. variety over $\mathbb{A}^1$, Y affine $= \operatorname{Spec} A$

$\downarrow w$ $\quad DB_\lambda(Y, w)$: objects are

$\mathbb{A}^1 \ni \lambda$ $\quad$ 2-periodic sequences

$$\{ \dots \to P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} P_1 \to \dots\dots \}$$

of projective A-modules

s.t. $\quad p_0 p_1 = p_1 p_0 = (w - 1_A \cdot \lambda) = (w - \lambda)$

Morphisms $\operatorname{Mor}(P^\cdot, Q^\cdot) =$ 2-periodic morphisms up to 2-periodic homotopies

$DB_\lambda(Y, w)$ is triangulated.

$$DB(Y, w) := \prod_{\lambda \in \mathbb{A}^1} DB_\lambda(Y, w)$$

- note $DB_\lambda(Y,w) = 0$ if Y_λ is smooth.

$\boxed{\text{Def. 2.}}$ Triangulated category of singularities.

Let X be a noetherian scheme

$D^b(\mathrm{coh}\, X) \supset \mathrm{Perf}(X)$ perfect complexes:

locally quasi isomorphic to a bounded complex of vector bundles.

<u>Def</u> $D_{sing}(X) = D^b(\mathrm{coh}\, X) / \mathrm{Perf}\, X$

<u>Theorem</u> $DB_\lambda(Y,w)$ (as defined above)

$\cong D_{sing}(Y_\lambda)$ singularities on fiber.

So $DB(Y,w) = \prod_{\lambda \in \mathbb{A}^1} D_{sing}(Y_\lambda)$

$\left[Y_\lambda \text{ smooth} \Rightarrow \text{all } D^b(\mathrm{coh}) \text{ are perfect} \right]$

In general case $D_{sing}(X)$ is not 2-periodic

- true for fiber of morphism on smooth variety.

Y with action of $G \Rightarrow$

$$D^G_{sing}(X) = D^b(\mathrm{coh}^G X) / \mathrm{Perf}^G X$$

$(X, w, B, w=0)$:

$$D^b(\mathrm{coh}\, X) \quad \text{vs} \quad DB(X, w=0) \; ?$$

consider trivial $\mathbb{C}^*$ action on X
& standard $\mathbb{C}^*$ action on $\mathbb{A}^1$

Fiber of
$$\begin{array}{c} X \\ \downarrow{\scriptstyle w=0} \\ \mathbb{A}^1 \end{array}$$
is the dg scheme $(X, \mathcal{P})$

$\mathcal{P}$ = sheaf of dgas

$$\mathcal{P} = \underset{\text{nontrivial } \mathbb{C}^* \text{ action}}{\mathcal{O}_X(-1)} \xrightarrow{d=0} \mathcal{O}_X$$

Thm $D^{b,\mathbb{C}^*}(\mathrm{coh}\, \mathcal{P}^\bullet) / \mathrm{Perf}^{\mathbb{C}^*}(\mathcal{P}^\bullet)$ bounded $\mathcal{P}^\bullet$-modules

$$= D^b(\mathrm{coh}\, X)$$

So (X,ω,B) σ-model is the same as LG-model with $W=0$ & $\mathbb{C}^* \curvearrowright X$

Non $\mathbb{C}^*$ equivariant version
$$D^b(\mathrm{coh}\, P)/\mathrm{Perf}(P) = D^{\mathbb{Z}/2}(\mathrm{coh}\, X)$$
2-periodic version.

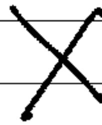
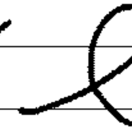
X noetherian, separated, finite Krull dim, with enough locally free sheaves (ie every $\mathcal{F}$ coherent has v.b $\twoheadrightarrow \mathcal{F}$)
– eg X quasiprojective.

$\Rightarrow$ perfect complexes on X are <u>globally</u> quasi-isom to bounded complexes of vector bundles.

<u>Fact</u> $U \xrightarrow{j} X$ s.t. $\mathrm{Sing}\, X \subset U$
Zariski open

$\Rightarrow j^* : D^b(\mathrm{coh}\, X) \to D^b(\mathrm{coh}\, U)$ induces
$$D_{sing}(X) \xrightarrow{\sim} D_{sing}(U)$$

However analytically isomorphic singularities <u>don't</u> have same $D_{sing}(X)$:

e.g. $D_{sing}($  $) \not\cong D_{sing}($  $)$:

$D_{sing}(X)$ is not necessarily idempotent complete:

$$A \xrightarrow{p} A \quad \text{with } p^2 = p,$$

can ask if $A = A_0 \oplus A_1$ with $p_0 = 0$ & $p_1 = id$.

Can take Karoubian envelope, adding all kernels of projectors

$\Rightarrow$ $\overline{D_{sing}(X)}$, triangulated category (triangulated)

<u>Theorem</u> (Thomason) Let T be a triangulated category [essentially small] $\Rightarrow$ $\exists$ 1-1 correspondence between strictly full triangulated subcategories $A \subset T$ with Karoubian envelope $\overline{A} = T$, & subgroups $H \subset K_0(T)$

[$H = \mathrm{Im}\ K_0(\mathcal{A}) \longrightarrow K_0(T)$,

$\mathcal{A}$ = objects with class in H]

e.g. $E \oplus E[1]$ for ~~any~~ $E \in T$ ...

So to know $D_{sing}(X) \Longleftrightarrow$

to know $\overline{D_{sing}(X)}$ & $K_0(D_{sing}(X)) \subset K_0(\overline{D_{sing}(X)})$

Let $\hat{X}$ = formal completion of X at the singular locus.

Theorem X_1, X_2 as above & $\hat{X}_1 \simeq \hat{X}_2$

$\Longrightarrow \overline{D_{sing}(X_1)} \simeq \overline{D_{sing}(X_2)}$

idea $Z \hookrightarrow X$ closed

$$\begin{array}{ccc} D^b_Z(\mathrm{coh}\, X) & \subset & D^b(\mathrm{coh}\, X) \\ \cup & & \cup \\ \mathrm{Perf}_Z(X) & \subset & \mathrm{Perf}(X) \end{array}$$

: cohomological support on Z

proof is sequence of lemmas:

a. Lemma $A \in D_Z(\mathrm{coh}\, X)$ belongs to $\mathrm{Perf}_Z(X)$

$\iff \forall B \in D^b_Z(X)$, $\mathrm{Hom}(A, B[i]) = 0$

except for finitely many $i \in \mathbb{Z}$.

b. Lemma: $D^b_Z(X) \hookrightarrow D^b(X)$ induces

fully faithful $D_Z(X)/\mathrm{Perf}_Z(X) \hookrightarrow D_{sing}(X)$

c. Lemma $Z_1 \subset X_1$, $Z_2 \subset X_2$ s.t.

$\hat{X}_{1,Z_1} \simeq \hat{X}_{2,Z_2} \Rightarrow D^b_{Z_1}(X_1) \simeq D^b_{Z_2}(X_2)$

d. Lemma Any object in $D_{sing}(X)$ is a direct summand of an object in

$D^b_{Z=sing}(X) / \mathrm{Perf}_{Z=sing}(X)$, so the latter

has Karoubi completion $\overline{D_{sing}(X)}$.

$\Rightarrow$ categories $D^b_{sing}(X)/\mathrm{Perf}(X) \subset D_{sing}(X) \subset \overline{D_{sing}(X)}$

... don't have to agree in general!

Let S be a smooth scheme, $\mathcal{E}$ vector bdl,
$s \in \Gamma(S, \mathcal{E})$.

$X \subset S$ zero subscheme of s, assume regular ($\dim X = \operatorname{rk} \mathcal{E}$)

s gives section of $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E}^\vee)$

$\Rightarrow$ can take its zero section Y

$$\begin{tikzcd} Z = \mathbb{P}(\mathcal{E}^\vee|_X) \arrow[r, hook, "i"] \arrow[d, "p"] & Y \arrow[r, hook] & \mathbb{P}(\mathcal{E}^\vee) \arrow[d] \\ X \arrow[rr, hook] & & S \end{tikzcd}$$

$\varphi_Z := Ri_* p^* : D^b(X) \longrightarrow D^b(Y)$

induces $\overline{\varphi_Z} : D_{sing}(X) \xrightarrow{\sim} D_{sing}(Y)$

<u>Theorem</u> $\overline{\varphi_Z}$ is an equivalence

[though $D_{sing}(-)/\mathrm{perf}(\,)$ not nec. equivalent!]

<u>Application</u> S smooth variety, f, g functions

$D = \{g = 0\}$ smooth

$\Rightarrow$ two LG models

$$\begin{array}{ccc} D & & T = S \times \mathbb{A}^1_x \\ \downarrow f|_D & & \downarrow w = f + xg \\ \mathbb{A}^1 & & \mathbb{A}^1 \end{array}$$

<u>Theorem</u> $DB(D, f|_D) \simeq DB(T, w)$
equiv of LG models

<u>Example</u> Knörrer periodicity:

$$D_{sing}(f = 0) \simeq D_{sing}(f + x^2 + y^2 = 0)$$

<u>Remark</u>

$$\begin{array}{ccc} Y & & Y' \\ \downarrow w & & \downarrow w' \\ \mathbb{A}^1 & & \mathbb{A}^1 \end{array}$$

Suppose given $D^b(Y) \simeq D^b(Y')$

given by an object $\mathcal{E}$ on $\underset{\mathbb{A}^1}{Y \times Y'}$

$\Rightarrow$ functors $D^b(Y_\lambda) \xrightarrow{\sim} D^b(Y'_\lambda)$

$\Rightarrow$ $D_{sing}(Y_\lambda) \xrightarrow{\sim} D_{sing}(Y'_\lambda)$

... e.g. consider a flop inside a fiber, get an equivalent LG model.

Mirror symmetry for genus 2 curves:

C genus 2 $\subset \mathbb{P}^1 \times \mathbb{P}^1$ degree $(2,3)$

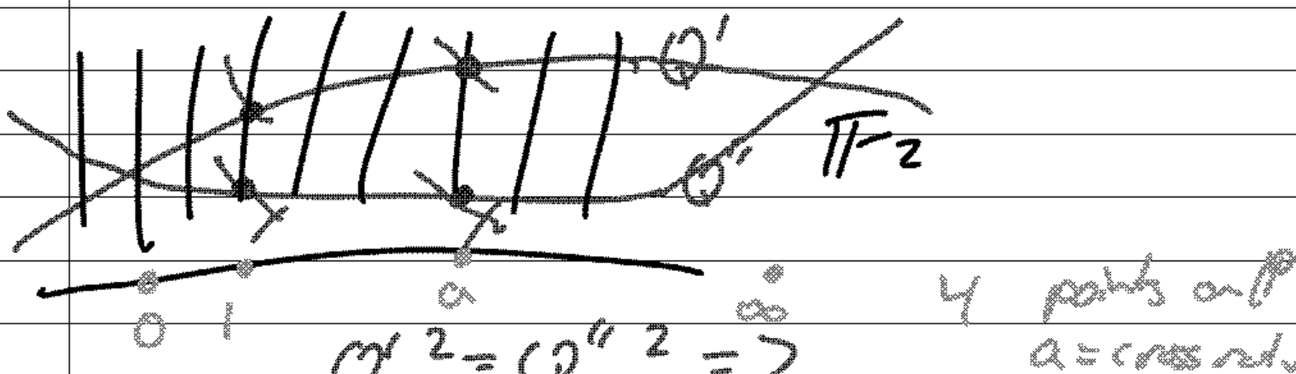
$\rightsquigarrow$ (Hori-Vafa procedure) affine variety with superpotential (& partial completion)

$\rightsquigarrow$ $\begin{array}{c} Y \\ \downarrow w \\ \mathbb{A}^1 \end{array}$ $\dim Y = 3$, Y_λ are K3s, general Y_λ has Picard number 18

Fiber over 0:

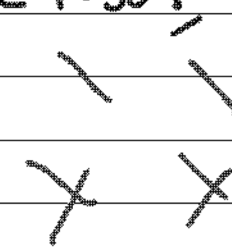
$Y_0 = S_0 \cup S_1 \cup S_2$

union of three isomorphic surfaces, each a blowup of Hirzebruch $\mathbb{F}_2$ six times



$Q'^2 = Q''^2 = 2$

: : = 4 points in 2 fibers
blow up six times



$Q_i' = Q_{i+1}''$ i mod 3

$\Rightarrow$ banana w/ 3 peels

$Y_0 =$ 

Conj $\overline{D\mathrm{sing}(Y_0)} = D\mathrm{Fuk}(C)$

Remark: $D_{sing}(Y_0(a))$ depends on a
= crossratio

$$K_0(D_{sing}\, Y_0(a)) = \underset{H^1(C,\mathbb{Z})}{\underset{\shortparallel}{\mathbb{Z}^4}} \oplus \begin{cases} \mathbb{Z} & \operatorname{ord}(a) = \infty \\ \mathbb{Z}/2n & a^n = 1 \end{cases}$$

Parameter a appears as class of ω, B in Fukaya category.

<u>Conjecturally</u> $\overline{D_{sing}(Y_0(a))}$ doesn't depend on a

$(a \neq -1)$ & $K_0(\overline{D_{sing}}) = \mathbb{Z}^4 \oplus \mathbb{Z}/2 \oplus \underset{a}{\underset{\shortparallel}{\mathbb{C}^*}}$

independent of a

Fukaya category also independent:
can add any multiple of <u>canonical class</u>
to $B + i\omega$ without affecting Fuk
(conjecturally)

'' V. bundles on C genus 2

Let $X = Q_1 \cap Q_2 \subset \mathbb{P}^5$ Fano

$\longleftrightarrow$ (Hori–Va (-a))

$$\begin{array}{c} Z \\ \downarrow w \\ \mathbb{A}^1 \end{array}$$

K3 fibration with generic Pic # = 19

$Z_0 = Y_0 \ (a = -1)$

$D\,\mathrm{Fuk}(X) \simeq \langle D_{sg}(Z_0)\} / \langle \mathrm{Vect}^1_{\mathbb{C}} \rangle, \langle \mathrm{Vect}\ \mathbb{C} \rangle$

<u>Theorem</u> $D^b\mathrm{Coh}\ Q_1 \cap Q_2 = \langle D^b\mathrm{Coh}\ C, \mathcal{O}, \mathcal{O}(1) \rangle$

semiorthogonal