Note Title 3/10/2008

R. Rouquier - Higher Representation Theory

4d TQFT (Crane-Frenkel etc)

2-representation theory

Study of categories
with actions & their relations

Variety $X \rightsquigarrow$ moduli space of sheaves $\rightsquigarrow$ numerical invariants

Cat. of sheaves on $X \stackrel{?}{\Longrightarrow}$ Cat. of sheaves on moduli

Can we purely algebraically go from the category of sheaves on X to the category of sheaves on moduli? cut out geometry altogether!

Fix k alg. closed field
$\mathfrak{g}$ complex semisimple (or symmetrizable Kac-Moody) Lie algebra
Can construct a monoidal category $\mathcal{A}(\mathfrak{g})/_k$ generated by objects E_i, F_i
and with $\mathbb{C} \otimes K_0(\mathcal{A}(\mathfrak{g})) =: K_{\mathbb{C}}\mathcal{A}(\mathfrak{g}) \simeq U\mathfrak{g}$
split Grothendieck group

Conjectural properties: Let $\mathcal{V}$ be a triangulated category acted on by $\mathcal{A}(\mathfrak{g})$, i.e. have monoidal functor $\mathcal{A} \longrightarrow \text{Triang.Fun}(\mathcal{V}, \mathcal{V})$, & assume $V = K_{\mathbb{C}}(\mathcal{V})$ is locally finite as a rep of $U\mathfrak{g}$

Then 1. $\exists\ \mathcal{V} = \bigoplus_{\lambda} \mathcal{V}_{\lambda}$ corresponding to the weight space decomposition of V

2. $\exists$ braid group action on $\mathcal{V}$ inducing usual action of Weyl group (or rather extended by an elementary 2-group...)

3. $\forall\ L$ irr. f.d rep of $\mathfrak{g}$, $\exists!$ minimal $\mathcal{V}(L)$ acted on by $\mathcal{A}$, with $K_{\mathbb{C}} \mathcal{V}(L) = L$

4. $\exists$ exhaustive filtration (canonical) $0 = \mathcal{V}\{0\} \subset \mathcal{V}\{1\} \subset \dots \subset \mathcal{V}$ by sub $\mathcal{A}$-modules (thick triangulated subcategories) with $K_{\mathbb{C}}(\mathcal{V}\{i\} / \mathcal{V}\{i-1\})$ isotypic

5. If V is a multiple of L simple $\Rightarrow$
$V \simeq$ "$\mathcal{M} \otimes V(L)$", $\mathcal{M}$ = "multiplicity category" with trivial action of $\mathcal{A}$.

6. There exists a tensor product of $\mathcal{A}$-modules making rep$(\mathcal{A})$ into a braided monoidal 2-category. $\underset{?}{\rightsquigarrow}$ 4d TQFT

Quantum version: category should be graded, with $K_{\mathcal{C}} = U_q\mathfrak{g}$:

For abelian categories expect abelian version $V_{ab}(L)$ with $D(V_{ab}(L)) = V(L)$

a) Kashiwara operators:
S simple object $\rightsquigarrow$ take composition factors of $E_i * S$, get crystal operators

b) If char $k=0$ then $V_{ab}(L)$ is graded & $\{[S]$ S simple$\}$ is the canonical basis & $V_{ab}(L)$ is Koszul

Construction in type A

$S \subset k$. Construct a quiver with vertices S & edge between $i \bullet\!\!-\!\!\bullet\, i+1$ when both in S

$\Rightarrow \mathfrak{sl}_S = \bigoplus \mathfrak{sl}_I$ $\quad I$ connected components

Depending on the type of intervals S get

$\mathfrak{sl}_\infty$, $\mathfrak{sl}_n$, $\hat{\mathfrak{sl}}_p$ when char $k = p$.

[Multiplicative version: $q \in k \setminus \{0,1\}$

$S \subset k \setminus \{0\}$ $\quad i \bullet\!\!-\!\!\bullet\, q \cdot i$ get $\mathfrak{sl}_n$ or $\hat{\mathfrak{sl}}_n$]

Def Let V be an abelian category /k

Assume every object has a finite composition series. A 2-representation of $\mathfrak{sl}_S$ on V is the data of [principal $\mathfrak{sl}_2$]

1. E, F exact functors $V \longrightarrow V$

2. (E,F) made into an adjoint pair

3. $X \in \operatorname{End} E$, $T \in \operatorname{End} E^2$

Let E_i $(i \in I)$ to be the generalized i-eigenspace of X on E (split $E(-)$ as sum of eigenspaces, $E = \oplus E_i$)
& same for F

Conditions

1. $\{i : E_i \neq 0\} \subset S$
2. $[E_i], [F_i]$ on V give a loc finite actn of $\mathfrak{sl}_S$
3. F is isomorphic to a left adjoint of E (the other adjunction – but not fixed isomorphism)
4. $[S]$ is a weight vector $\forall S$ simple
5. $$\begin{tikzcd}[column sep=small] & EEE \arrow[dl, "ET"'] \arrow[dr, "TE"] & \\ EEE \arrow[d, "TE"'] & \circlearrowleft & EEE \arrow[d, "ET"] \\ EEE \arrow[dr, "ET"'] & & EEE \arrow[dl, "TE"] \\ & EEE & \end{tikzcd}$$
6. $(T+1)(T-q)=0$ mult. case (additive case: $T^2=1$)
7. $T \circ (EX) \circ T = XE - T$ (add.) $q(XE)$ (mult.)

Note: get map from affine (mult. case) or degenerate affine (add. case) Hecke algebra $H_n \longrightarrow \operatorname{End} E^n$

T = usual braiding on tensor product, making it commutative

$X \longleftrightarrow$ "twist" of some kind

<u>Examples</u>

1. N f.d vector space / $k = \mathbb{C}$

Look at category $\mathfrak{gl}(N)$-mod $= \mathcal{V}$

Functor $E = N \otimes$ ——

$M \in \mathfrak{gl}(N)$-mod $\quad \underset{\substack{\shortparallel \\ N^* \otimes N}}{\mathfrak{gl}(N)} \otimes M \longrightarrow M$

$\Rightarrow$ get endomorphism $X(M) : N \otimes M \longrightarrow N \otimes M$

$T : N \otimes N \otimes M$

eg $S = \mathbb{C}$ $\quad \mathcal{V}$ = BGG category $\mathcal{O}$

Degenerate affine Hecke algebra action here is due to Arakawa-Suzuki, explained via Schur-Weyl.

Example 2 $\quad V = \bigoplus_{n\geq 0} \overline{\mathbb{F}_p} S_n\text{-mod}$

$E = \bigoplus \mathrm{Ind}_{S_n}^{S_{n+1}}, \quad X = (1n) + \dots + (n-1, n)$

Jucys-Murphy element of group algebra

T comes from S_2 commuting with induction
$S_n \longrightarrow S_{n+2} \Rightarrow \boxed{\widehat{sl_p}}$ action

(Ariki, Grojnowski, ...)

Lascoux-Leclerc-Thibon conjecture on Canonical bases

Example 3 $\quad G = SL_n \mathbb{C}$

$V = \hat{\bigoplus}_{i=0} D^b_B(Gr(i, \mathbb{C}^n))$

action of E_i, F_i by correspondences, $\quad V = (\mathbb{C}^2)^{\otimes n}$

X is c_1 (line bundle) ... get sl_2 action

Case of sl_2:

<u>Theorem</u> (Chuang–Rouquier)

For V a 2-rep of sl_2

$\exists\ \Theta : D^b(V) \circlearrowleft$ categorifying

$[\Theta] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. ($\rightsquigarrow$ solve problems in rep theory of symmetric group.

filtration & minimality of $V(L)$ hold in this case)