# P. Seidel - Fukaya categories & $A_n$ quiver algebras

Note Title 3/12/2008

<u>Example</u>

$$X_m^n = \{ p(x_0) + x_1^2 + \cdots + x_n^2 = 0 \} \subset \mathbb{C}^{n+1}$$

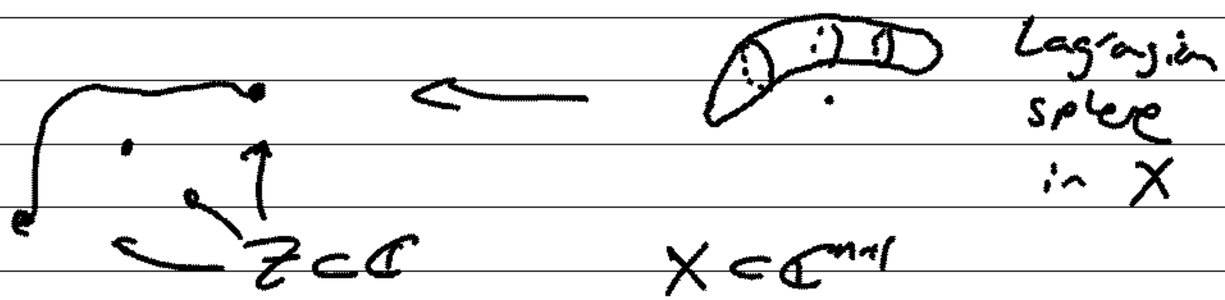
$p$ is a generic polynomial of deg $m+1 \geq 3$

Let $Z = p^{-1}(0) \subset \mathbb{C}$.

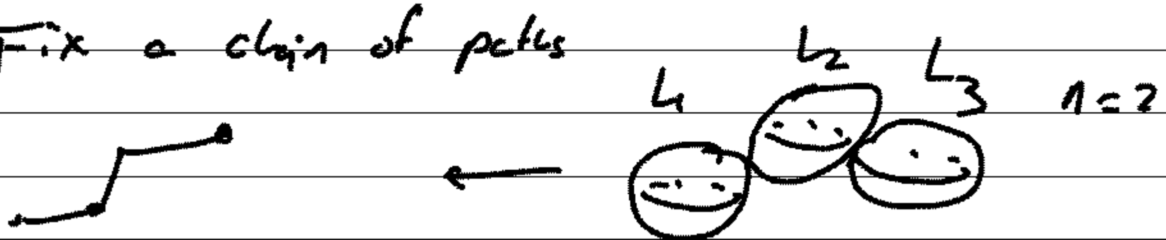
For $n=1$ this is the double branched cover of $\mathbb{C}$ branched along $Z$

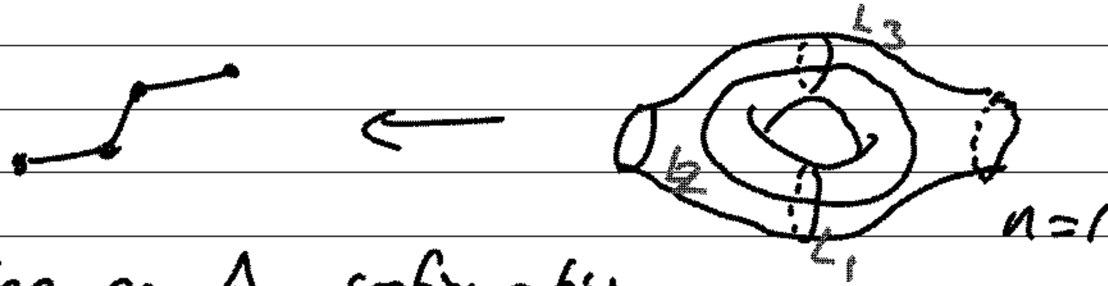
Pencil of conics becoming singular along $Z$
— Milnor fiber of $n$-dim $A_m$ singularity

A sketch of the symplectic geometry:




Fix a chain of paths






See an $A_m$ configuration of spheres over this chain of paths in $\mathbb{C}$.

Let $\mathcal{F}(X_m^n)$ be the Fukaya category

$\cup$

$\mathcal{F}_m^n$ = full subcategory with objects $L_1, \dots, L_m$

<u>Lemma</u> The $L_i$ are idempotent generators of $\mathcal{F}(X_m^n)$

$\Rightarrow$ have full & faithful functor

$$\mathcal{F}(X_m^n) \xrightarrow[\text{Yoneda}]{} \operatorname{mod} \mathcal{F}(X_m^n) \xrightarrow[\text{restrict}]{} \operatorname{mod}(\mathcal{F}_m^n)$$

mod = functors to chain complexes

– ie can see all $\mathcal{F}(X_m^n)$ using $\mathcal{F}_m^n$.

For completeness we temporarily pass to the cohomological level

$$H(\mathcal{F}_m^n) = \bigoplus_{i,j=1}^{n} HF^*(L_i, L_j)$$

... we're identifying categories with $m$ objects as algebras over $R_m = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{m}$

Moreover, the restriction functor maps $L$ to $\mathcal{M}_L$ such that

$$H^*(\mathcal{M}_L) = \bigoplus_{i=1}^{n} HF^*(L_i, L)$$

[note $HF^*(L, L) =$ ordinary $H^*(L)$

$L_1 \pitchfork L_2 \Rightarrow HF^*$ computed by intersection points,

one intersection point $\Rightarrow$ no room for differentials ! etc.]

Even more explicitly, take $A_m^n$ to be the quiver algebra with relations

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend right, "0"'] & \underset{2}{\bullet} \arrow[l, bend right, "n"'] \arrow[r, bend right, "0"'] & \cdots \arrow[l, bend right, "n"'] \arrow[r, bend right, "0"'] & \underset{m}{\bullet} \arrow[l, bend right, "n"'] \end{tikzcd}$$

$\overset{n}{\curvearrowleft}$ : degree $n$ arrow

$\underset{0}{\curvearrowright}$ : deg 0 relations : $(i, i+1, i+2) = 0$

$$(i, i-1, i-2) = 0$$

$$(i, i+1, i) = \pm(i, i-1, i)$$

- ie can't go two steps in any direction etc

<u>Lemma</u> $H(F^n_m) \cong A^n_m$ :

eg dependence on $n$ comes only through the <u>degrees</u>.

<u>Remaining question</u> Determine $F^n_m$ among the quasi isom classes of $A_\infty$ algebras with cohomology $A^n_m$.

Old theory to solve such classification problems - problem in <u>deformation theory</u>.

Given $\mathcal{A}$ $A_\infty$ algebra, $H(\mathcal{A}) = A$, can find a quasi isom $A_\infty$ algebra with vanishing differential

This can be thought of as living <u>on</u> <u>A itself</u>, so start with ordinary product

$$\mu^2 : A \otimes A \longrightarrow A \qquad (\deg 0)$$

+ have higher order terms

$$\hbar \mu^3 : A^{\otimes 3} \longrightarrow A[-1], \text{ etc.}$$

with quadratic relations

$$\hbar^2 \mu^4 : A^{\otimes 4} \longrightarrow A[-2]$$

$\Rightarrow$ formal $\hbar$ deformation (label maps by cohomological degree)

so governed by $HH^*(A,A)$:

first order deformations governed by

$$\prod_{d=3}^{\infty} HH^d(A, A[2-d])$$

(this is $H^1$ of corresponding cplx)

**Lemma** If $n \geq 2$

$$HH^d(A_m^n, A_m^n[2-d]) = 0 \quad \forall d > 2$$

Hence $\mathcal{F}_m^n$ is necessarily formal.
(i.e. quasi isomorphic to $A_m^n$ as $A_\infty$ algebra)

– can gauge away all higher order products.

**Corollary** For $n \geq 2$, $\mathcal{F}(X_m^n) \hookrightarrow \mathrm{mod}(A_m^n)$

[ old story : Khovanov-Seidel '97,
Seidel-Thomas '98 ]

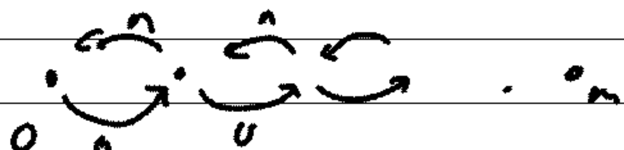
What about double cover $n=1$?
plenty of stuff in the corresponding Hochschild homology, so deformation theory abstractly doesn't solve everything.

Let's introduce a slightly larger object. [Khovanov]
Geometrically add noncompact $L_0$ going off to $\infty$

$\rightsquigarrow$ leads to $\hat{F}_m^n$,

$$H(\hat{F}_m^n) = \bigoplus_{i,j=0}^{m} HF^*(L_i, L_j)$$

Algebraically take quiver $\hat{A}_m^n$



& extra relation $(0,1,0)=0$

So $H(\hat{F}_m^n) = \hat{A}_m^n$.

__Lemma__ $\hat{A}_m^n$ is Koszul.

For any augmented algebra have a spectral sequence $\mathrm{Ext}^*_{\hat{A}}(\hat{R},\hat{R}) \otimes_{\hat{R}\otimes\hat{R}} \hat{A} \Rightarrow HH^*(\hat{A},\hat{A})$

(filter Hochschild complex by length of path)

$\hat{R} = \mathbb{C}^{m+1}$

Since $\hat{A}$ is Koszul $\rightsquigarrow$ explicit quiver description of the Ext algebra

Lemma Take $n=1$, $m+1 \geq 3$.
Then $HH^d(\hat{A}, \hat{A}[2-d]) = \begin{cases} & d=3 \text{ or } d \geq 5 \\ \mathbb{C} & d=4 \end{cases}$

so 1-dim Zariski tangent
Space to the deformation space
Hence up to quasi isom $\exists$ at most one
nontrivial $A_\infty$ structure (due to rescaling
symmetry in $t_1$).

So only two choices for our $\hat{F}_m^n$ $\Longrightarrow$
just need to calculate a product of degree 4.

[Priddy Koszulity : independent of grading on algebra]

Prop $\hat{F}_m^n$ : $n=1$ is not formal.
For $m \geq 5 \Rightarrow F_m^n$ itself is also not formal.

The nontrivial $4^{th}$ order product:
Take an $A_\infty$ structure on $A_m'$ $m \geq 5$
quasi isom to $F_m^n$ s.t $\mu^3 = 0$, then

$$\mu^4(32,21,12,23)$$
$$-\mu^4(32,23,32,23)+\mu^4(32,23,34,43)$$
$$+\mu^4(34,43,32,23)-\mu^4(34,43,34,43)$$
$$+\mu^4(34,45,54,43)=\text{const.}(3)$$

(unique gauge invariant linear combination!)

- Can count polygons when have a bunch of transverse curves, but here can't arrange transversality.
- can't compute this geometrically! need resolutions etc...

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What does this mean geometrically?

From now on set $n=1$ $m=5$
$S=S_3$ the simple module associated to the middle vertex. Taking the trivial $A_\infty$ structure on $A$, what is the classification

of $A_\infty$ modules $\zeta$ with $H^*(\zeta) = S$ ?

One choice : $S$ itself in trivial way,

Deformations! governed by $Ext^d_A(S, S[1-d])$ $d \geq 2$,

which is 1-dimensional but obstructed.

$\mathfrak{g}$ dg Lie algebra controlling deformations,

$$\mathbb{C} \cong H^1(\mathfrak{g}) \xrightarrow[{[,]}]{\sim} H^2(\mathfrak{g}) \cong \mathbb{C}$$

$\leadsto$ fat point, has no nonconstant discs.

So all such $\zeta$'s are mutually isomorphic.

When we add the nontrivial $\mu^4$,

algebra is no longer augmented, paths

of positive length no longer form an ideal,

So for $\mathcal{F} = \mathcal{F}^n_m$ $(m = 5, n = 1)$

the corresponding moduli space (deform $S$

into an $\mathcal{F}$-module) is roughly

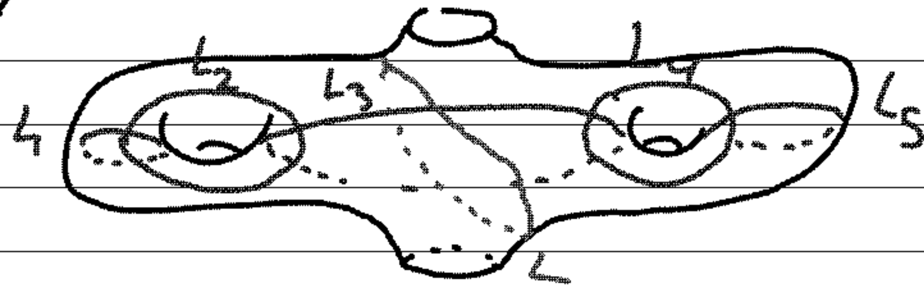
$\mathbb{C}[q]/q^2-1$ : no trivial solution but have

two solutions!

Geometrically:

$X$

genus 2 branched over $\mathbb{C}$ at 6 points




To $L$ we associate the module $\mathcal{M}_L \in \mathrm{Ob} \bmod \mathcal{F}_m^n$

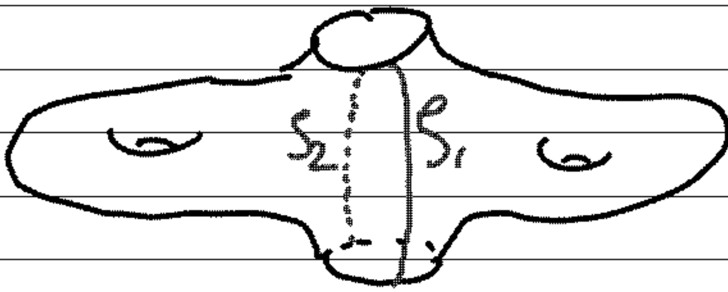
$$H(\mathcal{M}_L) = S_3 \oplus S_3[-1]$$

(two intersection points with $L_3$)

<u>If</u> $\mathcal{F}_m^n$ were formal, $\mathcal{M}_L$ would have to be an extension of $S$ by itself

in immediate contradiction to the fact that $H(\hom(\mathcal{M}_L, \mathcal{M}_L)) = HF^{\to}(L, L)$
$= H^*(S^1, \mathbb{C})$

What actually happens is that $\mathcal{M}_L$ is an extension of one module by the other




Mapping class group acts in dim $n=1$, bigger than braid group action which always exists for any dim (at least once genus is big enough)

- genus 2 with 2 punctures is not hyperelliptic

$\Rightarrow$ see more than hyperelliptic MCG = braid group. Have extra symmetries in dim $=1$ !

<u>Remark</u> There is a similar geometric picture for D, E type diagrams.

For $E_6$ Wajnryb proved for $X$ (genus 3 surface w/ boundary to $E_6$ config)

$$Br(E_6) \longrightarrow \pi_0(\mathrm{Diff}\, X)$$

is <u>not</u> injective

Again this appears to be due to nonformality
of the Fukaya $A_\infty$ structure