B. Toën - Grothendieck group of dg categories

Note Title 3/10/2008

(w/ G. Vezzosi)

Main objective: To study K_0(dg categories) as one studies $K_0(k\text{-mod}^{perf})$
K groups of perfect complexes

$$K_0(\;X\;) \xrightarrow[\text{Chern character}]{Ch} HC_0^-(X)$$

= negative cyclic homology

$$K_0(\text{dg-cat}/X) \xrightarrow{?} ?$$

Objective: theory of sheaves of dg categories..

<u>Easy example</u> $X = BG$ G finite, $/\mathbb{C}$
V vector bundle on $X \Longleftrightarrow$ linear rep. of G

Pull back to loop stack

$$\begin{array}{ccc} \mathrm{Hom}(B\mathbb{Z}, X) = IX & \xrightarrow{\pi} & X \\ \parallel & & \parallel \\ G/G & & BG \end{array}$$

$\pi^* V$ is equipped with a natural automorphism α:

monodromy action $\alpha : \pi^* V \circlearrowleft$

Via evaluation map:

$\mathcal{I}X \times B\mathbb{Z} \longrightarrow X$ is a self homotopy of the projection $\pi \Rightarrow$ automorphism of π^*.

$\mathrm{Tr}(\alpha) \in \mathcal{O}(\mathcal{I}X) = \mathbb{C}[G]^G = \overset{HC_0}{\text{class function}},$

$\dashrightarrow$ the [Chern] character of a representation

$K_0(BG) \longrightarrow \mathcal{O}(\mathcal{I}X)$

We want to think of Ch this way for general X (scheme, alg. stack etc)

& dg categories: an autoequivalence is a bimodule & trace is Hochschild homology ....

X scheme, $\mathcal{I}X = \mathrm{Hom}(B\mathbb{Z}, X) = X$

$\rightsquigarrow$ no nonconstant loops on X.

But now consider as a derived stack

$\rightsquigarrow$ consider $\mathbb{R}\mathrm{Hom}(B\mathbb{Z}, X)$ derived loop spaces

2 main ingredients:

1. derived loop spaces
2. Theory of rigid $(1,\infty)$ $\otimes$-cat

(use to prove $Tr(\alpha)$ is S^1-invariant)

<u>Plan</u>

1. LX derived loops
2. Rigid $(1,\infty)$ $\otimes$-cats
3. Ch.

1. X is a scheme (or alg stack, or any derived stack ...) / k comm. base ring

$$X \in Ho(dSt(k))$$

Def. The derived loop stack of X is

$$LX = \mathbb{R}Map(B\mathbb{Z}, X) \qquad S^1 := B\mathbb{Z}$$
$$\in Ho(dSt(k))$$

<u>Remark</u> • $X = Spec\, A$, $S^1 = * \amalg^h_{* \amalg^h *} *$ 

$$LX = X \times^h_{X \times X} X = \text{self intersection of the diagonal}$$

$$= R\mathrm{Spec}\left(A \overset{\mathbb{L}}{\underset{A \otimes_k A}{\otimes}} A\right) = R\mathrm{Spec}(HH_\cdot(A))$$

ie $\mathcal{O}(LX) \simeq HH_\cdot(A)$ functions are Hochschild chains

- S^1 acts on LX

$\Rightarrow S^1 \circlearrowright \mathcal{O}(LX)$ = circle action on Hochschild chains giving de Rham operator / Connes cyclic differential, making $HH_\cdot(A)$ into a cyclic complex.

$\Rightarrow \mathcal{O}(LX)^{hS^1} \simeq HC^-(A)$ for $X = \mathrm{Spec}\, A$.

ie $\pi_i\left(\mathcal{O}(LX)^{hS^1}\right) \simeq HC^-_i(A)$

negative cyclic homology.

For X any scheme

$\mathcal{O}(LX)^{hS^1} \simeq HC^-(X)$

Chern character: V vector bundle on X

$LX \xrightarrow{\pi} X$, $LX \times B\mathbb{Z} \xrightarrow[e = ev_1]{} X$

$e^*(V)$ is a vector bundle on $LX \times B\mathbb{Z}$

$\iff$ vect bundle $\pi^* V$ on LX + an autoequivalence $\alpha \circlearrowright \pi^*(V)$ (monodromy)

$\Rightarrow$ $Tr(\alpha) \in \mathcal{O}(LX) = HH(X)$

& in fact has a natural S^1 equivariant version $\longrightarrow$ lift to $\mathcal{O}(LX)^{hS^1}$

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2. Rigid $\otimes$ $(1,\infty)$-category:

- X scheme $\Rightarrow$ $Vect(X)$ is a rigid $\otimes$-category
- X derived scheme, $A \in$ sc Alg

A-mod = simplicial A-modules $\supset$ vector bundles

$M_\bullet$ s.t. $\pi_0(M)$ v. bundle on $\pi_0(A)$

$$\pi_i(M) \simeq \pi_0(M) \otimes_{\pi_0(A)} \pi_i(A)$$

We have to invert weak equivalences of simplicial A-modules $\rightsquigarrow$ higher categories

<u>Def</u> A $(1,\infty)$ category is an $\mathcal{S}$-category, ie a category enriched in simplicial sets

$f : T \to T'$ is an <u>equivalence</u> of $\mathcal{S}$-categories if it induces a weak equivalence on the Hom spaces $T(x,y) \to T(fx, fy)$

& $[T] \longrightarrow [T']$ is essentially surjective

(category of connected components, $[T](x,y) = \pi_0 T(x,y)$.

- Ho ($\mathcal{S}$-cat) has internal Homs RHom (strict model for lax functors)

- Localizations exist :

$T \qquad W \subseteq [T]$

$\downarrow$ universal in $Ho(\mathcal{S}\text{-cat})$ inverting W.

$L_W T \qquad (\Longleftrightarrow$ Dwyer-Kan simplicial localization$)$

Monoidal structures:

Γ = category of pointed finite sets

<u>Def</u> A symmetric $\otimes$ $(1,\infty)$ cat. is a functor $\Gamma \xrightarrow{T_\bullet} \mathcal{S}\text{-Cat}$

st. • $T_{\{\phi\}} = *$

• $T_1 \xrightarrow{\sim} T_1 \times \dots \times T_1$ equivalence

$n = \{0, \dots, n\} \longrightarrow \{0,1\}^n$

- gives monoidal structure up to weak equivalence:

$$\begin{tikzcd} T_1 & T_2 \arrow[l] \arrow[r] & T_1 \times T_1 \arrow[ll, dashed, bend left, "\otimes"] \end{tikzcd}$$

$Ho(\mathcal{S}\text{-}Cat^{\otimes}) \hookrightarrow Ho(\mathcal{S}\text{-}Cat^{\Gamma})$
full subcategory

<u>Def</u> T $\otimes$ $(1,\infty)$-cat $\Rightarrow$
$[T]$ symmetric monoidal category (as $[T_1] \leftarrow [T_2] \to [T_1 \times T_1]$) "$[T_1]$". T is <u>rigid</u> if $[T]$ is rigid (ie $\exists$ duals)

<u>Ex</u> $1Bord$: $(1,\infty)$ category of 1-bordism
Objects : 0-dim manifolds (oriented)
Homs : moduli of 1-dim manifolds with boundaries.
$\otimes = \coprod$, so $1Bord \in Ho(\mathcal{S}\text{-}Cat^{\otimes})$

Theorem (Hopkins-Lurie)
$$Map_{\otimes}(1\text{-}Bord, T) \cong Map_{\mathcal{S}\text{-}Cat}(*, T)$$
for $T \in Ho(\mathcal{S}\text{-}Cat^{\otimes})$

Relative version: X topological space
$\Rightarrow$ fundamental groupoid is a $(1,\infty)$ category
$T_\infty(X) \in Ho(\mathcal{S}\text{-Cat})$

Relative version of Bord: $1\text{-Bord}(X)$
maps to X
$Map_\otimes(1\text{-Bord}(X), T) \simeq Map_{\mathcal{S}\text{-Cat}}(T_\infty(X), T)$

<u>Consequence</u> $T \in Ho(\mathcal{S}, Cat^\otimes)$

$Map_\otimes(1\text{-Bord}(S^1), T) \simeq Map(B\mathbb{Z}, T)$

$= \text{pairs } (x, h), \quad x \in T \quad h : x \circlearrowleft$

$End(\phi) \ni S^1 \xrightarrow{id} S^1 \Rightarrow$ trace

$$\begin{tikzcd}
Map_\otimes(1\text{-Bord}(S^1), T) \arrow[r, "\simeq"] \arrow[d] & Map(B\mathbb{Z}, T) \arrow[dl, "Tr"] \\
End_T(1) & S^1\text{-equivariant}
\end{tikzcd}$$

get S^1 equivariant lift of trace map.

Chern character

$T : \mathrm{stAlg} \longrightarrow \otimes\,(1,\infty)\text{-cat}$, rigid

T is a "derived stack of rigid $(1,\infty)$ categories"
... e.g. vector bundles, or dg categories

Theorem $\forall X$ derived stack

$$\exists\ Ch : T(X)^{\mathrm{int}} \longrightarrow \underline{\mathrm{End}}_{T(LX)}(1)^{hS^1}$$

$\underline{\mathrm{int}}$: sub $(1,\infty)$ category of $\underline{\text{isomorphisms}}$
(i.e. functorial just for isomorphisms)

$$\begin{tikzcd} T^{\mathrm{int}}(X) \arrow[r] & T^{\mathrm{int}}(LX \times B\mathbb{Z}) = \mathrm{Map}(B\mathbb{Z}, T(X)) \arrow[d, "Tr"] \\ & \mathrm{End}_{T(LX)}(1) \end{tikzcd}$$

$\circlearrowleft$ $S^1 \times S^1$

map equalizes the two circle actions

equivariant.

- $Ch : K_0(X) \longrightarrow HC^-(X)$
- $Ch : dg\,cat \longrightarrow QC(LX)^{S^1}$