G. Vezzosi – Background in DAG

Note Title 3/10/2008

2 approaches
- "ringed spaces" (ringed ∞-topoi) J. Lurie
- "functor of points" B. Toën–Vezzosi

Plan: · what are derived spaces?
· geometric/algebraic derived spaces
· tangents & cotangents

k = comm. ring (for today usually field of char 0)

k-alg := commutative k-algebras

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\begin{tikzcd}
k\text{-alg} \arrow[r, "\text{schemes, alg. spaces}"] \arrow[dr, "\text{1-stacks}"] \arrow[ddr, "\text{higher stacks}"] \arrow[dd] & \text{Sets} \\
& \text{groupoids} \arrow[u, "\pi_0"'] \\
sk\text{-alg} \arrow[r, "\text{derived stacks}"] & \text{SSets} \arrow[u, "\pi_1"']
\end{tikzcd}
$$

SSets = simplicial sets

sk-alg = simplicial k-algebras

Think of SSet, sk-alg as ∞-categories
– concretely, as model categories

AG	DAG
$(k\text{-mod}, \otimes)$	• sk mod or $Ch^{\leq 0}(k\text{-mod})$ (if char $k = 0$) [model category with $\otimes$)
• Commutative ring objects $\mathrm{Comm}(k\text{-mod}, \otimes) = (\mathrm{Aff}_k)^{opp}$: commutative rings $\longleftrightarrow$ affine schemes	• $\mathrm{Comm}(sk\text{-mod}, \otimes) := d\mathrm{Aff}_k^{opp}$, a model category; simplicial k-algebras $\longleftrightarrow$ nonpos. comm. dgas.
• Prestacks: $PSt(k) = \{\mathrm{Aff}_k^{op} \longrightarrow SSet\}$ model category (projective model structure)	• Derived prestacks $dPSt(k) := \{d\mathrm{Aff}_k^{op} \longrightarrow SSet\}$ model category: add weak equivalences from <u>source</u> $h_X \to h_Y$ weak eq on representable functors when $X \longrightarrow Y$ is a weak equiv

[Remark: fibrant objects in $dPSt(k)$ are
$F : dAff_k^{op} \longrightarrow SSets$ which
1. are objectwise fibrant
2. preserve weak equivalences

i.e. $\underbrace{Ho(dPSt(k))}_{\substack{\text{full subcategory} \\ \text{satisfying 2.}}} \subset Ho(\underbrace{dAff_k^{op} \longrightarrow SSet}_{\substack{\text{with projective} \\ \text{model structure (ie} \\ \text{measure on target)}}})$]

• τ Grothendieck topology on Aff_k (eg τ = étale topology)	• τ = "model" topology on $dAff_k$ $\Longleftrightarrow$ Grothendieck topology on $Ho(dAff_k)$ eg τ = strong étale topology

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Remark: For $A \in sdg_k$, one should think of A as "given" (up to weak equivalence) by an comm. k-algebra $\pi_0(A)$ + "derived nilpotents" $\pi_{>0}(A)$

$\Rightarrow$ consider $\mathrm{Spec}\, \pi_0 A \hookrightarrow \mathbb{R}\mathrm{Spec}\, A$ (closed embedding) as a derived formal thickening.

"Derived geometry on A" =
"usual geometry on $\pi_0(A)$ + deformation theory from A"

__Example__ $A \longrightarrow B$ map of salg_k, it is an isomorphism in the homotopy category iff $\pi_0(A) \xrightarrow{\sim} \pi_0(B)$ and the relative cotangent complex $\mathbb{L}_{B/A} \simeq 0$ is trivial.

__Def__ $A \longrightarrow B$ is strongly [étale / flat / smooth / Zariski open immersion] if

1. $\pi_0(A) \longrightarrow \pi_0(B)$ is [étale / ...]
2. $\pi_i(A) \underset{\pi_0(A)}{\otimes} \pi_0(B) \xrightarrow{\sim} \pi_i(B)$ $\forall i$

__Def__ $\{A \to A_i\}$ is a strongly étale covering family if
i) $A \to A_i$ strongly étale

ii) $\{\operatorname{Spec} \pi_0 A_i \to \operatorname{Spec} \pi_0 A\}$ is an étale cover.

<u>Examples</u> Let's work with nonpos. graded comm. dgas $A^\bullet \in \mathrm{cdga}_k^{\leq 0}$ (char $k=0$)

1. Standard strongly étale maps:

$$A^\bullet \longrightarrow B^\bullet = \frac{A^\bullet[x_1, \dots x_n]\langle \xi_1, \dots \xi_n \rangle}{d\xi_i = f_i \in A_0[x_1, \dots x_n]}$$

$\deg x_i = 0$, $\deg \xi_i = -1$

\+ Jacobian criterion: $\det\left(\frac{\partial f_i}{\partial x_j}\right) \in \frac{H^0(A^\bullet)[x_1 \dots x_n]}{(f_1, \dots f_n)}$ is a unit.

Then $A^\bullet \longrightarrow B^\bullet$ is strongly étale, & Zariski locally any strongly étale map is of this form. (analog of standard local description of étale maps)

2. $A[x]\langle\xi\rangle / d\xi = x a_0 - 1$

$a_0 \in A^0$ ... this is a strong Zariski open immersion: on H^0 this is the principal open corresponding to a_0.

• τ-stacks:

$St_\tau(k)$: change model structure on $PSt(k)$ so as to invert all hypercovers (left Bousfield localization)

$F \to G$ is a weak eq iff $\pi_i(F) \xrightarrow{\sim} \pi_i(G)$ isom of π_i sheaves (for any choice of basepoint)

$\Rightarrow$ stacks are the fibrant objects: $Ho(St(k))$ is the full subcat. of $Ho(PSt(k))$ such that

• derived stacks wrt strong version of τ, $dSt_\tau(k)$ - again change model structure on $dPSt_k$ so as to invert homotopy hypercovers.

$F \to G$ is again a weak eq iff

as stacks on $(Ho(dAff_k), \tau)$

$\Rightarrow$ derived stacks are prestacks satisfying sheaf condition

$\forall\, U_\bullet \to X$ τ hypercov $F(X) \xrightarrow{\sim} \operatorname{holim} F(U)$ iso in $Ho(SSets)$ – ie sheaf condition	"
$\exists$ left adjoint $a : Ho(PSt) \to Ho(St)$ stackification functor	"

$Ho(St(k))$ | "
has holims & hocolims

- $\rightsquigarrow$ $F \searrow H \swarrow G$ $\Rightarrow$

  $F \times^h_H G$ still a stack
  homotopy fiber product

e.g. $A \in st\text{-}dg$
$\mathbb{R}Spec\, A : St\text{-}dg \to SSet$
$B \longmapsto Map_{s\text{-}dg_{St}}(A, B)$
(~Yoneda ..)

- have internal Homs
  $\mathbb{R}Hom(F, G)$ stack, ie
  $Map(F \times G, H)$
  $\simeq Map(F, \mathbb{R}Hom(G,H))$

$$\mathrm{Aff}_k \hookrightarrow d\mathrm{Aff}_k$$

$$\Downarrow$$

adjunction $\mathrm{Ho}(\mathrm{St}_\tau(k)) \underset{t_0 = \text{truncation}}{\overset{i}{\rightleftarrows}} \mathrm{Ho}(d\mathrm{St}_{\text{strong-}\tau}(k))$

i is fully faithful

$i(\mathrm{Spec}\, R) = \mathbb{R}\mathrm{Spec}\, R$

$t_0(\mathbb{R}\mathrm{Spec}\, A) = \mathrm{Spec}\, \pi_0(A)$

- t_0 preserves holim / hocolim & $\mathbb{R}\mathrm{Hom}$
- i does not preserve holim or $\mathbb{R}\mathrm{Hom}$!

Notion of geometric stacks

- always have a cotangent complex
- its dual evaluated at a point is the tangent space $\mathbb{R}\mathrm{Hom}(k[\varepsilon]/\varepsilon^2, X)$
- gives geometric meaning to a cotangent complex of a scheme as dual to tangent space of $i(X)$ - ..

- t_0, i preserve geometricity