Edward Witten - Duality from Six Dimensions

Note Title 2/7/2008

Duality - for physicists - comes from QFT:
a range of things from mirror symmetry
to geometric Langlands

Very difficult mathematically: much analysis
+ vast algebraic machinery

In turn duality is linked to & from string theory:
embedding QFT in string theory extremely useful
to understand the former

QFT ⊃ CFT: conformally invariant QFT
- formulated on manifolds given only conformal
structure...
in fact we'll need SUSY CFT — superconformal
field theory:

SCFT : CFT :: differential forms : functions

Why CFT? play fundamental role:

CFT : QFT :: simple Lie gps : all Lie gps

- basic building blocks appear as limits of any QFT ....

In n dimensions: $\mathbb{R}^{1,n-1}$ standard conformal
($n>2$) space, conformal group $SO(2,n)$
or $Spin(2,n)$ when have fermions.

$n=2$: $\quad Spin(2,2) = SL_2\mathbb{R} \times SL_2\mathbb{R}$

$$\begin{array}{ccc} SL_2\mathbb{R} & \times & SL_2\mathbb{R} \\ \cap & & \\ \mathrm{Diff}\,S^1 & \times & \mathrm{Diff}\,S^1 \end{array}$$

For superconformal field theory:
Supergroup of symmetries of super version of $\mathbb{R}^{1,n-1}$

Bosonic part: $Spin(2,n) \times H$ with H compact
(can prove symmetry of any CFT is of this form)

Fermionic part = sum of copies of spin rep(s) which is real (might force restrictions on which sum of copies).

Nahm translated the classification of symmetries into the following math result:
simple such supergroups exist in dimension $n \leq 5$ only.

e.g. have $OSp(p|2m)$ with bosonic part $O(p) \times Sp(2m)$ — but fermions transform as vectors not spinors of $O(p)$

But for $p=8$ have triality of $Spin(8)$ & mod triality the vector & spinor are equivalent
$\Rightarrow$ $OSp(2,6|2m)$ solves our problem, & is maximal such solution
[triality is the last of the exceptions in Lie theory...]

$n=5$ $\exists$ exceptional supergroup due to Kac ...

$n=4$: use another low-dim coincidence, this time for $SU(p|q)$:
$Spin(2,4) = SU(2,2)$

$SL(p|q)$: $\left(\begin{array}{c|c} Sl_p & * \\ \hline * & Sl_q \end{array}\right)$ fermions in fundamental rep

Spinor of $Spin(2,4)$ = fundamental 4d rep

$\Rightarrow SU(2,2|q) \supset \underbrace{S(U(2,2)\times U(q))}_{\text{even part}}$

--- simple for generic q.

Now set $q=4$: $p=q \Rightarrow Str(Id)=0$
so should remove identity, get
$PSU(2,2|4)$, most exceptional group
in dim $n=4$.

[In 3 dim use fact that $Spin(3,2) = Sp(4,\mathbb{R})$
2 dim $Spin(2,2) = SL_2 \times SL_2$
... get solution in every dimension ≤ 6]

To reduce dimension: conformal group of
$\mathbb{R}^{1,n-1} \times S^1$ doesn't give conformal transformations
of $\mathbb{R}^{1,n-1}$ so need exceptions to

produce examples in low dimensions, not just dim reduction.

5, 6 dim cases of field theories not known at time of Nahm's theorem, so the 6 was considered a curiosity.... but not so now!

In 6d have $Osp(2,6|2m)$ for $m=1$ or 2
... today consider most SUSY one, $m=2$.

In 4 dim: $PSU(2,2|q)$ $q \leq 4$
- we'll consider the most SUSY one, $q=4$.

4dim $PSU(2,2|4)$ "$N=4$ Supersymmetry"

Known theories are classified by the choice of a compact Lie group G (which we'll take to be simple) & a parameter $\tau \in \mathbb{H}$ upper half plane.

Write $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$ $e>0$, $\theta \in \mathbb{R}$

Fields: F = curvature of a connection A on a G-bundle on $\mathbb{R}^{1,3}$ (or any 4-manifold..)

Action $I = \left(\frac{1}{2e^2} \int \mathrm{Tr}\, F \wedge * F + \frac{\theta}{8\pi^2} \int \mathrm{Tr}\, F \wedge F + \ldots\ldots \right)$

.... : rest of terms determined uniquely by requiring $N=4$ SUSY.

2 Trivial Statements

- Standard methods let one compute everything one wants to compute for $e^2 \ll 1$ ie $\mathrm{Im}\,\tau \gg 1$ [as asymptotic expansion & understand qualitative properties .... eg for particle moving on a manifold: can't compute full spectrum of Laplacian but can compute kernel — topological part! — & an asymptotic expansion]

- Also elementary: Theory (G,τ) = Theory $(G,\tau+1)$: $\theta \longmapsto \theta + 2\pi$ action changes by $2\pi c_2$. But quantum theory depends only on $\exp(iI)$ so doesn't feel this change! [θ only appears in this one term!]

2 Nontrivial statements

1. Montonen-Olive duality:

$$\text{Theory}(G, \tau) \simeq \text{Theory}\left({}^L G, -\frac{1}{n_G \tau}\right)$$

[$n_G = 1$ simply laced, otherwise 2 or 3]

${}^L G$ = Goddard-Nuyts-Olive-Langlands dual group.

G simply laced $\Rightarrow n_G = 1$,
get symmetry under $\tau \mapsto \tau + 1$ & $\tau \mapsto -1/\tau$
so get $SL_2\mathbb{Z}$: so there is no
strong coupling regime: only one cusp for
$\mathbb{H}/SL_2\mathbb{Z}$, which is the weak coupling
expansion! so initially believed the theory
is already "understood"... turns out to be wrong.

2. $G = A, B, C, D$ (classical groups) have
another nontrivial phenomenon discovered
by Maldacena

beyond SCFT have string theories in 10 dimensions/ M theory in 11 dimensions

... e.g type IIB theory in 10 dim,

can have supergroups a symmetries –

so can seek theory with $PSU(2,2|4)$ symmetry.

Classically: look for 10 manifold with this symmetry which is solution of classical theory:

find $M = PSU(2,2|4) / SO(2,3) \times Sp(4)$

[we're using $SU(4) = SO(6)$

$Sp(4) \subset SU(4)$, $SU(4)/Sp(4) = S^5$]

Even part: $SU(2,2) \times SU(4) / Sp(2,2) \times Sp(4)$

$$= AdS_5 \times S_5$$

$CAdS_5 \Longleftrightarrow H_5$ hyperbolic 5-space:)

Lorentzian Euclidean

This classical solution depends on an integer $N > 0$.

Let's think in Euclidean signature:

S^4 = conformal boundary (at infinity) of $\mathbb{H}^5$.

In quantum gravity asymptotic behavior at infinity doesn't fluctuate, get well defined QFT on S^4. Under this conformal transformation the finite radius S^5 gets shrunk to a point:

quantum gravity of $\mathbb{H}_5 \times S_5 \rightsquigarrow$ QFT on S_4, four dimensional theory!

Maldacena:

$$\text{Theory}(A_{N-1}, \tau) \simeq \mathrm{II}_B \text{ on } AdS_5 \times S^5$$

with parameters τ, N

versus for type B, C, D:

$AdS_5 \times \mathbb{RP}^5$ with a pair of gerbes of order two, determining group.

Consequences: (1) enables us to compute for fixed G with $\mathrm{Im}\,\tau \to 0$

– and implies geometric Langlands duality via a sequence of more elementary statements

(2) enables us to compute for A_n with $n \to \infty$ & $g^2 n \gg 1$.

Our goal: to explain why (1) is true – many explanations, all coming from higher dimensional contexts in string theory...
ie putting the story in a more complicated story! we'll focus on a picture in six dimensions which doesn't explicitly mention string theory

[Note: all of the superconformal theories we've mentioned can be defined on any superconformal manifold of the appropriate type.]

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6 dimensions (Known) theories with maximal ($OSp(2,6|4)$ symmetry) depend on choice only of $G = A, D, E$ group (no other parameters)

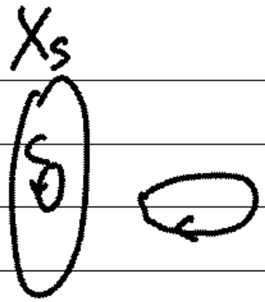
... not a gauge theory! G won't arise as symmetry group of theory.
The theory can't be described classically:
no parameter $\tau \Rightarrow$ no classical limit!
(unless $G = U(1)$ where have a subtle classical description)

... a nontrivial (i.e. non Gaussian) CFT.

--- has observables associated with Riemann surfaces (analogs of loop operators in gauge theory)
- trying to be trace of holonomy of a nonabelian gerbe, & is this in case $G = U(1)$ when there is a gerbe theory.

Can be formulated on any conformal spin manifold X_6.
We'll take $X_6 = X_5 \times S^1$. Give this a conformal structure from a (pseudo) Riemannian metric, where circumference $(S^1) = 2\pi r$.

Now consider surfaces in here:



Can consider $\Sigma = C \times S^1$
$C \subset X_5$ one-manifold
$\Rightarrow$ so get operators associated
to circles in X_5 $\Rightarrow$

On $X_5 \times S^1$ get a G-gauge theory on X_5:

These surface operators become Wilson loops in 5 dims. Keep r fixed & make very large scale calculations $\rightsquigarrow$ get action $\underline{I} = \mathcal{C}(r)\int_{X_5} Tr(F \wedge * F + \dots)$

on $X_6 = \mathbb{R}^{1,4} \times S^1$ <u>don't</u> have 5d superconformal symmetry but do have lots of symmetry.

6d conformal symmetry $\Rightarrow$ theory invariant under rescaling X_5 & S^1 simultaneously.

So need to fix coefficient $\mathcal{C}(r) = \frac{1}{r}$ to preserve 6d conformal symmetry..

If we had a classical action in 6d $\rightsquigarrow$

would get a classical action in 5d by integrating over S^1 $\Longrightarrow$ would have factor of r, not $\frac{1}{r}$!

Now take $X_5 = X_4 \times S^1_s$. Now we can integrate action over the fiber $\rightsquigarrow$

$$I = \frac{s}{r} \int_{X_4} \mathrm{Tr}\,(F \wedge * F) + \dots$$, rest determined

by just supersymmetry (not superconformal symmetry) in 6 dim! our 6d theory was already unique just as a SUSY QFT!

$\rightsquigarrow$ find that in 4d our theory becomes in fact the superconformal $N=4$ theory with $G = $ ADE group & $\tau = i\frac{r}{s}$.

$$X_6 = X_4 \times S^1_s \times S^1_r$$

But could take the two circles in opposite order! $\Longrightarrow$ so the 4d theory has $\tau \longrightarrow -1/\tau$ symmetry.

We only had $\tau \in i\mathbb{R}$ by taking square tori $S^1_s \times S^1_r$ but could take any torus:

General claim If $X_6 = X_4 \times \mathbb{C}/\Lambda = \langle 1, \tau \rangle$
$\Rightarrow$ get $\mathcal{N}=4$ SYM on X_4
with parameters G, τ.:
6d theory is manifestly invariant under
$$\tau \longmapsto \frac{a\tau + b}{c\tau + d}$$

Note: global effects in the 6d theory depend on some discrete data which in 4d reduces to second Stiefel-Whitney number!
So can really see group G, not just its Lie algebra.

Other surface operators: in 5d can look at operators given by surfaces in X_5 factor of $X_5 \times S^1$ ...
these are the 't Hooft operators! gauge fields with codim 3 singularities $\Longrightarrow$ surfaces in 5d theory!

[The 6d theory has $Out(G)$ as its symmetries — so can twist by an outer automorphism of G, get twisted form of theory ... see non simply laced groups as fixed points!]

Where does the 6-dimensional theory come from?

In ten dimensions we have IIA & IIB superstring theories, which are close consts. Let our 10d spin manifold be of the form

$X_{10} = X_9 \times S^1_r$ [All our S^1's have the nonbounding spin structure]

(depends on metric!)

(• IIB on $X_9 \times S^1_r \iff$ IIA on $X_9 \times S^1_{1/r}$

($S^1_{1/r} \longleftrightarrow$ flat line bundles on S^1_r)

... abelian duality, can be proven explicitly

Now study $\Gamma \subset SU_2$ finite subgroup ($\longleftrightarrow ADE$) acting on $\mathbb{R}^4$

2. IIA on $X_6 \times \mathbb{R}^4/\Gamma$ (or really have on X_{10} with a singular locus $X_6 \subset X_{10}$ with a transversal ADE singularity)

- find G gauge theory appearing on X_6.

Action $I = \int_{X_{10}} \sqrt{g}\,(R + \ldots) + \int_{X_6} \mathrm{Tr}(F \wedge * F + \ldots)$

string theory effective action for such on X_{10}, get extra fields appearing on the singular locus.

[idea: water waves are excitations that appear localized on interface between water & air]

[What is relation of the ADE singularity & G symmetry? physics near singularity sees resolution of singularity etc... usual classical story.

Another explanation for this:

IIA on $X_6 \times K3 \Longleftrightarrow$ heterotic string on $X_6 \times T^4$

ADE singularity ⟷ ADE gauge symmetry]

- IIB on $X_6 \times \mathbb{R}^4/\Gamma_G$ now gives us the 6-dimensional superconformal theory on X_6!

$\Rightarrow$ Take $X_6 = X_5 \times S^1_s$ & $r = \frac{1}{s}$

IIB on $X_5 \times S^1_s \times \mathbb{R}^4/\Gamma_G$ = 6 - superconformal theory on $X_5 \times S^1_s$

$\Updownarrow$

IIA on $X_5 \times S^1_r \times \mathbb{R}^4/\Gamma_G$ = G-gauge theory on $X_5 \times S^1_r$

$$I = \int_{X_5 \times S^1_r} \mathrm{Tr} \wedge * F$$

Now take 5-dim effective action $I = r \int_{X_5} \mathrm{Tr}\, F \wedge * F$

So at long distances on X_5 get in IIB side

$I = \frac{1}{s} \int_{X_5} \mathrm{Tr}\, F \wedge * F$: $\frac{1}{s}$ suggests conformal invariance in 6-dimensions.

Note: In string theory has a definite unit of lengths so it makes sense to say $s = \frac{1}{r}$ with both s, r lengths

M-theory: an 11d theory with gravity has solutions $AdS_7 \times S^4$ type A

$AdS_7 \times \mathbb{RP}^4$ type D

with $OSp(2,6|4)$ superconformal symmetry

Holographic dual of this (Maldacena) is a theory without gravity on the conformal boundary of AdS_7, which is a conformal compactification of $\mathbb{R}^{1,5}$

$\Longrightarrow$ construction of the A, D 6d superconformal theories.

To get other X_6: look for 11-manifolds M_{11} that at infinity look like $X_6 \times S^4$ or $X_6 \times \mathbb{RP}^4$:

think of X_6 as conformal boundary of some M_7

— do a path integral over every 7-manifold with this boundary!

[If we compactify the 6d theory on
$X_4 \times \Sigma$ Σ Riemann surface
get an $\mathcal{N}=2$ gauge theory in four
dimensions, with matter fields, on large scales
... too many hypermultiplets]