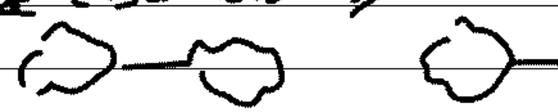
Spencer Bloch : Graphs

Note Title 5/19/2007

Common ideas in graph studies :

1. Outer space (Vogtmann + collaborators)
2. Graph homology (Kontsevich)
3. Coordinates on Teichmüller space (Strebel)
4. Feynman amplitudes, arising as algebro-geometric periods

... all involve common structures :

(1PI) Bestvina-Feighn (Invent. 140 2000)
notion of core graph :
a noncore graph can be disconnected by cutting an edge : e.g 

G is <u>core</u> if $\forall e \in G \quad h_1(G \setminus e) < h_1(G)$

G core graph, $\Delta_G = \{(a_1, \dots, a_N) \mid \sum a_i = 1\}$

$a_i \leftrightarrow e_i \in G$

$\Delta_G^\times = \Delta_G - \Sigma_G$ remove edges that support a loop

(want nonzero lengths for loops)

$$\Sigma_G = \bigcup F$$

Face $F : A_{i_1} = \dots = A_{i_p} = 0$

$h_1(e_{i_1} \cup \dots \cup e_{i_p}) > 0$

$G' = e_{i_1} \cup \dots \cup e_{i_p}$

$F = \Delta_{G//G'}$

contract all connected components

$B_\bullet$-$F_\bullet$: Compactify Δ^*_G by chains

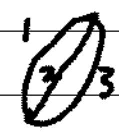
$\emptyset \subset G_1 \subsetneq \cdots \subsetneq G_p = G$ $\quad$ G_i all core

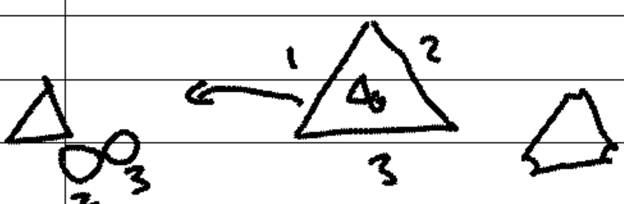
Given chain ch: $\Delta(ch) := \prod \Delta_{G_i/G_{i-1}}$ "assoc. graded"

$\Delta(ch)^* = \prod \Delta^*_{G_i/G_{i-1}}$

Blow up faces in Δ^* to obtain

$\Delta^c_G = \coprod_{\text{chains}} \Delta(ch)^*$

Δ^c_G is compact. e.g. 



Not allowed to shrink loops - so corners are not allowed

BF: get hexagon

$\Delta^*_{2\bigcirc 3} \times \Delta^*_8$ tadpole

blowup of corners.

1. Connes-Kreimer Hopf algebra
2. Kontsevich graph homology

1. : $H = \bigoplus \mathbb{Q}[G]$ vector space of isom classes of graphs

$H \otimes H \longrightarrow H$ given by disjoint union

$\Delta : H \longrightarrow H \otimes H$ $\qquad \Delta[G] = \sum_{G' \in \mathcal{S}(G)} [G'] \otimes [G/\!/G']$

Connes-Kreimer take for $\mathcal{S}(G)$ the set of subgraphs that are divergent for the problem at hand.

$\mathcal{G} = \operatorname{Spec} H$ group scheme:
physics associates to graph $[G]$ number $FA(G)$ Feynman amplitude
$\Rightarrow$ should get "Feynman point" $F \in \mathcal{G}$ ....

Our purposes: take $\mathcal{S}(G)$ = core graphs in G
satisfies coassociativity conditions $\rightsquigarrow$
Hopf algebra, H. Have a natural differential on H given by Kontsevich graph homology:

$\partial [G] = \sum (-1)^i [G/e_i]$

(have to orient graph carefully)

- not allowed to contract loops -

really takes place in

$H_*(\Delta_G; \Sigma_G)$ or really union

over all graphs of Δ_G with suitable attachment.

<u>Theorem</u> H is a differential Hopf algebra (almost)

More precisely $H / H \cdot O$ is diff Hopf algebra

(quotient out ideal generated by a loop)

d preserves loop number, lowers # edges.

(Joint w. D. Kreimer, K. Yeats)

Feynman amplitudes are certain periods

<u>Most interesting graphs</u>: log divergent.

G core, 2n edges, $h_1(G) = n$

-- <u>just</u> divergent. coefficient of linear

term in renormalization is the interesting prod!

$$\int_{\Delta_G} \frac{\Omega_{2n-1}}{\sqrt{\psi_G^4}} \qquad \Omega = \sum \pm A_i \, dA_1 \ldots \widehat{dA_i} \ldots dA_{2n}$$

Identify $\Delta_G = \{ (a_1, \dots a_{2n}) \in \mathbb{P}^{2n-1}(\mathbb{R}) : a_i \geq 0 \}$

--- chain in $\mathbb{P}^{2n-1}(\mathbb{R})$

Point of $\mathbb{P}^{2n-1}(\mathbb{R})$: metric on graph up to scaling $\Rightarrow$ graph Laplacian

$$\Psi_G = A_1 \cdots A_{2n} \cdot \det(\text{Laplacian}_A(G))$$

$$= \sum_{\substack{T \subset G \\ \text{spanning tree}}} \prod_{e \notin T} A_e$$

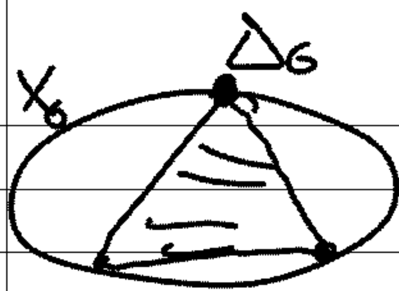
Why write Ψ_G^2 as $\sqrt{\Psi_G^4}$?

Should think of Gaussian integral with $\Psi_G = \det$ of Laplacian. 4 comes from dim = 4.

--- Feynman rules associate quadric $\sum_1^4 x_i^2$

These integrals turn out (Broadhurst - Kreimer) to usually be multiple zeta numbers.

Draw $X_G = \{\Psi_G = 0\}$ hypersurface in $\mathbb{P}^{2n-1}$



$\Rightarrow$ get chain

$$\eta_G \in H^{2n-1}_{DR}(\mathbb{P}^{2n-1} - X_G)$$

Problems arise when this chain hits X_G.

So from motivic point of view we're studying $H^{2n-1}(\mathbb{P}^{2n-1} \setminus X_G, \Xi \setminus X_G \cap \Xi)$

where $\Xi : \prod A_i = 0$

Problem in defining this motive exactly comes from where X_G hits Δ_G

— this is where the problems in renormalization arise...

To get well defined motive must blow up Δ_G

$$\Delta_G \cap X_G = \bigcup_{L \subset X_G} L(\mathbb{R})^+$$

: L coordinate linear space $A_{i_1} = \dots = A_{i_p} = 0$

$L(\mathbb{R})^+ : a_j \geq 0$

When is $L : A_1 = \dots = A_p = 0 \subset X_G$?

$\Longleftrightarrow h_1(e_1 \cup \dots \cup e_p) \geq 1$

Blow up! $\mathbb{P}^{n-1}$ of L

$$\begin{tikzcd} P \arrow[d] & & & E \cong L \times \mathbb{P}^{p-1}_{A_1,\dots,A_p} \arrow[d] \\ \mathbb{P}^{2n-1} & \supset X_G & \supset & L \cong \mathbb{P}^{2n-p-1} \end{tikzcd}$$

(exceptional)

$G' = e_1 \cup \dots \cup e_p \implies X_{G'} \subset \mathbb{P}^{p-1}$

$X_{G/\!/G'} \subset \mathbb{P}^{2n-p-1}$

$$\begin{array}{l} P \supset E \cup Y = \text{strict transform of } X_G \\ \downarrow \qquad Y \cap E = X_{G/\!/G'} \times \mathbb{P}^{p-1} \cup \mathbb{P}^{2n-p-1} \times X_{G'} \\ \mathbb{P}^{2n-1} \end{array}$$

$\implies$ get Gysin sequence

$$\begin{tikzcd} H^{2n-1}(\mathbb{P}^{2n-1} \setminus X_G) \arrow[r, "\partial"] \arrow[d, equal] & H^{2n-p-1}(\mathbb{P}^{2n-p-1} \setminus X_{G/\!/G'}) \otimes H^{p-1}(\mathbb{P}^{p-1} \setminus X_{G'}) \arrow[d, equal] \\ M(G) \arrow[r, "\partial"] & M(G/\!/G') \otimes M(G') \end{tikzcd}$$

motive of graph

See Connes-Kreimer configurations arising:

C-K motive of $G \iff$ Bestvina-Feighn compactification

both control multiloops

[Full renormalization data : $\mathrm{Spec}\, \mathbb{C}[[t]] \to \mathcal{G}$]