Kevin Costello - Renormalization in BV Formalism

Note Title 5/18/2007

M : compact oriented 4-manifold
with a conformal class of metric

$$A = \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d_+} \Omega^2_+(M)$$

$$\xrightarrow{Id} \quad \Omega^2_+(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \Omega^4$$

degrees: 0 1 2 3

A is a differential graded commutative algebra
with differential Q, & $Tr : A \to \mathbb{C}$:
integration $\int : \Omega^4(M) \to \mathbb{C}$ (degree -3)

Pairing $Tr(ab)$ has no kernel, so
A is Frobenius

- has same relation to Yang-Mills connections
as de Rham does to flat connections.

V a vector bundle on M

$$A(V) = A \otimes_{C^\infty(M)} \Gamma(V)$$

$A(V)$ is an A-module.

Q: What structure on V makes $\mathcal{A}(V)$
a dg $\mathcal{A}$-module?

A: A connection on V satisfying YM

$$\Gamma(V) \xrightarrow{d_A} \Omega^1(V) \qquad d_A \text{ connection on } V$$
$$\Gamma(V) \xrightarrow{B} \Omega^2_+(V) \qquad B \in \Omega^2_+(\mathrm{End}\, V)$$

$d^2 = 0 \iff B = d_{A_+} \circ d_A = F(A)_+$ self dual part of curvature

$d_A B = 0 \iff d_A F(A)_+ = 0$, SD Yang-Mills.

Build 3d CY category with objects Yang-Mills connections....
($\mathcal{A}$-mod ?)

Yang-Mills action:
V vector bundle w/ YM connection

$\mathcal{A}(\mathrm{End}\, V) = \mathrm{End}_{\mathcal{A}}(\mathcal{A}(V))$ dg $\mathcal{A}$-module
$\Downarrow$
has a differential Q
So $\mathcal{A}(\mathrm{End}\, V)$ is a dga with a trace:

$\mathrm{Tr}: \mathcal{A}(\mathrm{End}\, V) \xrightarrow{\mathrm{Tr}_V} \mathcal{A} \xrightarrow{\mathrm{Tr}} \mathbb{C}$

$\Rightarrow$ non commutative Frobenius algebra

Let $\mathcal{E} = \mathcal{A}(\mathrm{End}\, V)[1]$

Following structure:

1. odd symplectic pairing $\mathrm{Tr}(\alpha\beta)$
2. differential Q preserving pairing
3. $S_{YM}: \mathcal{E} \longrightarrow \mathbb{C}$

   $S_{YM}(\alpha) = \mathrm{Tr}(\alpha^3)$ an even function

Claim: $\mathcal{E}$, with action $\frac{1}{2}\langle e, Qe\rangle + S_{YM}(e)$ is the BV odd symplectic manifold constructed from YM gauge theory.

(Chern-Simons: same story for flat vector bundle)

BV formalism: integrate over Lagrangian, or isotropic, subspace in this odd symplectic manifold $\mathcal{E}$.

Pick metric on M & on V

$\rightsquigarrow$ hermitian metric on $\mathcal{E}$,
& $Q^\dagger$ adjoint of Q.

$$\begin{array}{cccc} \Omega^0(\mathrm{End}\,V) & \Omega^1(\mathrm{End}\,V) & \Omega^2_+(\mathrm{End}\,V) & \Omega^4(\mathrm{End}\,V) \\ & \Omega^2_+(\mathrm{End}\,V) & \Omega^3(\mathrm{End}\,V) & \end{array}$$

Ghosts — Lie algebra of gauge

Fields (usual + auxiliary)

antifields

antighosts

$\rightsquigarrow$ cotangent of fields / gauge ...

Want to construct
$$Z(e) = \int_{X \in \mathrm{Im}(Q^\dagger)} \exp\left(\frac{1}{2\hbar} \langle X, QX \rangle + \frac{1}{\hbar} S_{\mathrm{int}}(X, e)\right)$$
$e \in \mathcal{E}$

Moral of the story will be
1. If we use functional integrals of this form, renormalization becomes relatively easy; counterterms are unique + local
2. These functional integrals play a special role in BV formalism.

$\rightsquigarrow$ higher genus homotopical perturbation lemma:

perturbs between solutions of master equation.

<u>Notation</u> $\mathcal{O}(\mathcal{E}) = \prod_{n \geq 0} \mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{C})_{S_n}$

continuous linear maps   completed tensor product   coinvariants

$\mathcal{O}(\mathcal{E})$ is an algebra

$H = [Q, Q^\dagger] : \mathcal{E} \longrightarrow \mathcal{E}$ positive 2^{nd} order elliptic

$K_t \in \mathcal{E} \otimes \mathcal{E}$ heat kernel for H.

Let $P(\varepsilon, T) = \int_\varepsilon^T (Q^\dagger \otimes 1) K_t \, dt$

Any element $P \in \mathcal{E} \otimes \mathcal{E}$ defines a 2^{nd} order differential operator $\partial_P : \mathcal{O}(\mathcal{E}) \circlearrowleft$ by contraction:

$\mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{C})_{S_n} \longrightarrow \mathrm{Hom}(\mathcal{E}^{\otimes n-2}, \mathbb{C})_{S_{n-2}}$

Formal identity:

$Z(e) = \lim_{\varepsilon \to 0} \left(e^{\hbar \partial_{P(\varepsilon, \infty)}} e^{S_{ym}/\hbar} \right)(e)$

– ie in finite dimensions this is easy identity, in ∞ dimensions want to use this as a definition....

Notation: $P \in \mathcal{E} \otimes \mathcal{E}$, $S \in \mathcal{O}(\mathcal{E})$

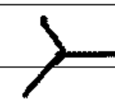
$\Gamma(P,S) = \hbar \log\left(e^{\hbar \partial_P} e^{S/\hbar}\right) \in \mathcal{O}(\mathcal{E})$

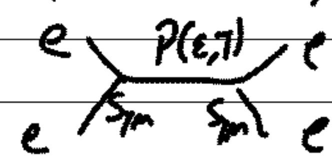
[log: want connected graphs]

Write $\Gamma(P(\varepsilon,T), S_{ym}) = \sum \hbar^i \Gamma_{i,k}(P, S_{ym})$

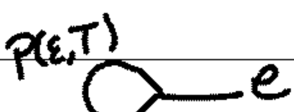
where $\Gamma_{i,k}$ is homogeneous of deg k
as a function on $\mathcal{E}$

$\Gamma_{0,2}, \Gamma_{0,1}, \Gamma_{0,0} = 0$

$\Gamma_{0,3} = (P(\varepsilon,T), S_{ym})(e) = S_{ym}(e)$ 

$\Gamma_{0,4}\ (P(\varepsilon,T), S_{ym})\ (e)$: 

$\Gamma_{0,5}, \Gamma_{0,n}$ = Sum. over trivalent trees. ..

$\Gamma_{1,1}$: 

Theorem If $S \in \mathcal{O}(\mathcal{E})[[\hbar]]$ is any local function

$\Rightarrow \exists$ a small ϵ asymptotic expansion

$$\Gamma_{i,k}(P(\epsilon,T), S)(\epsilon) \sim \sum f_r(\epsilon)\Phi_r(T,\epsilon)$$

where $f_r(\epsilon) \in \mathcal{R} \subset An((0,\infty))$

$\mathcal{R}$ = universal subalgebra of analytic functions on the line, with a countable basis

$$\Phi_r(T,\epsilon) : \mathcal{E}^{\otimes k} \longrightarrow C^\infty((0,\infty])$$

Def Let $R_{\geq 0} \subset R$ be the set

$$\{f \in R, \lim_{\epsilon \to 0} f(\epsilon) \text{ exists}\}$$

A renormalization scheme is a complementary subspace $R_{<0} \subset R$, $R = R_{<0} \oplus R_{\geq 0}$

— contractible space of such!

Theorem Let $S \in \mathcal{O}(\mathcal{E})[[\hbar]]$ be any local functional. Then there exists a unique series $S^{CT}_{i,k}$ (counterterms) which are local functionals, in $\mathcal{O}(\mathcal{E}) \otimes R_{<0}$, homogeneous of degree k as functions of e, such that

$$\lim_{\epsilon \to 0} \Gamma_{i,k}\left(P(\varepsilon, T), S - \sum \hbar^i S^{CT}_{i,k}\right)$$

exists.

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idea $\Gamma_{0,k}(P(\epsilon, T), S)$ non singular $\epsilon \to 0$ limit

$\Gamma_{1,1}(P(\epsilon, T), S)(e) \sim \sum f_r(\epsilon) \Phi_r(T, e)$

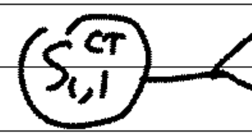
$\Rightarrow$ set $S^{CT}_{1,1}(e) =$ singular part of $\Gamma_{1,1}(P(\epsilon, T), S)(e)$:

ie project $f_r(\epsilon)$ onto $R<0$.

$S^{CT}_{1,2} =$ singular part of $\Gamma_{1,2}(P(\epsilon, T), S - \hbar S^{CT}_{1,1})$ etc.

Draw in case S cubic!

$S^{CT}_{1,2}$ = singular part of  + 

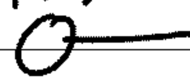
− 

$S^{CT}_{2,0}$ = singular part of
$\Gamma_{2,0}\left(P(\varepsilon,T),\ S - \sum \hbar S^{CT}_{i,k}\right)$

Hard part: verifying answer is local.

<u>Lemma</u> $S^{CT}_{i,k}$ is <u>independent</u> of T
(<u>singular</u> part doesn't change)

$\Longrightarrow$ $S^{CT}_{i,k}$ local: can take T as small as we like $\Longrightarrow$ $\int_\varepsilon^T$ (heat kernel) concentrated near diagonal.

Why? $\frac{d}{dT} S^{CT}_{1,1}$ $P(\varepsilon,T)$ 

$Q^{\dagger} K_T$: nosy!

$\Longrightarrow$ singular part of