Sasha Goncharov – Motives, graphs etc

Note Title 5/19/2007

Y regular projective variety $/\mathbb{C}$
(for simplicity)

$$\left\{\begin{array}{c}\text{mixed } \mathbb{R}\text{-Hodge structure} \\ \text{on rational homotopy} \\ \text{type of } Y\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Feynman} \\ \text{integrals} \\ \text{rel. } Y(\mathbb{C})\end{array}\right\}$$

LHS: M compact Kähler $\Rightarrow H^*(M,\mathbb{R}) \circlearrowleft \mathbb{C}^*$

$\longrightarrow H^*(M,\mathbb{C}) = \bigoplus_{p+q=\dim M} H^{p,q}$ ($z^p \bar{z}^q$ eigenspace)

– pure real Hodge structure.

Deligne

Def 1. $\mathbb{R}$-MHS: V real vector space

'70 $W_\bullet$ filtration on V, $F^\bullet$ on $V \otimes \mathbb{C}$
s.t. $gr_n^W V$ is a pure Hodge structure of wt $= n$.

'82 . $\mathbb{R}$-MHS abelian cat $\Rightarrow$
equivalent to category of representations of a proalgebraic group

$$0 \longrightarrow \underset{\text{prounipotent}}{U} \longrightarrow G_{H} \longrightarrow \mathbb{C}^* \longrightarrow 0$$

– in fact semidirect product

Fact: Lie(U) is <u>free</u> Lie algebra with one generator $N_{p,q}$ for p,q pos. integers

$\Rightarrow$ <u>Def 2</u> An $\mathbb{R}$-MHS is a vector space $V/\mathbb{R}$ &

- $V_{\mathbb{C}} = \bigoplus V^{p,q}$

- $N : V_{\mathbb{C}} \circlearrowleft$ operator,

$$N - \mathrm{Id} : V^{p,q} \longrightarrow \bigoplus_{\substack{p'<p \\ q'<q}} V^{p',q'}$$

strictly lowers both degrees

$(N \leftrightarrow \Sigma N_{p,q})$

- $\bar{N} = -N$, $\bar{\ }$ exchanges p,q.

Lie U has (other) canonical generators....

Rational homotopy type: Sullivan, Quillen, Morgan, Hain

$$\begin{array}{ccc} M \text{ manifold} & \rightsquigarrow & \text{DG Lie algebra} \\ & & L(M,x) \quad (0,-1,\dots) \\ \rightsquigarrow & \nearrow\swarrow & \\ \text{DG commutative algebra } A^\bullet(M) & & \end{array}$$

$L(M,x)$ is of motivic origin – in particular MHS.

Goal: (re)define this $L^{H}(M,x)$.

Need: $gr^W L^{H}(M,x)$ bigraded vector space

• Operator N. : Feynman integral

(for now only sure for curves that this is same as Hain....
get new construction of MHS on cohomology of noncompact varieties!)

$\underline{gr^W L^{H}(M,x)}$

$\mathbb{H} = H_*(Y)/H_0(Y)\ [-1]$ lives in deg $0, -1, \ldots$

reduced homology.

$\tilde{\delta} : \mathbb{H} \longrightarrow \Lambda^2 \mathbb{H}$ coproduct (dual to product in $H^*(X)$)

$T_{\mathbb{H}}$ tensor algebra of $\mathbb{H}$: dg algebra

$\delta : T_{\mathbb{H}} \longrightarrow T_{\mathbb{H}}$ differential generated by $\tilde{\delta}$

$T_{\mathbb{H}}, \delta \quad = \quad gr^W L^{\#}(Y, x)$

(this will be enveloping algebra of the rational homotopy Lie algebra)

<u>$N : T_{\mathbb{H}} \circlearrowleft$</u> Hodge version of Galois symmetry — derivation of tensor algebra.

Lie algebra $C_{\mathbb{H}} = T_{\mathbb{H}} / [T_{\mathbb{H}}, T_{\mathbb{H}}]$
(cyclic words)

$C(h_0 \otimes \cdots \otimes h_m)$ a cyclic word

Let $\mathcal{H} = H^{2n}(Y)[2]$

$\mathbb{H}^\vee \otimes \mathbb{H}^\vee \longrightarrow \mathcal{H}$ Poincaré duality pairing

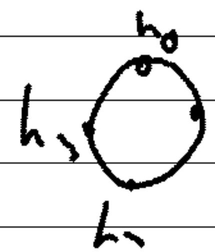
Claim: 1. $C_{\mathbb{H}} \otimes \mathcal{H}$ has dg Lie alg. structure

2. $\chi : C_{\mathbb{H}} \otimes \mathcal{H} \longrightarrow \mathrm{Der}(T_{\mathbb{H}}, d)$

$F \in C_{\mathbb{H}}$, $h \in \mathbb{H}$

$\chi_{F \otimes \mathcal{H}} : h \longmapsto \sum_i \frac{\partial F}{\partial p_i} \langle p_i \wedge \mathcal{H}, h \rangle$

$\mathbb{D} : C_{\mathbb{H}} \longrightarrow T_{\mathbb{H}} \otimes \mathbb{H}$ differential



$\longmapsto (h_1 \otimes h_2 \otimes h_3) \otimes h_0$ + permutation

$\frac{\partial F}{\partial h_0}$

This is $\mathbb{B} : HH_0(T_{\mathbb{H}}) \longrightarrow HH_1(T_{\mathbb{H}})$

Lie bracket: $\{F \otimes \mathcal{U}, G \otimes \mathcal{U}\}$

$$= \frac{\partial F}{\partial p_i} \otimes \frac{\partial G}{\partial q_j} \mathrm{Tr}_{ij}\left((p_i \cup \mathcal{U}) \cup (q_j \cup \mathcal{U})\right)$$

Hodge correlators

A linear map $C_{\mathbb{H}^\vee} \otimes \mathcal{U} \longrightarrow \mathbb{C}$

$\Longleftrightarrow$ $N \subset C_{\mathbb{H}} \otimes \mathcal{U}$, a MHS.

To build this use transcendental methods.

1. Green's current for $Y(\mathbb{C})$:

Hodge theory gives $H^{p,q}(Y(\mathbb{C}),\mathbb{C}) \to A^{*,*}(Y,\mathbb{C})$
... could take *any* such splitting of Dolbeault cohomology.

$Id : H^*(Y(\mathbb{C}),\mathbb{C}) \circlearrowleft$:

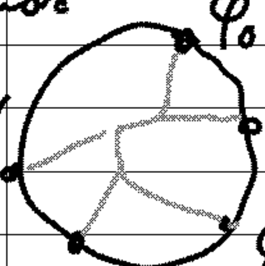
can write using delta-function δ_{Δ_Y} on diagonal as current on $Y(\mathbb{C})\times Y(\mathbb{C})$

Alternately take basis α_i of $H^{p,q}$
$\alpha_i^\vee$ dual basis

$$\Longrightarrow \quad \delta_{\Delta_Y} - \sum \alpha_i \otimes \alpha_i^\vee \quad \text{cohomologous to zero}$$
$$\| $$
$$\frac{1}{2\pi i}\partial\bar\partial G \quad (\text{by } \partial\bar\partial \text{ lemma})$$

G = Green's current.

w
Cyclic word in cohomology



$w = C(\varphi_0, \dots, \varphi_m)$

Draw trivalent tree T
Wanted! a distribution $\nu_T(w)$ on $Y(\mathbb{C})^{\#\text{internal vert. of } T}$

$$\omega_m : \Lambda^{m+1} \mathcal{A}^\bullet_{Y(\mathbb{C})} \longrightarrow \mathcal{A}^\bullet_{Y(\mathbb{C})}$$

polylinear degree m diff op

$$\varphi_0 \wedge \dots \wedge \varphi_m \longmapsto \mathrm{Alt}_{m+1}\Big(\sum_{i=0}^{m} (-1)^i \varphi_i\, \partial\varphi_1 \wedge \dots \wedge \partial\varphi_i \wedge \overline{\partial}\varphi_{i+1} \wedge \dots \overline{\partial}\varphi_m\Big)$$

internal edge $\longmapsto$ Green current G_E

$$\mathrm{Corr}_{\mathrm{Hdg}}(W) = \sum_{\Gamma} \int_{Y(\mathbb{C})^{\text{internal vert}}} \omega_v \left(G_{E_0} \wedge \dots \wedge G_{E_v}\right) \wedge \varphi_0 \dots \wedge \varphi_m$$

$\wedge$

$C_{IH^\vee}$

Open varieties : an example

Y Shimura variety , e.g. universal modular curve.

Langlands : H^* (Shimura varieties) picks up all pure motives.

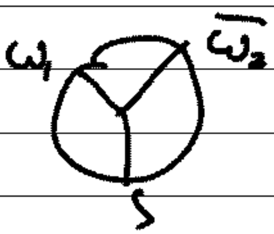
Look at rational motivic homotopy types of Shimuras minus Shimura subvarieties – suggest they carry all information on category of mixed motives !

$\mathbb{Q}$ or other number field $\rightsquigarrow$
{periods of all mixed motives / $\mathbb{Q}$}
countable collection: suggest they should all arise from complements of Shimura varieties

example: modular curve

$$\mathcal{M} : \varprojlim_N \mathbb{H} / \Gamma(N) \hookrightarrow \overline{\mathcal{M}}$$

$\overline{\mathcal{M}} - \mathcal{M} = S$ cusps



$$\int_{X(\mathbb{C})} G(s,x) \wedge \omega_1 \wedge \overline{\omega_2}$$

If ω_1, ω_2 are Hecke eigenforms
$\Rightarrow$ this is Rankin-Selberg integral
$= L(\omega_1 \otimes \omega_2, 2)$ + less interesting numbers

building blocks for Exts of mixed motives are given (via Beilinson conjectures) by such L values.

Feynman picture:

$$\psi : Y(\mathbb{C}) \longrightarrow \mathrm{Mat}_{N \times N}$$

$\psi \in$ Lie algebra valued in matrices

$\Rightarrow$ action $S(\psi) = \mathrm{Tr}\left(\partial\psi \wedge \bar{\partial}\psi + h\psi[\partial\psi, \bar{\partial}\psi]\right)$

Theorem $\int F_w(\psi)\, e^{S(\psi)} D\psi$

$$\underset{\substack{N \to \infty \\ h = \frac{1}{\sqrt{N}} \\ \text{leading term}}}{=\!=\!=} \text{Correlator}(W)$$

for some F_w labelled by cyclic word.