Nigel Higson: The Connes-Kasparov Conjecture

Note Title 5/21/2007

Connes-Kasparov: part of Baum-Connes about Lie groups

G locally compact group $\Rightarrow$ reduced C^* algebra

$$C^*_\lambda(G) = L^1(G) \text{ completed in } B(L^2 G)$$

Why C^* algebras?

- Gelfand-Naimark: comm. C^* algebras $\longleftrightarrow$ Hausdorff spaces
- vehicle for rep theory (I. Segal) $\rightsquigarrow$ GNS construction

Index theory: $M \circlearrowleft$ proper G action on manifold, cocompact action.

$$C_c^\infty(M) \rightsquigarrow L^2(M) \text{ Hilbert space}$$

$$C_c^\infty(M) \rightsquigarrow L^2_G(M) \text{ Hilbert } C^*_\lambda(G)\text{-module}$$

Hilbert module: have inner product with values in $C^*_\lambda(G)$ not just scalars

D Dirac-type operator on M

$\rightsquigarrow$ promote to selfadjoint operator on $L^2_G(M)$ [equivariant]

In favorable situations $\ker D$ is a projective fin. gen. $C^*_\lambda(G)$-module

— always possible to construct index

$$\mathrm{Index}_G(D) \in K_n(C^*_\lambda G)$$

[$n = \dim M$]

<u>Conjecture</u> (Baum-Connes)

M spin G-manifold, universal as proper G-space

[$\forall$ proper G-space $W \longrightarrow M$ map unique up to G-homotopy.

$\longleftrightarrow$ two projections $M \times M \rightrightarrows M$ homotopic

..... negative curvature condition on M]

$K_G(M) :=$ Grothendieck group of equivariant vector bundles.

Then
$$K_G(M) \xrightarrow{\mu^{\text{assembly}}} K_n(C^*_\lambda G)$$
$$E \longmapsto \mathrm{Index}_G(D_E)$$

is an isomorphism.

G complex semisimple group (Type I operator algebra)
π irreducible (tempered since we took L^2)

$\pi[C^*_\lambda G] \subseteq K(H_\pi)$ compact operators on the rep. space H_π
($C^*_\lambda G$ has "points" — homomorphism into $K(H_\pi)$ $\dashrightarrow L^\infty$)

$G = KB = KMAN$ K compact B Borel

$$\hat{G} = \{(\sigma,\lambda) \in \hat{M}\times\hat{A}\}/W \qquad \begin{pmatrix}\text{Gelfand-Naimark}\\ \text{Harish-Chandra}\end{pmatrix}$$
$$(\sigma,\lambda) \longmapsto \mathrm{Ind}_B^G\, \sigma\otimes\lambda$$

$\Longrightarrow$ $\underline{H}$ bundle of representation spaces over $\hat{M}\times\hat{A}$ (Hilbert space bundle)

- Each $H_{\sigma\lambda}$ is irreducible (Gelfand-Naimark, Wallach)
- The action of W on X lifts to $\underline{H}$ G-equivariantly (Kunze & Stein)

$$C^*_\lambda(G) \xrightarrow[\cong]{\pi} K(\underline{H})^W$$
W-equivariant compact endos of $\underline{H}$

& $K(\underline{H})^W \xrightarrow[C_0(X,\underline{H})^W]{\sim} C_0(X)^W$

Morita

Dirac operators: $\quad S$ spin

$L^2_G(M,S) \underset{C^*_\lambda G}{\otimes} H_{\sigma,\lambda}$ = fin dim Hilbert space

$\rightsquigarrow L^2_G(M,S) \underset{C^*_\lambda G}{\otimes} \underline{H}$ = vector bundle on X/W.

Dirac operator acts on this tensor product

$D \otimes Id_{\underline{H}} : L^2_G(M,S) \otimes \underline{H} \longrightarrow L^2_G(M,S) \otimes \underline{H}$

endomorphism of bundle

$\dashrightarrow$ vector bundle equipped with endomorphism

$\Rightarrow$ cycle for K theory

$K_G(M) \longrightarrow K(C^*_\lambda G)$:

$M = G/K$, $\quad K_G(M) = R(K)$ reps of K

$K(C^*_\lambda G) = K(X/W)$

Atiyah-Hirzebruch

Here $X/W = (\hat{M} \times \hat{A})/W = \bigcup_{\sigma \in \hat{M}/W} \hat{A}/W_\sigma$

$\Rightarrow$ $\{$ irreps of $K \}$ $\longrightarrow$ $\{$ Bott generators on $\bigcup \hat{A}/W_\sigma$ $\}$

(RHS is a bunch of vector spaces $\rightsquigarrow$ K theory given by Bott periodicity)

[Penington - Plymen, Wasserman ---- similar in real case]

<u>Example 2</u> $L \subseteq V$ lattice in vector space

$$C^*_\lambda(L) \underset{\text{Fourier}}{\cong} C(V^*/L^*)$$

$M = V$ universal proper space

$K_L(M) = K(V/L)$

Baum-Connes here! $K(V/L) \xrightarrow{\mu} K(V^*/L^*)$

Fourier - Mukai map is <u>involutive</u>.

<u>Kasparov</u> Generalization to lattices in Lie groups — built map going backwards, more or less inverse to Baum-Connes! $\mu^* \mu$ identity on $K_G(M)$ ... ⇝ this half of BC gives Novikov conjecture...

Bott periodicity argument (à la Atiyah with elliptic operators)

Lafforgue ⟹ BC holds for real & p-adic semisimple groups.

⟹ parametrization of discrete series: discrete points of spectrum, all detectable by Dirac operators ---- recover Atiyah-Schmid, Parthasarathy.

<u>Mackey Analogy</u> G Lie group ⟹ Cartan motion group

$$G_0 = K \ltimes (\mathrm{Lie}\, G / \mathrm{Lie}\, K)$$

Mackey: $\widehat{G_0}$ analogous to $\widehat{G}$.

Baum-Connes from deformation point of view:
have one parameter family deforming
G to G_0:

$\{G_t\}_{t\in[0,1]}$ $\quad G_0 = G_0, \quad G_t = G$ for $t\neq 0$

(Connes-Higson: specialization)

Asymptotic morphism
$$K(C^*_\lambda(G_0)) \xrightarrow{\nu} K(C^*_\lambda G)$$

<u>Fact</u> ν isomorphism $\iff$ BC map μ isom

So BC makes Mackey analogy precise on K-theory

$$\hat{G}_0 \xleftrightarrow[\text{method}]{\text{Mackey}} \{(\pi,\lambda) \in \hat{K}_\lambda \times \hat{V}\} / \text{conjugacy in } K$$

(Mackey description of reps of $K \ltimes V$,
let $V = \text{Lie } G / \text{Lie } k = p$ in Cartan decomp)

$$(\pi,\lambda) \longmapsto \text{Ind}_{K_\lambda V}^{KV} \pi\otimes\lambda$$

Up to conjugacy: $p \longrightarrow \mathfrak{a} \xrightarrow{\lambda} \pi$
λ factors through
$\mathfrak{a} = \text{Lie } A$

Typically $K_\lambda = M$. So typical representations are parametrized by $(\hat{M} \times \hat{A})/W$

In fact this is the <u>exact</u> parametrization of $\hat{G}$:

λ not regular $\Rightarrow$ isotropy $K_\lambda \supseteq M$ as normal form

K_λ connected $\rightsquigarrow$ describe reps

$$\hat{K}_\lambda = \hat{M}/W_\lambda .$$

So $$\hat{G}_0 = \coprod_{\lambda \in \hat{A}/W} \hat{M}/W_\lambda = (\hat{M} \times \hat{A})/W$$

$\hat{G}_0 = \hat{G}$ as sets — not as topological spaces!

Let $Z_{\sigma,t}$ = Reps in $\hat{G}_t$ with lowest K-type π_σ or less

$\bar{Z}_{\sigma,t}$ = Reps in $\hat{G}_t$, lowest K-type strictly less than π_σ

These correspond to ideals $J_{\sigma,t} \triangleleft \bar{J}_{\sigma,t} \triangleleft C^*_\lambda(G_t)$

Theorem $J_{\sigma,t} / J_{\sigma,t}^{-} \underset{\text{Morita}}{\cong} C_0(\hat{A}/W_\sigma)$

commutative C^* algebra, independent of t

Corollary $\{ C^*_\lambda(G_t) \}$ is built from constant fields by extensions, Morita equivalences & direct limits

$\Rightarrow$ $K(C^*_\lambda(G_t))$ is constant in t.