Ryszard Nest - Homological algebra & operator algebras

Note Title 5/23/2007

Γ discrete group, A : a Γ-C^*-algebra

$A \longrightarrow K^*_\Gamma(A)$ topological invariant nicely behaved

$\searrow K_*(A \rtimes_{red} \Gamma)$ reduced C^* algebra (harmonic analysis of Γ: appears in C^2 ...) complicated object!

Crossed product is not an exact functor in general.

Say Γ is exact if $A \longmapsto A \rtimes_{red} \Gamma$ is exact

Assembly map : $K^*_\Gamma(A) \xrightarrow{\mu} K_*(A \rtimes_{red} \Gamma)$

Baum-Connes conjecture: μ is an isomorphism

... really question is <u>when</u> is it an isom.

K^* defined very commutatively - in terms of classifying space ... what is a quantum / NC analog?

Both sides live in KK-category, Kasparov's homotopy category for C^* algebras

$KK_*^\Gamma(A,B) := \{$ equiv classes of Γ-equivariant $*$-homomorphisms $\phi: A \to B$, "up to homotopy" $\}$

$* = 0,1$

.... invert some $*$-homomorphisms

Cycles are very concrete here.

Typical cycle: H_B Hilbert C^*-module for B (scalar product valued in B)

$A \xrightarrow{\rho} \mathcal{L}(H_B)$ bounded operators on H_B

$F \in \mathcal{L}(H_B)$ s.t.

$\rho(a)(F^2-1)$ & $\rho(a)(F-F^*) \in K(H_B)$ compact operators.

(Spectral triple: same notion with F unbounded)

If $B = \mathbb{C}$ then such a triple is just a K-homology class

Intuition (Kasparov): ρ is a B-parametrized family of Dirac operators (except now base "Spec B" is noncommutative)

Facts

1. Bott periodicity: suspend A or B shifts grading

2. $KK^\Gamma(A,B) \times KK^\Gamma(B,C) \longrightarrow KK^\Gamma(A,C)$

product ... "familywise index"

KK^Γ : category with objects : Γ-C^*-algebras

& morphisms $KK_0^\Gamma(A,B)$

Functor $A \longmapsto A[-1] = \underset{\text{suspension}}{SA} = C_0(\mathbb{R}, A)$

Two sources of exact triangles:

1. remember $*$-homomorphisms give KK elements

any KK^Γ-hom is equivalent to a $*$-hom

$\phi : A \to B$ $*$ homomorphism

$\Rightarrow$ mapping cone triangle

$$\begin{tikzcd} A \arrow[r, "\phi"] & B \arrow[d, "{[1]}"] \\ & C_\phi \arrow[ul] \end{tikzcd}$$

$C_\phi = \{ (f,a) : f \in C_0((0,1], B),$
$a \in A \ , f(1) = \phi(a) \}$

$\cup$

SB suspension contained as elements vanishing at 1.

Theorem KK^Γ is a triangulated category
(for Γ discrete group or quantum group)

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modification: look at algebras + lattices of ideals:

$(X, \leq)$ poset

$\mathrm{Ob}_X = \{ A \ C^*\text{-algebra} + X \longrightarrow \text{ideals}(A) \}$

$\mathrm{Mor}_X = KK^X(-,-)$: preserve prescribed lattice structure.

Shift $A \to S(A)$

Exact triangles: not enough $*$-homomorphisms to give all needed triangles

Recall every element of $KK_1(A,B) = KK_0(SA,B)$ is an extension $0 \to B \otimes K \to E \to A \to 0$

K = compact operators.

Want to make such into exact triangles.

$\Rightarrow$ triangulated categories.

<u>Applications</u> 1. KK^Γ for Γ a discrete group has a localizing subcategory generated by "compactly trivial" actions:

i.e. $\{A : \forall H \subset \Gamma \text{ finite } A = 0 \text{ in } KK^H\}$

$\Rightarrow$ quotient category

$KK^\Gamma / \langle$compactly trivial$\rangle$

projective objects: P s.t. $KK^\Gamma(P, A) = 0$
$\forall A$ compactly trivial
...eg objects with proper group actions.

Assembly map: $A \longmapsto K_*(A \rtimes_{red} \Gamma)$
has a left derived functor $K^*_\Gamma(A)$
& the natural map from the derived functor
to $K_*(A \rtimes_{red} \Gamma)$ is the assembly map!

Why? 1. Check: $\exists$ a projective approx. to
$\mathbb{C}$ (with trivial Γ-action)
$D \in KK^\Gamma(P, \mathbb{C})$ Dirac operator
$\forall$ Q projective, $KK^\Gamma(Q, P) \xrightarrow{\sim} KK^\Gamma(Q, \mathbb{C})$

$A \in KK^\Gamma$, $P \otimes A \xrightarrow{D \otimes 1_A} A$
is a projective approximation to A

$$K_*((P \otimes A) \rtimes \Gamma) \longrightarrow K_*(A \rtimes_{red} \Gamma)$$

$$\overset{\shortparallel}{K^*_\Gamma(A)}$$

If D is invertible $\Rightarrow$ $P \otimes A \longrightarrow A$ is an equivalence, so everything equivalent to projectable (proper action)
— e.g. Γ amenable, hyperbolic .....

So for example, $K^*_\Gamma(A) \longrightarrow K^*(A \rtimes_{red} \Gamma)$

- version for local cyclic cohomology (version of HC suitable to C^* algebras)

(2) Γ "discrete" quantum group:
... not obvious what to use for subgroups
--- more homog. spaces than subgroups!
so have to decide what we mean by compactly contractible objects — what to use as "torsion".

But for any choice of subcategory get quotient derived category & an assembly map
"qgr": quotient by torsion ...

Examples: Γ = dual $\hat{K}$ for K connected compact group (torsion free discrete quantum group)

$$C^*(K) \xrightarrow{\Delta} \mathcal{M}(C^*(K) \otimes C^*(K))$$

$C^*(K)$ is a $\hat{K}$-algebra

Theorem 1. $\mathbb{C} \in \langle C^*(K) \rangle$: trivial action lives in category generated by module $C^*(K)$ universal proper action

2. $KK^{\hat{K}} \xrightarrow{\sim} \langle C^*(K) \rangle$
(assembly map is an isomorphism)

[e.g for $\Gamma = \widehat{SU_q(2)}$ $0 < q < 1$]

$A \in KK^{\Gamma}$ $B \in KK^{\Gamma \bowtie \hat{\Gamma}}$ Drinfeld double

$\Rightarrow$ can still define twisted tensor product $A \otimes_s B \in KK^{\Gamma}$, use it to repeat commutative arguments.

$A \longmapsto A \rtimes \Gamma \bowtie \hat{\Gamma} \underset{\Gamma\text{-Morita}}{\sim} A$ ... get action of double ...