Dmitri Nikshych - Isotropic Subcategories of Modular Categories

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Main goal: understand p-tensor categories (prime power dimension)
Methods: categorified linear algebra
(notion of Morita pair)

$\{\text{metric Lie algebras}\} \underset{\text{classical limit}}{\leftsquigarrow} \{\text{modular categories}\}$

|| Lie algebra + nondegen. invariant form

k alg. closed field of char. 0

$\mathcal{C}$ - finite semisimple k-linear tensor category

Let $\{X_1, \ldots X_n\}$ be simple objects in $\mathcal{C}$
$\Rightarrow \forall X \in \mathcal{C} \quad X \otimes X_i \cong \bigoplus_j N_{ij}^X X_j$

So $X \longleftrightarrow \mathbb{Z}_{\geq 0}$ matrix $\{N_{ij}^X\} = N^X$

Def $\dim X$ = Frobenius-Perron eigenvalue of N^X (largest positive eigenvalue)

$\dim \mathcal{C} = \sum \dim(X_i)^2$

$\rightsquigarrow$ cyclotomic integer.

$\mathcal{C}$ is modular if it has a nondegenerate braiding & a twist:

$c_{V,W}: \; V \otimes W \xrightarrow{\sim} W \otimes V$

nondegeneracy:

$\{ V : c_{V,W} \circ c_{W,V} = \mathrm{id}_{W \otimes V} \text{ all } W \} = \langle 1 \rangle$

$\Longleftrightarrow$ nondegeneracy of S-matrix.

Twist: automorphism of identity $\theta_V : V \to V$

s.t. $(\theta_V \otimes \theta_W) \, c_{W,V} \circ c_{V,W} = \theta_{V \otimes W}$

Example ("commutative")

G an abelian group, Vec_G = G-graded vector spaces, a tensor category.

Braiding on Vec_G $\Longleftrightarrow$ bilinear form $B: G \times G \longrightarrow k^*$ (simple objects are just 1-dim graded vector spaces)

θ is associated quadratic form.

$\theta_x = B(x,x)\, \mathrm{id}_x$.

Vec_G is <u>modular</u> if this quadratic form is nondegenerate.

[Can also introduce an associator Vec_G^ω to get more examples]

<u>Drinfeld doubles</u>

$\mathcal{C}$ tensor category $\Rightarrow$ modular category $Z(\mathcal{C})$, with $\dim Z(\mathcal{C}) = \dim(\mathcal{C})^2$.

e.g. G abelian, take dual group & add, get nondegen quadratic form $\to$ modular category.

<u>Manin pair</u> $(\mathfrak{g}, \mathfrak{l})$ $\mathfrak{g}$ = metric Lie algebra

$\mathfrak{l}$ Lagrangian subalgebra $\mathfrak{l}^\perp = \mathfrak{l}$

Drinfeld: $\mathfrak{l}$ is a quasi-Lie bialgebra & $\mathfrak{g} \simeq$ double of $\mathfrak{l}$.

Metric Lie algebras $\underset{\text{classical limit}}{\leftsquigarrow}$ quasi Hopf algebras

Categorify Main pairs:

$c_{X,Y} \circ c_{Y,X} \rightsquigarrow$ "scalar product"

$c_{X,Y} \circ c_{Y,X} = id_{X \otimes Y} \rightsquigarrow$ "orthogonality"

<u>Def</u> $\mathcal{C}$ modular. A isotropic subcategory is a full tensor subcategory $\mathcal{L} \subset \mathcal{C}$ st $\theta|_{\mathcal{L}} = id$.

If $\mathcal{L}$ is isotropic $\Rightarrow c_{X,Y} \circ c_{Y,X} = id \ \forall X, Y \in \mathcal{L}$

So $\mathcal{L}$ is symmetric $\xRightarrow[\text{(Deligne)}]{} \mathcal{L} \simeq \mathrm{Rep}\, G$ for G a group.

<u>Def</u> An isotropic subcategory is Lagrangian if $\mathcal{L} = \mathcal{L}^{\perp}$. (in this case $\dim \mathcal{C} = (\dim \mathcal{L})^2$)

<u>Theorem</u> Let $\mathcal{C}$ be a modular tensor category. There is a bijection of sets

$$\left\{ \begin{matrix} \text{Lagrangian} \\ \text{subcategories of } \mathcal{C} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{braided tensor} \\ \text{equivalences} \\ \mathcal{C} \xrightarrow{\sim} Z(\mathrm{Vec}_G^{\omega}) \end{matrix} \right\}.$$

Here Vec_G^{ω} = G-graded vector spaces with associator

$\omega \in Z^3(G, k^\times)$. (We're looking at such equivalences up to equivalences of Vec_G).

Neither G nor ω are uniquely defined...

Morita equivalence

Def (Müger) Let $\mathcal{D}, \mathcal{E}$ be tensor categories. Then $\mathcal{D}, \mathcal{E}$ are categorically Morita equivalent if $\exists$ an algebra A in $\mathcal{D}$ s.t. $\mathcal{E} \xrightarrow{\sim} A$-bimodules in $\mathcal{D}$.

<u>Theorem</u> (D. Naidu, D. Nikshych)

Vec_G^ω, $Vec_{G'}^{\omega'}$ are Morita equivalent iff $Z(Vec_G^\omega) \xrightarrow{\sim} Z(Vec_{G'}^{\omega'})$ as braided tensor categories.

<u>Example</u> Let H be a group acting on an abelian group A. Then $Vec_{A \rtimes H}$ & $Vec_{\hat{A} \rtimes H}$ are Morita equivalent

Application to nilpotent tensor categories

$\mathcal{C}$ a tensor category

Def The adjoint subcategory $\mathcal{C}_{ad} \subset \mathcal{C}$ is the full tensor subcategory of $\mathcal{C}$ generated by all $X \otimes X^*$, X simple.

Ex 1. Let $\mathcal{C} = \mathrm{Rep}\, G$, G finite group

$$\mathcal{C}_{ad} = \mathrm{Rep}\,(G/Z(G))$$

2. $(\mathrm{Vec}_G)_{ad} = \langle 1 \rangle$

Define a central series of $\mathcal{C}$:

$$\mathcal{C}^{(0)} = \mathcal{C}, \quad \mathcal{C}^{(n)} = (\mathcal{C}^{(n-1)})_{ad} \quad n \geq 1$$

$$\mathcal{C} = \mathcal{C}^{(0)} \supset \mathcal{C}^{(1)} = \mathcal{C}_{ad} \supset \ldots\ldots \supset \mathcal{C}^{(n)}$$

Sequence must stabilize since $\mathcal{C}$ finite.

Def $\mathcal{C}$ is *nilpotent* if $\mathcal{C}^{(n)} = \langle 1 \rangle$ some n.

[$\mathrm{Rep}\, G$ nilpotent $\iff$ G nilpotent; Vec_G always nilpotent]

C_{ad} = neutral component of universal grading of C.

Theorem (Etingof - Ostrik - N.)
If $\dim C = p^n$, p a prime $\Rightarrow$
C is nilpotent.

$$\left[\ \dim \operatorname{Rep} G = |G| = \sum_{\substack{V_i \\ \text{irrep}}} (\dim V_i)^2\ \right]$$

Classical limit hint: any nilpotent metric Lie algebra & $\dim \mathfrak{g} = 2n \Rightarrow$
$\mathfrak{g}$ has an abelian Lagrangian ideal $\mathfrak{a}$.
(Koth, Olbrich)

Let C be a modular p-category, $\dim C = p^n$.
When is C equivalent to $Z(\mathrm{Vec}_G^\omega)$ G p-group?

Theorem Let C be a modular category
Then $C \simeq Z(\mathrm{Vec}_G^\omega)$ as a braided $\otimes$cat with G a p-group iff:

1. $\dim C = p^{2n}$ $n \in \mathbb{Z}_+$
2. All objects of C have integral dimension
3. The central charge $\xi(C)$ is 1.

Here central charge

$$\xi(C) := \frac{1}{\sqrt{\dim C}} \sum \theta_{X_i} \dim(X_i)^2$$

... "Gauss sum" ... always root of unity.

__Corollary__ Let C be a tensor category of prime power dimension. Assume that objects of C have integer dimension (automatic if $p>2$)
$\Rightarrow$ C is Morita equivalent to some Vec_G^ω .... i.e. C is group theoretical
(G a p-group)

__Corollary__ Let C be a braided nilpotent category
$\Rightarrow$ $C \cong$ direct product of braided p-categories.
(Sylow decomposition)