Tony Pantev - $*$-quantization of F-M transforms

Note Title 5/21/2007

with D. Arinkin & J. Block
- quantize analytic FM transforms
- deformation theory of $*$-structures
- naturality

Recall If X, Y complex analytic spaces
$K \in D^b_{coh}(X \times Y)$ proper over $X, Y \Rightarrow$

$$\phi_K : D_{qcoh}(X) \longrightarrow D_{qcoh}(Y)$$
$$F \longmapsto p_{Y*}(p_X^* F \otimes K)$$

FM transform: when this is an equivalence

Q: If we deform X in some direction, will Y, K propagate along?

Algebraic result:

Theorem (Arinkin '06) X, Y quasicompact quasisep/$\mathbb{C}$ & smooth, & supp K proper over X, Y
If $\mathbb{X} \longrightarrow \mathbb{D} = \mathrm{Spf}\ \mathbb{C}[[\hbar]]$
formal 1-parameter quantization of X
$\Rightarrow \exists!$ quantization & $\tilde{K} \in D(\mathbb{X} \times \mathbb{Y}^{op})$
st $\phi_{\tilde{K}}$ is an equivalence

Want to generalize this to analytic spaces

$*$-structures & $*$-deformations

Def M complex manifold, P coherent sheaf.
A $*$-deformation of P (as a sheaf of vector spaces)
is a pair $(\widetilde{P}, I_{\widetilde{P}})$ where

- $\widetilde{P}$ is a sheaf of $\underbrace{\mathbb{C}[[\hbar]]}_{=R}$-modules
  $\widetilde{P}/\hbar = P$
- $I_{\widetilde{P}} \subset \underline{\mathrm{Isom}}_R(P \hat{\otimes}_{\mathbb{C}} R, \widetilde{P})$

  which is a torsor over

$\underline{\mathrm{Diff}}_0(P \otimes R, P \otimes R) = \left\{ \sum D_i \hbar^i : D_0 = 1, D_i \in \mathrm{Diff}(P,P) \right\}$

There is a natural version of a $*$-local
map of $*$-deformations.

Def A neutralized $*$-quantization of M
is a $*$-deformation of $\mathcal{O}_M$ + a product
$m : \mathcal{O}_{\widetilde{M}} \otimes_R \mathcal{O}_{\widetilde{M}} \longrightarrow \mathcal{O}_{\widetilde{M}}$, $m/\hbar =$ usual product,
which is $*$ local.

If $P \in \mathrm{Coh}(M)$, a $*$-quantization of P is a $*$-deformation $(\tilde{P}, I_{\tilde{P}})$ + a $*$-local action of $\mathcal{O}_{\tilde{M}}$.

<u>Def</u> A $*$-quantization of M is an $\mathbb{R}$-linear stack of algebroids on M, $\mathbb{M}$, s.t. $\forall\, U \subset M$, $\alpha, \beta \in \mathbb{M}(U)$, have specified an $\mathcal{O}_U$-module structure on $\mathcal{L}_{\alpha\beta} = \underline{\mathrm{Hom}}(\alpha, \beta)/\hbar$.

+ a structure of $*$ deformation of $\mathcal{L}_{\alpha\beta}$ on $\underline{\mathrm{Hom}}(\alpha, \beta)$, so that composition gives an $\mathcal{O}$-linear map mod $\hbar$.
Also want $\mathcal{O}$-module structure on $\mathcal{L}_{\alpha\beta}$ + $*$-structure compatible with restriction.

$QM :=$ stack of neutralized local $*$ quantizations of M, a stack of groupoids over M_{an}.

QM has a topology s.t. $U \subset M$, $\tilde{U} \in QM(U)$
$\rightsquigarrow$ can look at covering families
$\{ \tilde{V}_i \longrightarrow \tilde{U} \}_{i \in I}$ s.t. $U = \bigcup_i V_i$

$\rightsquigarrow$ site QM_{an}, ringed site

$\mathcal{O}_{QM_{an}}(\tilde{U}) = \mathcal{O}_{\tilde{U}}$

QM comes equipped with an inertial action of $\mathcal{O}^{\times}_{QM_{an}}$:

we have $ad: \mathcal{O}^{\times}_{QM_{an}} \longrightarrow I_{QM}$

-- nonconnected version of crossed modules

$\rightsquigarrow$ construct a quotient 2-stack $[QM/\mathcal{O}^{\times}]$

Theorem (ABP) $[QM/\mathcal{O}^{\times}] =$ 2-stack of $\star$-quantization (not just inertial ones)

If Sh_R = stack of R-linear categories which parametrizes sheaves of R-modules on M

$\Rightarrow$ we'll define a sheaf on a $\star$-quantization $\mathbb{M}$ to be a 1-morphism $F: \mathbb{M} \to Sh_R$

(agrees with Polesello-Schapira definition)

- "sheaves of $\mathcal{O}$-modules" on $\mathbb{M}$, even though there's no sheaf $\mathcal{O}$....

Theorem (ABP) If X, Y complex manifolds

$\Gamma \underset{i_\Gamma}{\subset} X \times Y$ closed submanifolds,

$V \longrightarrow \Gamma$ holomorphic vector bundle,

$P = i_{\Gamma *} V$ F-M kernel

$\Rightarrow$ for any $*$-quantization $\mathbb{X}$ $\exists$ $(\mathbb{Y}, \tilde{P})$
with $\mathbb{Y}$ a $*$-quantization of Y,
$\tilde{P}$ $*$-module on $\mathbb{X} \times \mathbb{Y}^{op}$ defines FM kernel,
& $(\mathbb{Y}, \tilde{P})$ unique up to a 1-isomorphism
which is unique up to a unique 2-isomorphism.

Proof using deformation theory

Q: what controls the $*$-quantizations of (Y, P)
that are compatible with $\mathbb{X}$?

Notation If $Z \xrightarrow{p_X} X$ smooth map
& $P \in \mathrm{Coh}\, Z$

$\mathrm{Diff}_Z(\mathcal{O}_X, \ldots \mathcal{O}_X, P; P) := \mathrm{Diff}_{p^{-1}\mathcal{O}_X}(p^{-1}\mathcal{O}_X, \ldots p^{-1}\mathcal{O}_X, \mathrm{Diff}_Z(P, P))$

Can define a bar complex

$$\mathcal{C} : C^0 \to C^1 \to C^2 \to \cdots$$

$$C^0 = \underline{\mathrm{Diff}}_Z(P,P) \qquad C^1 = \mathrm{Diff}(\mathcal{O}_X, P; P)$$

$$C^k = \mathrm{Diff}(\mathcal{O}_X, \ldots \mathcal{O}_X, P; P)$$

with bar differential

<u>Theorem</u> This complex controls the $*$-quantizations of (Y, P) compatible with $\mathbb{X}$, with $Z = X \times Y$.

<u>Theorem</u> $\forall\, U \subset Y,\ Rp_{U*}\, \mathcal{C} \simeq \mathcal{O}_U$

Proof: <u>Step 1</u> $\mathcal{C} \simeq i_{\Gamma*}\, \mathrm{Diff}_{\Gamma/X}(V,V)$

<u>Step 2</u> $Rp_{U*}\, i_{\Gamma*}\, \mathrm{Diff}_{\Gamma/X}(V,V) = \mathcal{O}_U$.