Yan Soibelman - Quantization of CY varieties

Note Title 5/17/2007

Consider $(\mathbb{C}^*)^2$ with $\omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}$

($\longleftrightarrow$ $\{x_1, x_2\} = x_1 x_2$ quadratic
Poisson bracket)

Look at birational automorphisms of $\{,\}$:

e.g. $(x_1, x_2) \longmapsto (x_1(1+\frac{1}{x_2}), x_2)$

$= (\exp\{Li_2(x_2^{-1}), x_1\}, x_2)$

ie exponentiate a Hamiltonian field H_f
associated to dilogarithm $Li_2(t) = \sum_{n\geq 1} \frac{(-1)^n}{n^2} t^n$.

Given any angle $V \subset \mathbb{R}^2$
build pronilpotent group
containing expressions $\exp\{Li_2(x_1^a x_2^b), \cdot\}$
$(a,b) \in V \in \mathbb{Z}^2$

<u>Quantization</u> replace $\mathbb{C}^*$ by k^*, $k = \mathbb{C}((t))$
nonarchimedean field

$q \in k^*$, $|q| = 1$ $\quad (|f| = \exp(-\mathrm{ord}_{t=0} f))$

quantization $\xi\eta = q\eta\xi$:

$\forall\, U \subset \mathbb{R}^2$ open, connected $\rightsquigarrow$
produce an algebra $\mathcal{O}_q(U)$

$$= \left\{ \sum_{m,n\in\mathbb{Z}} c_{m,n}\, \xi^m \eta^n : c_{m,n} \in k \;\&\; \begin{array}{l} \log|c_{m,n}| + mx + ny \to -\infty \\ \text{when } |m|+|n| \to \infty \end{array} \right\}$$

$(x,y) \in U$

nonarchimedean convergent series.

e.g. $q = 1$: $(\mathbb{G}_m^{an})^2 \xrightarrow{\pi} \mathbb{R}^2 \supset U$

nonarchimedean analytic torus

π = tropicalization $\pi(z_1, z_2) = (\log|z_1|, \log|z_2|)$

The analytic space is described by

$\pi_* \mathcal{O}_{(\mathbb{G}_m^{an})^2}$ direct image sheaf,

which is what we wrote above for all q

[π is Stein]

Our birational automorphisms above can be quantized, to an automorphism of our quantized stuff.

$|q|=1$, $\xi\eta = q\eta\xi$,

$$(\xi,\eta) \longmapsto (\xi(1+\eta^{-1}),\eta)$$

$$= \left(\exp\left(\frac{\mathrm{Li}_{2,q}(\eta^{-1})}{q-1}\right)\cdot \xi \exp\frac{\mathrm{Li}_{2,q}(\eta^{-1})}{q-1},\ \eta\right)$$

conjugation by exp of quantum dilogarithm:

$\mathrm{Li}_{2,q}$ uniquely determined from formula

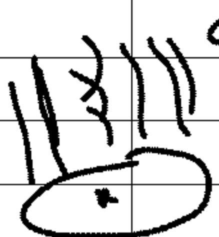
$$\exp\frac{\mathrm{Li}_{2,q}(-x)}{q-1} = (x;q)_\infty := \prod_{n\geq 0}(1-xq^n)$$

$$\parallel$$

$$\sum_{n\geq 0} (-1)^n \frac{q^{\frac{n(n-1)}{2}} x^n}{\prod_{i=0}^{n}(1-q^{i+1})}$$

— Not a cluster transformation..

Recall maximal degenerations of CY varieties



$\dim_{\mathbb{C}} X_t = n$ $\quad \mathcal{X} = (X_t)_t$ family over t, degenerates at $t=0$

$T : H^n(X_t)$ monodromy operator: require to be maximally unipotent —

$\exists\, v$ s.t. $(T - id)^n v \neq 0$, $(T - id)^{n+1} = 0$

(v = holomorphic volume form)

To such associate a polytope, the skeleton.

Think of $\mathcal{X}$ as variety $/ \mathbb{C}((t))$ ---

non archimedean CY analytic space.

$sk(\mathcal{X})$ PL manifold, real variety of dim n.:

choose divisor with normal crossings compactification & basis

incidence polytope of the components

... inside here sub-polytope

which is a birational invariant (indep of blow up) – only components detected by volume form approximately....

$X^{an} \xrightarrow[\underline{\text{Stein}}]{\pi} Sk(X)$, so can describe X^{an} in terms of $\pi_* \mathcal{O}_{X^{an}}$.

$Sk(X)$ carries an integral affine structure, singular at codimension ≥ 2.

e.g. standard degeneration of elliptic curves $\operatorname{Im} \tau \to i\infty$:

Take curve $\longrightarrow Sk(X) = S^1 = \mathbb{R}/\mathbb{Z}$

For any abelian variety $Sk(X)$ is nonsingular.

For $X = K3$, $Sk(X) = S^2$

with integral affine structure singular at (generically) 24 points.

— real part of this comes from elliptically fibered K3 ..... SLag fibration with 24 singular fibers

Now let's use these singular torus fibrations to quantize CYs!

eg $n=2$ (K3) : generically (outside codim 2) the fibration is isomorphic to $(\mathbb{G}_m^{an})^2 \longrightarrow \mathbb{R}^2$

[.... e.g. Drinfeld UHP, skeleton = building
UHP $\longrightarrow$ building is singular torus fibration]

Outside the 24 singular points can quantize using above $\mathbb{G}_m^2$ version...

Try some ansatz at the singular points.

K3 case monodromy of the integral affine structure is everywhere standard ~ conjugate to $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$

$\Rightarrow$ local model for singularity is

$$\begin{matrix} X = K3 \\ \downarrow \pi \\ S \end{matrix} \qquad S = \{ (\alpha\beta - 1)\,\gamma = 1 \}$$

$\rightarrow$ need to quantize $\pi_* \mathcal{O}_f$.

Problem: our sheaves of algebras actually <u>don't</u> glue ...

Let's look on mirror, $\begin{matrix} X^\vee \\ \downarrow \\ S^2 \end{matrix}$ mirror K3

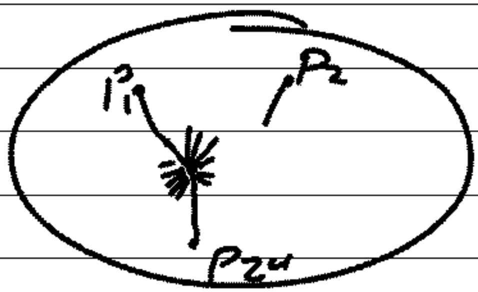
Problem is that tor. fibers have holomorphic discs hanging on them: m_1 operators..

they must pass through singular points

– mirror dual get invariant lives on S^2 for the monodromy operators, fixed by affine structure, so don't have coherent sheaves on them! {since mirror can't
no Fukaya object}

$\Rightarrow$ must use standard symplectomorphism (birational maps) given above to glue the structure sheaves $T_X\mathcal{O}$ we want.

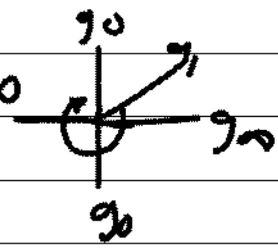
What are these lines?



collisions of two such lines generates ∞ many ones
$\Rightarrow$ infinite tree
(edges in every rational direction)

$T_\ell = \exp(f_\ell(x_1^{n_1} x_2^{n_2}),$

Why dilogarithm? g_∞



the elements
we get are $\in$ some pronilpotent group
& must satisfy $g_0 g_\infty = g_\infty g_1 g_0$

$\rightarrow$ see 5 term identity for dilogarithm

Gross-Siebert on degenerations should imply similar quantization in any dimension!

Donaldson-Thomas story

Fukaya-sib! count SLags s with fixed class in $K(\mathrm{Fuk}\, X)$

... more generally fix a stability condition & count stable objects in fixed K-class..

$\Rightarrow$ connection on Stab $\mathcal{C}$ with coeffs = DT invariants.

Stab contains codim 1 walls like above, along which stable objects change
For every wall have transformation expressing change in # stable objects.
w/ Kontsevich: guess formula for these!
Symplectomorphism of $\mathrm{Im}(K(\mathcal{C})) \subset HP(\mathcal{C})$
symplectic