Boris Tsygan - Fukaya Category

Note Title 5/17/2007

- (M,ω) symplectic $\rightsquigarrow$ Fukaya A_∞ category.
- Microlocal analysis! $M = T^*X \rightsquigarrow \mathcal{D}_X$-mod
  $m \longmapsto$ microsupport(m) involutive
  ... Lagrangians are "noncommutative points"

General M: can look at deformation quantization

$\mathbb{A}_M = C^\infty(M)[[\hbar]] \quad f*g = fg + \frac{i\hbar}{2}\{f,g\} + (i\hbar)^2 P_2(f,g) + \dots$

sheaf of algebras

L Lagrangian $\rightsquigarrow$ module V_L

$$\mathrm{Ext}^\bullet_{\mathbb{A}_M}(V_{L_1}, V_{L_2}) = \bigoplus_{p \in L_1 \cap L_2} \mathbb{C}[n]$$

Exts are calculated locally at intersection points .... no Maslov index, no global A_∞ products etc..

Try to bridge gap between $\mathbb{A}_M$-modules & Fukaya category!

0) <u>Formal geometry</u>.

M manifold $x \in M \rightsquigarrow J_x$ Jets at x
(jets of functions $M \to \mathbb{R}$)

In local coords $J_x \cong \mathbb{C}[[\hat{x}_1, \ldots, \hat{x}_n]] = J$
Under coordinate change $x \longmapsto g(x)$
$\hat{x} \longmapsto g(x+\hat{x}) - g(x)$ $\quad J \xrightarrow{\sim} J$

$\Rightarrow$ vector bundle of commutative algebras J_M
with fiber J & transition functions
$U_i \cap U_j \longrightarrow G = \mathrm{Aut}\, J$ = formal diffeos of 0.

J_M has flat connection ∇_{can}:
$$\nabla_{can} = \sum \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial \hat{x}_i} \right) dx_i$$

& $C^\infty(M) = \Gamma(M, J_M)^{\nabla_{can}}$ flat sections

Given (M, ω) & $*$ deformation quantization
$J_M[[\hbar]]$ bundle of noncommutative algebras

$(M, \omega) = \mathbb{R}^{2n}, d\xi \wedge dx$ have standard
Moyal-Weyl $*$, which is $Sp(2n)$-invariant

$[\xi_k, x_l] = i\hbar\delta_{kl}$ $\quad [\xi_k, \xi_l] = [x_k, x_l] = 0$

$W := \mathbb{C}[[\hat{x}, \hat{\xi}, \hbar]]$ with Moyal $*$

$J_M[[\hbar]]$ is a bundle of algebras on M, fiber $= W$
∇_{can} flat $*$-preserving
$g_{ij} : U_i \cap U_j \longrightarrow G := \mathrm{Aut}\, W$

$G \longrightarrow Sp(2n, \mathbb{R})$

$\hbar \to 0 \searrow$ formal symplectomorphisms of $\mathbb{R}^{2n}$ at 0 $\nearrow$ linear part

Splitting $Sp(2n, \mathbb{R}) \longrightarrow G$ comes from Sp-invariance
of Moyal product.

$\mathfrak{g} = \mathrm{Der}\, W = \left\{ \frac{1}{i\hbar}[F, ?] \right\}$ inner up to scale

$\mathfrak{g} = \frac{1}{i\hbar} W \Big/ \frac{1}{i\hbar}\, \mathbb{C}[[\hbar]]$

$\Rightarrow$ Central extension

$$0 \longrightarrow \frac{1}{i\hbar}\,\mathbb{C}[[\hbar]] \longrightarrow \tilde{\mathfrak{g}} = \frac{1}{i\hbar}W \longrightarrow \mathfrak{g} \to 0$$

Grading $|\hat{x}| = |\hat{\xi}| = 1$, $|\hbar| = 2$

$$\mathfrak{g} = \prod_{i \geq -1} \mathfrak{g}_i, \qquad \tilde{\mathfrak{g}} = \prod_{i \geq -2} \tilde{\mathfrak{g}}_i$$

& $\tilde{\mathfrak{g}}_{-2} = \frac{1}{i\hbar}\mathbb{C}$, $\mathfrak{g}_{-1} = \tilde{\mathfrak{g}}_{-1} = \underbrace{\frac{1}{i\hbar}\langle \hat{x}_i, \hat{\xi}_i \rangle}_{\text{linear functions}}$

$$\tilde{\mathfrak{g}}_0 = \underset{\substack{\shortparallel \\ \frac{1}{i\hbar}\text{ quadratic expressions}}}{\mathfrak{sp}}(2n) \oplus \mathbb{C}\cdot 1$$

Lie(G): starts from $\mathfrak{g}_0$ & move up,

$\mathfrak{g}_{-1}$ moves points.

On group level have nontrivial extension

$$0 \longrightarrow \mathbb{Z} \to \widetilde{Sp(2n)} \longrightarrow Sp(2n) \longrightarrow 1$$

$$G = Sp(2n) \ltimes \exp(\mathfrak{g}_{\geq 1})$$

$$\tilde{G} = \widetilde{Sp(2n)} \ltimes \exp(\tilde{\mathfrak{g}}_{\geq 1})$$

$(M,\omega), * \rightsquigarrow J_M[[\hbar]], *$

∇_{can} is $\mathfrak{g}$-valued : $d+A$, $A \in \Omega^1 \otimes \mathfrak{g}$
$\nabla^2 = 0$

Fedosov: ∇_{can} comes from
a connection $\tilde{\nabla}_{can}$ which is $\tilde{\mathfrak{g}}$-valued

$$\tilde{\nabla}^2 = \frac{1}{i\hbar}\cdot\left(\omega + i\hbar\,\theta_0 + (i\hbar)^2\theta_1 + \dots\right) = \theta$$
$$\in \Omega^2\left(M, \tfrac{1}{i\hbar}\mathbb{C}[[\hbar]]\right) \text{ closed}$$

<u>Fedosov's Theorem</u>
$*$ products for $\longleftrightarrow$ $[\theta] \in H^2\left(M, \frac{1}{i\hbar}\mathbb{C}[[\hbar]]\right)$

$\Rightarrow$ Take the deformation quantization with
$$\theta = \frac{1}{i\hbar}\,\omega.$$

1. Assume the transition functions lift to $\tilde{G}$:

$$\begin{tikzcd} & \tilde{G} \arrow[d] \\ g_{ij} : U_i \cap U_j \arrow[r] \arrow[ur, dotted] & G \end{tikzcd}$$

- happens iff $2c_1(TM)=0$
(Calabi-Yau condition)

2. (Nest-Tsygan 2004) L Lagrangian
$\rightsquigarrow$ V_L A_M-module.

Will give V_L from W-module $\hat{V} = \mathbb{C}[[\hat{x}, \hbar]](\hbar^{-1})$
where $\hat{\xi} \sim i\hbar \frac{\partial}{\partial \hat{x}}$, $\hat{x} \sim \hat{x}$ complete wrt grading
-- $\tilde{\mathfrak{g}} = \frac{1}{\hbar} W$ acts on $\hat{V}$.

Does this integrate to $\tilde{G}$?
$$\begin{array}{ccc} \tilde{P} & \to & \tilde{G} \\ \downarrow & \nearrow & \downarrow \\ P & \subset & G \longrightarrow Sp(2n) \end{array}$$
No, but it integrates to $\tilde{P}$

$P = p \in G$ preserving the Lagrangian $\hat{\xi}=0$.

$\tilde{P}$ = preimage of P.

$\hat{V}$ is a $(\tilde{\mathfrak{g}}, P)$-module with compatible W-module structure.

Structure group of $\mathbb{J}_m[[\hbar]]/_L$
reduces to $\widetilde{P}$ $\Rightarrow$ use this to glue
a bundle of $\mathbb{J}_m[[\hbar]]$-modules $\mathbb{V}_L$
with connection $\widehat{\nabla}^2_{can} = \frac{1}{i\hbar}\omega = 0$ on L.

$\rightsquigarrow$ take its flat sections, get module V_L.

Now $G_{x,y} = \widetilde{\mathrm{Iso}}(\mathbb{A}_x, \mathbb{A}_y)$
cover of groupoid of isomorphisms of $\mathbb{A} = \mathbb{J}[[\hbar]]$
(cover corresponding to fundamental groupoid)
$\widetilde{P}_{xx} \subset G_{x,y}$.

Speculation: correct object to consider is

$$\Upsilon_L = \bigsqcup \widetilde{G}_{y,x} \underset{\widetilde{P}_{xx}}{\times} \mathbb{V}_L$$

induce $\mathbb{V}_L$ to full groupoid :

- $L = \{\hat{\xi} = 0\}$

$\Rightarrow$ Y_L = expressions of form

$$e^{(a\frac{\hat{x}^2}{2} + b\hat{x} + c)/i\hbar} f(\hat{x}, \hbar)$$

+ Fourier transforms.

- Seems to be meaningful for torus T^2 (produces Θ-functions)