Tom Bridgeland - Joyce's work on counting invariants and wall-crossing

Note Title 3/31/2007

Work of Dominic Joyce
(today: joint discussions with Valerio Toledano-Laredo)

$\mathcal{D}$ triangulated category $\rightsquigarrow$ Stab $\mathcal{D}$
complex manifold: "space of stability conditions"

in particular construct holomorphic functions $\rightsquigarrow$ should carry interesting geometric structure

Baby model: $\mathcal{A}$ abelian category
R finite dim k-algebra - eg path algebra of a quiver
$\mathcal{A} = \mathrm{Mod}_{fd}(R)$

$\mathcal{A} \rightsquigarrow \mathrm{Stab}\,\mathcal{A}$

Recall the Grothendieck group
$$K(\mathcal{A}) = \mathbb{Z}\cdot \mathrm{Ob}\,\mathcal{A} \Big/ \begin{array}{l}[B]=[A]+[C] \\ \text{when } 0\to A\to B\to C\to 0\end{array}$$

In our case $K(\mathcal{A})$ = free abelian group on simple R-modules$/\sim$.

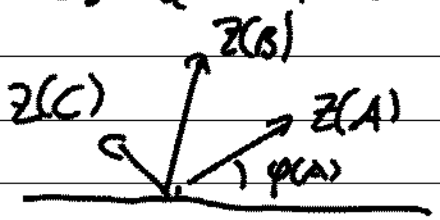
$$K(\mathcal{A}) = \mathbb{Z}^{\oplus n} = \bigoplus \mathbb{Z}[S_i]$$

Positive cone $\quad K_{>0}(\mathcal{A}) = \{[A] : 0 \neq A \in \mathcal{A}\}$

$$= \left\{ \sum k_i [S_i] : \begin{array}{l} k_i \geq 0, \\ \text{not all} = 0 \end{array} \right\}$$

Def A stability condition on $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ s.t. $Z(K_{>0}(\mathcal{A})) \subset \mathbb{H}$ upper half plane

[general D: take t-structure & glue together such Z's on heart...
Today we're just studying a local model for Stab D, which is all we need to describe functions]



Write Stab$(\mathcal{A})$ for the set of stability conditions

$$\text{Stab}\,\mathcal{A} \simeq \mathbb{H}^n \quad \text{via} \quad Z \mapsto (Z(S_1) \dots Z(S_n))$$

An object $0 \neq E \in \mathcal{A}$ is <u>semistable</u> wrt $Z \in \mathrm{Stab}\,\mathcal{A}$ if

$$0 \neq A \subsetneq E \implies \varphi(A) \leq \varphi(E)$$

where the phase $\varphi(E) = \frac{1}{\pi} \arg Z(E) \in (0,1)$

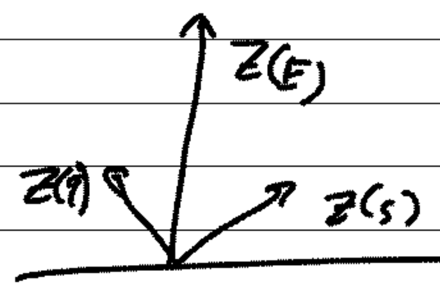
<u>Example</u> $\mathcal{A} = \mathrm{Rep}\,(\bullet \longrightarrow \bullet)$

$\mathcal{A}$ has two simple objects $S = (\mathbb{C} \to 0)$, $T = (0 \to \mathbb{C})$

& one other indecomposable: $E = (\mathbb{C} \xrightarrow{id} \mathbb{C})$

& $0 \to T \to E \to S \to 0$

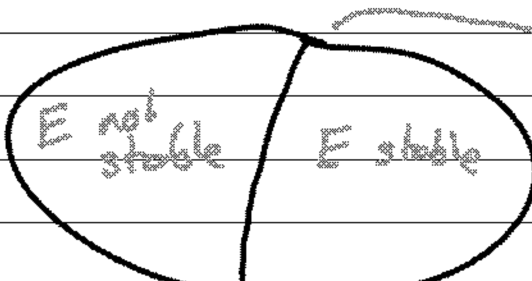
$$Z(E) = Z(S) + Z(T)$$



S, T always semistable

E semistable $\iff \varphi(T) \leq \varphi(S)$

$\mathrm{Stab}\,\mathcal{A}$ is $\mathbb{H}^2$



real codim 1 wall: $Z(S)/Z(T) \in \mathbb{R}_{>0}$

Aim: Want to make holomorphic functions
$\mathrm{Stab}\,\mathcal{A} \to \mathbb{C}$ by "counting" semistable objects & forming a generating function. (weighted by Z)

Obvious problem: Counting invariants are discontinuous due to wall crossing behaviour

Won't discuss how to count (Donaldson-Thomas theory) but rather how to deal with wall crossing.

[• $\mathcal{A}$ = modules for a f.d algebra $\Rightarrow$ stability conditions do come from GIT stability]

<u>Notes</u> i) $\exists$ good moduli space $M_{\mathcal{A}}(\alpha)$ parametrising semistable objects of type $\alpha \in K(\mathcal{A})$

ii) Harder - Narasimhan property:
$\forall\, 0 \neq E \in \mathcal{A}$ $\exists!$ filtration
$0 = E_0 \subset E_1 \subset \dots \subset E_n = E$ with $F_i = E_i/E_{i-1}$ semistable & $\varphi(F_1) > \dots > \varphi(F_n)$.

Hall algebras

$\boxed{\text{General idea:}}$ $\mathcal{M}$ = stack of all objects of $\mathcal{A}$

$\mathcal{M}'$ = stack of short exact sequences in $\mathcal{A}$

$$\begin{tikzcd} \{0\to A\to B\to C\to 0\} \quad \mathcal{M}' \arrow[r] \arrow[d] & \mathcal{M} \ \{B\} \\ \mathcal{M}\times\mathcal{M} \ \{A\}\ \{C\} & \end{tikzcd}$$

correspondence.

Applying a cohomology theory gives a convolution product

$$H^*(\mathcal{M})\otimes H^*(\mathcal{M}) \longrightarrow H^*(\mathcal{M})$$

Second isomorphism theorem $\Rightarrow$ this is associative.

One way to make this idea work:

Assume k finite field

$\mathcal{M}$ = set of isom classes of objects

$$= \coprod_{\alpha\in K(\mathcal{A})} \mathcal{M}_\alpha .$$

Each $\mathcal{M}_\alpha$ is a finite set!

Define $H(\mathcal{A}) = \left\{ \begin{array}{c} \text{functions } f : \mathcal{M} \to \mathbb{C} \text{ with} \\ \text{finite support} \end{array} \right\}$

$$= \bigoplus_{\alpha \in K(\mathcal{A})} \left\{ \begin{array}{c} \text{functions} \\ \mathcal{M}_\alpha \to \mathbb{C} \end{array} \right\}$$

Product: $f * g(E) = \sum_{A \subset E} f(A)\, g(E/A)$ (Ringel)

associative, $f * g * h(E) = \sum_{A \subset B \subset E} f(A)\, g(B/A)\, h(E/B)$

Ringel: $\mathcal{A} = \mathrm{Rep}\, Q$ Q quiver with no loops

$\Rightarrow H(\mathcal{A}) = U_q \mathfrak{n}_+$ $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$

($\mathbb{C}[q]$-algebra) Kac-Moody (Q)

Define characteristic functions for $\alpha \in K(\mathcal{A})$

$$1_\alpha \in H(\mathcal{A}), \quad 1_\alpha(E) = \begin{cases} 1 & [E] = \alpha \\ 0 & \text{otherwise} \end{cases}$$

$$1^{ss}_\alpha(Z) \in H(\mathcal{A}), \quad 1^{ss}_\alpha(Z)(E) = \begin{cases} 1 & [E]=\alpha,\ Z\text{-semistable} \\ 0 & \text{otherwise} \end{cases}$$

ie $1^{ss}_\alpha : \mathrm{Stab}\,\mathcal{A} \longrightarrow H(\mathcal{A})$

Harder - Narasimhan property :

$$1_\alpha = \sum_{n \geq 1} \sum_{\substack{\alpha_1 + \dots + \alpha_n = \alpha \\ \varphi(\alpha_1) > \dots > \varphi(\alpha_n)}} 1^{ss}_{\alpha_1}(Z) * \dots * 1^{ss}_{\alpha_n}(Z)$$

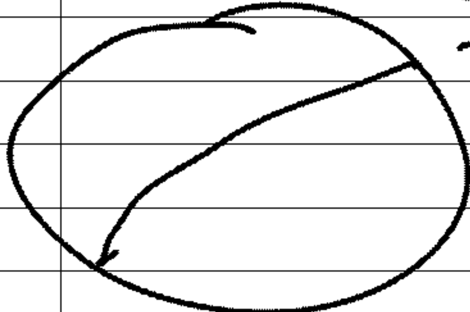
Reineke : inverted this recursion explicitly

$$1^{ss}_\alpha(Z) = \sum_{n \geq 1} \sum_{\substack{\alpha_1 + \dots + \alpha_n = \alpha \\ \varphi(\alpha_1 + \dots + \alpha_n) > \varphi(\alpha)}} (-1)^{n-1} 1_{\alpha_1} * \dots * 1_{\alpha_n}$$

So we know exactly how $1^{ss}_\alpha(Z)$ jumps as we cross walls in phase-space.

Consider $1^{ss}_\alpha(Z)$ on either side of a wall in $\mathrm{Stab}\,\mathcal{A}$:

i.e $\alpha_1 + \alpha_2 = \alpha \in K(\mathcal{A})$



$\frac{Z(\alpha_1)}{Z(\alpha_2)} \in \mathbb{R}_{>0}$ $\qquad [A_i] = \alpha_i$, $[A] = \alpha$

$$0 \to A_1 \to A \to A_2 \to 0$$

we're looking at how A switches from semistable to unstable.

Wall crossing: $1_\alpha^{ss} \longmapsto 1_\alpha^{ss} \pm [1_{\alpha_1}^{ss}, 1_{\alpha_2}^{ss}]$

commutator in $H(\mathcal{A})$

- lose extensions going $A_1 \to \_ \to A_2$
  but gain extensions $A_2 \to \_ \to A_1$!

Joyce: To make a holomorphic function

$$f_\alpha : \mathrm{Stab}\,\mathcal{A} \longrightarrow H_\alpha(\mathcal{A})$$

should write

$$f_\alpha(Z) = 1_\alpha^{ss}(Z) + \frac{1}{2\pi i}\sum_{\alpha_1+\alpha_2=\alpha} \log\left(\frac{Z(\alpha_1)}{Z(\alpha_2)}\right) 1_{\alpha_1}^{ss}(Z) * 1_{\alpha_2}^{ss}(Z)$$

Correct for jump with a log which branches in same way. (branch of log jumping on $\mathbb{R}_{>0}$)

This just takes care of first order stuff
— one term filtration! but have
deeper intersecting of walls to account for!

Joyce writes

$$f_\alpha(Z) = \sum_{n \geq 1} \sum_{\substack{\alpha_1 + \dots + \alpha_n = \alpha \\ \alpha_i \neq 0}} J_n\left(Z(\alpha_1), \dots, Z(\alpha_n)\right) 1^{ss}_{\alpha_1}(Z) * \dots * 1^{ss}_{\alpha_n}(Z)$$

for some functions with branch cuts!

$J_1(z) = 1 \qquad J_2(z_1, z_2) = \frac{1}{2\pi i} \log\left(z_1/z_2\right)$

J_n's have branch loci chosen to balance the jumping behaviour of $1^{ss}_\alpha(Z)$
— universal functions for any $\mathcal{A}$.

Here in abelian category world get only finite sums $\sum \alpha_i = \alpha$ --- in triangulated world gets much worse!

Surprising fact: J_n is essentially unique with this property.

& satisfy differential equation

$$dJ_n(z_1,\dots z_n) = \sum_{i=1}^{n-1} J_i(z_1,\dots z_i)\, J_{n-i}(z_{i+1},\dots z_n) \cdot d\log\left(\frac{z_1+\dots+z_i}{z_{i+1}+\dots+z_n}\right)$$

(Bridgeland)

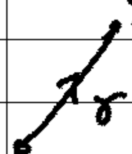
In fact if we invert the basis change

$$1_\alpha^{ss}(Z) = \sum_{\substack{n\geq 1\\ \alpha_1+\dots+\alpha_n=\alpha}} K_n(Z(\alpha_1)\dots Z(\alpha_n))\, f_{\alpha_1}(Z)*\dots*f_{\alpha_n}(Z)$$

Then the K's are given explicitly by polylogarithms / iterated integrals

$$K(z_1,\dots z_n) = \frac{1}{2\pi i}\int_\gamma \frac{dt}{t-s_1}\circ \dots \circ \frac{dt}{t-s_{n-1}}$$

$S_n = z_1+\dots+z_n$



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$s_i = z_1+\dots+z_i$

$$= \frac{1}{2\pi i}\int_{\gamma(\Delta^{n-1})} \frac{dt_1}{t_1-s_1}\wedge\dots\wedge\frac{dt_{n-1}}{t_{n-1}-s_{n-1}}$$

integral on a simplex

<u>Analogy</u> Consider a meromorphic connection

$$\nabla = d + \sum_{i=1}^{n} \frac{A_i \, dz}{z - u_i}$$

on a trivial $\mathbb{C}^n$ bundle over $\mathbb{P}^1$, poles at u_i, ∞.

Riemann-Hilbert: $A_1, \ldots A_n \longmapsto M_1, \ldots M_n$ monodromy matrices

$$\{ (u_1, \ldots, u_n) : u_i \neq u_j \} \sim \text{Stab } \mathcal{A}$$

$$A_i \sim f_\alpha$$

$$M_i \sim 1_\alpha^{ss}$$

Analogy: isomonodromy: need to deform A's in special way with parameters to keep M_i locally constant. (Schlessinger equation)

If <u>fix paths</u> rather than following them around will get jumps ...

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Using f_α can write connection
$$\nabla = d + \left(\frac{Z}{t} + \frac{f}{t^2}\right)dt$$
connection on $H(\mathcal{A})$ bundle over Stab $\mathcal{A}$.

Stokes matrices of this are the $\underline{1}_\alpha^{ss}$.

[Dynkin case: Stab $\mathcal{A}$ is a complexified Weyl chamber. have t-structures corresponding to Weyl chambers, need to glue these together.]

ie ∇ is really characterized by the property that the $\underline{1}_\alpha^{ss}$ are its Stokes matrices!
Joyce was implicitly explicitly inverting the Riemann-Hilbert correspondence!

Laplace transform for ∇ is a Fuchsian connection on $\mathbb{P}^1$:

$$c' + \sum_{\alpha} \frac{\pi_\alpha \circ f}{t - Z(\alpha)} \, dt \qquad \text{(poles at countably many points!)}$$

where $\pi_\alpha : H(A) \longrightarrow H_\alpha(A)$

The Stokes matrices become in this picture certain transition functions:
take a solution near $Z(\alpha)$ which lies in H_α & see its poles upon continuing to point $Z(\beta)$