Lawrence Ein - Valuations via Arcs

Note Title 3/31/2007

X complex variety

$X_\infty = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Spec}\, \mathbb{C}[[t]], X)$
space of arcs

$X_m = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Spec}\, \mathbb{C}[[t]]/t^{m+1}, X)$

A $\mathbb{C}$-algebra
an A point of $X_\infty \leftrightarrow \mathrm{Hom}(\mathrm{Spec}\, A[[t]], X)$.

e.g. $X = \mathbb{C}^2$ $\quad \gamma \in X_m : \gamma \in \mathbb{C}[x,y] \to \mathbb{C}[t]/t^m$

$\gamma \in X_\infty : \gamma \in \mathbb{C}[x,y] \to \mathbb{C}[[t]]$

$X_m \cong \mathrm{Spec}\, \mathbb{C}[x_1, x_2, x_1', x_2', \ldots, x^{(m)}, x_2^{(m)}]$

$X_\infty \cong \mathrm{Spec}\, \mathbb{C}[x_1, x_2, x_1', x_2', \ldots\ldots]$
has natural derivation
$\delta(x_1^{(m)}) = x_1^{(m+1)}$

$$\begin{array}{l} X_m \text{ is an affine } \mathbb{C}^{2m}\text{-bundle.} \\ \downarrow \pi_m \\ X \end{array}$$

e.g $C: x^2 - y^3 = 0$ planar curve.

$C_m : \mathbb{C}[x_1, x_2, \dots, x_1^{(m)}, x_2^{(m)}] / f, f', \dots f^{(m)}$

$f = x^2 - y^3 = 0.$

X smooth of dimension n
locally $X \xrightarrow{\varphi} \mathbb{C}^n$ étale over $\mathbb{C}^n$
& arcs are given by cartesian product

$$\begin{array}{ccc} X_m & \xrightarrow{\pi_m} & X \\ \downarrow & & \downarrow \\ (\mathbb{C}^n)_m & \longrightarrow & \mathbb{C}^n \end{array} \implies \begin{array}{l} X_m \xrightarrow{\pi_m} X \\ \text{is a locally trivial} \\ \mathbb{C}^{nm} \text{ fibration } / X. \end{array}$$

Singular case: e.g isolated planar curve singularity
$C: f(x,y) = 0$ in $\mathbb{C}^2$, multiplicity $m \neq 1$ at origin

$$\begin{array}{ccc} C_m & \hookrightarrow & X_m \\ p_m \downarrow & & \downarrow \pi_m \\ C & \hookrightarrow & X \end{array}$$

$p_m^{-1}(0) = \pi_m^{-1}(0) \cong \mathbb{C}^{2m}$

$(C_{\mathrm{sm}})_m = \mathbb{C}^n$ fibration $/ C_{\mathrm{sm}}$

... have components of different dimension, not irreducible.

Kolchin: X irreducible $\Rightarrow X_\infty$ is irreducible

Nash:
$$\begin{tikzcd} X_\infty \arrow[d, "\varphi"'] & \supset & \varphi^{-1}(S) \arrow[d] & \supset S_\infty \\ X & \supset & S = \mathrm{Sing}\, X & \end{tikzcd}$$

$\varphi^{-1}(S)$ has finitely many irreducible components

$X = \mathrm{Spec}\, A$ affine, $\gamma \in X_\infty$

$\Rightarrow$ have vanishing order

$$f \in A \mapsto \mathrm{ord}_\gamma f \in \mathbb{Z}_{\geq 0} \cup \infty$$

If W is an irreducible component of $\varphi^{-1}(S)$ not containing S_∞

$$\Rightarrow \mathrm{ord}_v f = \min_{\gamma \in W} \mathrm{ord}_\gamma f$$

divisorial valuation for A

(e.g. Y smooth $\supset E$ smooth irred divisor

$Y \xrightarrow{\mu} X$ birational (resolution) $\Rightarrow \mathrm{val}_E f = \mathrm{ord}_E f$)

Resolutions
$$\begin{array}{ccc} Y & \supset & \mu^{-1}(S) \text{ divisor} \\ {\scriptstyle\mu}\downarrow & & \\ X & \supset & S \end{array}$$

Valuations coming from Nash construction <u>always</u> appear (but some others may too) [Kollár, Ishii]

X smooth. $X_\infty \xrightarrow[\psi_m]{} X_m$ truncation.

Cylinder sets = sets of form
$\psi_m^{-1}(C)$, $C \subset X_m$ constructible

X smooth $\Rightarrow$ $X_{m+1} \xrightarrow[\psi_{m+1,m}]{} X_m \supset C$
flat & surjective $\Rightarrow$
$\operatorname{codim}(C, X_m) = \operatorname{codim}(\psi_{m+1,m}^{-1}(C), X_{m+1})$

$\rightsquigarrow$ $= \operatorname{codim}(\psi_m^{-1}(C), X_\infty)$
- so cylinders are finite codimensional.

C irreducible cylinder $\Rightarrow$ valuation Val_C:

$$f \in A \longmapsto \mathrm{Val}_C f = \min_{\gamma \in C} \{ \mathrm{ord}_\gamma f \}$$

discrete valuation for the function field of A.

.... can't have $\mathrm{Val}_C f = \infty$:

if Y is 0-locus of Y,

arcs along Y $Y_\infty \subset X_\infty$ has infinite codimension, but $\mathrm{ord}_\gamma f = \infty \iff \gamma \in Y_\infty$.

$\mathfrak{a} \subseteq A$ ideal. Y closed subscheme defined by $\mathfrak{a}$

Contact: $\mathrm{Cont}^m(\mathfrak{a}) = \{ \gamma : \mathrm{ord}_\gamma Y = m \}$

locally closed cylinder in X_∞

$\mathrm{Cont}^{\geq m}(\mathfrak{a}) = \{ \mathrm{ord}_\gamma Y \geq m \}$ closed cylinder.

$$\begin{array}{ccc} \text{smooth } Y & \xrightarrow[\text{proper, birational}]{\mu} & X \\ \cup & & \\ E & & \end{array}$$

E smooth irred divisor

$\mathrm{Cont}^m(E)$ contact set defines valuation

$\mathrm{Val}_{\mathrm{Cont}^m E} = m \, \mathrm{Val}_E$

so recover this way all divisorial valuations

$$Y_\infty \xrightarrow{\mu_\infty} X_\infty$$

$$\overline{\mu_\infty((\mathrm{Cont}^m(E))} = W(m\,\mathrm{Val}_E) \subset X_\infty$$

is an irreducible closed cylinder set.

... largest cylinder defining same valuation:
if C is any closed cylinder, irred.
s.t. $\mathrm{Val}_C = m\,\mathrm{Val}_E \implies C \subset W(m\,\mathrm{Val}_E)$

[with Lazarsfeld, Mustata]

W is an <u>irreducible contact locus</u> if W is the closure of an irreducible component of $\mathrm{Cont}^m(\alpha)$

C cylinder not dominating $X \implies$
$\exists$ irreducible closed contact set W s.t.
$C \subseteq W$ & $\mathrm{val}_C = \mathrm{val}_W$.

Moreover any contact locus determines a divisorial valuation val_W.

$$m\,\mathrm{Vol}_E \longmapsto W(m\,\mathrm{vol}_E)$$

gives a 1-1 correspondence between divisorial valuations & irreducible closed contact sets

Motivic Integration

X, Y smooth, $Y \xrightarrow{\mu} X$ birational proper

$$Y_m \xrightarrow{\mu_m} X_m$$

$K_{Y/X}$ relative canonical divisor

$\gamma \in Y_m$, $\mathrm{ord}_\gamma(K_{Y/X}) = e$

Assume $m \geq e$

$\Rightarrow \mu_m^{-1}\mu_m(\gamma) \simeq \mathbb{C}^e$.

E divisor, Discrepancy $k_E = \mathrm{ord}_E K_{Y/X}$

multiplicity q

Theorem (Ein Lazarsfeld Mustata)

$$\operatorname{Codim}\left(W(q\,\mathrm{val}_E)\right) = q\,\mathrm{val}_E\, K_{Y/X} + q$$
$$= q\,(k_E + 1)$$

ie discrepancy translates into codimension!

$X = \mathbb{C}^n$ $\quad V$ = divisorial valuation at 0

$\quad v(x_n) = a_n$

Then $\quad$ 1. $\sum a_n \leq q\,(k_E + 1)$ $\quad$ ie discrepancy big if a_i big

$\quad$ 2. If $\sum a_n = q(k_E + 1) \implies$

$\quad\quad V$ is the toric valuation with $v(x_n) = a_n$.

– ie extremal valuations are toric.

Toric valuations: $\quad V = q\,\mathrm{val}_E \quad q = \gcd(a_1, \ldots a_n)$

$$W(q\,\mathrm{val}_E) = \{\, \gamma \in X_\infty : \mathrm{ord}_\gamma(x_i) \geq a_i \,\}$$

[with de Fernex & Ishii]

Singular X of dim $= n$

Nash blow up: $X \hookrightarrow M$ smooth

$$\begin{tikzcd} Gr(\Omega^1_M|_X, n) \arrow[d] & \\ X & \supseteq X_{sm} \arrow[ul, bend right, "\sigma"'] \end{tikzcd}$$

n-dim quotients

Section σ is defined by $\Omega^1_M|_X \twoheadrightarrow \Omega^1_X$ over smooth locus.

Nash blowup is $\overline{\sigma(X_{sm})} \subset Gr(\Omega^1_M|_X, n)$.

Canonically: take sheaf Ω^1_X

$$\begin{tikzcd} \mathbb{P}(\Lambda^n \Omega^1_X) \arrow[d] & \\ X & \supset X_{sm} \arrow[ul, bend right, "\sigma"'] \end{tikzcd}$$

, take closure $\overline{\sigma(X)}$: independent of an embedding!

Fitting ideal $\operatorname{Fitt}(\Omega^1_X) = \mathfrak{a}$

Lipman: $X' \xrightarrow{f} X$ proper birational

$\mathfrak{a} \in \mathcal{O}_{X'}$ is locally principal

$\iff$ 1. $f^* \Omega^1_X \to Q \to 0$ locally free rk n

2. hom. dim $f^* \Omega^1_X \leq 1$.

..... blowing up Fitting ideal factors through the Nash blowup (usually not the same).

$\hat{X}$ = Nash blowup

$$\begin{tikzcd}
 & Y \ni E \arrow[dl] \arrow[d, "\mu"] \\
\hat{X} \arrow[r] & X
\end{tikzcd}$$

factor through Nash

$\Rightarrow$
$$\begin{tikzcd}
\mu^* \Omega^1_X \arrow[rr] \arrow[dr] & & \Omega^1_Y \\
 & \Omega^1_{\hat{X}} \arrow[ur] &
\end{tikzcd}$$

define Nash

$\Rightarrow$ canonical class

$\hat{K}_{Y/X}$: compare to $\hat{X}$, ...

<u>Codimension formula</u>: Codim$(W(q\,\mathrm{Val}_E, X)$)

$= q\,\mathrm{Val}_E\, \hat{K}_{Y/X} + q$

Always have natural map

$$\Lambda^n \Omega^1_X \xrightarrow[\eta]{} \omega_X \quad \text{for any } X.$$

X complete intersection $\Rightarrow$ η defines an ideal $\operatorname{Im}\eta = \mathfrak{b}\,\omega_X$

$\mathfrak{b}$ = Jacobian ideal.

$\Rightarrow$ (with Mustata) adjunction formula for X lci.

- For any surface singularity a sequence of Nash blowups & normalizing gives a smooth variety (Hironaka & Spivakovsky).