Note Title 3/31/2007

Yi Hu: Compactification of the space of maps from $\mathbb{P}^1$ to $\mathbb{P}^n$

(with Lin & Shao)

$$\mathring{M}_d(\mathbb{P}^n) = \text{all algebraic maps from } \mathbb{P}^1 \text{ to } \mathbb{P}^n$$
$$= \left\{ [f_0(w_0,w_1), \ldots, f_n(w_0,w_1)] : \begin{array}{l} \deg f_i = d, \\ \text{no common factor} \end{array} \right\}$$

Good compactification $M_d(\mathbb{P}^n)$

1. should be smooth
2. $M_d(\mathbb{P}^n) - \mathring{M}_d(\mathbb{P}^n) = \bigcup D_i$ divisor with normal crossings
3.* $M_d(\mathbb{P}^n)$ is also a moduli space.

Known compactifications:

- $N_d(\mathbb{P}^n)$: allow common zeros (linear σ-model)
  $= \mathbb{P}\left(H^0(\mathbb{P}^1, \mathcal{O}(d))^{n+1}\right)$
  .. satisfies 1 but not 2.

- Stable map moduli spaces
  $M_{0,0}(d, \mathbb{P}^n)$ or $M_{0,0}(1\times d, \mathbb{P}^1 \times \mathbb{P}^n)$
  $\qquad\qquad\qquad\qquad\qquad \cup$
  satisfies 1-3 in $\qquad \mathring{M}_d(\mathbb{P}^n)$
  orbifold sense.

• Quot compactification for $\mathring{M}_{\beta}(Gr(k, \mathbb{P}^n))$

$N_d(\mathbb{P}^n) \setminus \mathring{M}_d(\mathbb{P}^n)$ has a filtration by subvarieties

$$Z_{d,0}(\mathbb{P}^n) \subset Z_{d,1} \subset \dots \subset Z_{d,d-1}(\mathbb{P}^n) = N_d \setminus \mathring{M}_d$$

$$Z_{d,k} = \left\{ [f_0, \dots f_n] \in N_d : \begin{array}{l} f_0, \dots f_n \text{ have a} \\ \text{common factor of degree} \geq d-k \end{array} \right\}$$

$Z_{d,i}$ are singular for $i > 0$ but
can resolve by inductively blowing up
lower strata! i.e the naive hope works.

Let $V_d = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \simeq k^{d+1}$ k - a field

$$W_k = \bigoplus_{i=0}^{n} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(k))$$

$$\rho_{f_0, \dots f_n; k} : W_k \longrightarrow V_{d+k}$$
$$(u_0, \dots u_n) \longmapsto \sum_{i=0}^{n} u_i f_i$$

$\Longrightarrow$

defines a nowhere zero homomorphism

$$\rho_{d,k} : W_k \longrightarrow V_{d+k} \otimes \mathcal{O}_{N_d}(1)$$

ie $\rho_{d,k} \in \mathrm{Hom}(W_k, V_{d+k}) \otimes \mathcal{O}_{N_d}(1)$

$$\Lambda^{2k+2}\rho_{d,k} \in \mathrm{Hom}(\Lambda^{2k+2} W_k, \Lambda^{2k+2} V_{d+k}) \otimes \mathcal{O}_{N_d}(2k+2)$$

& $Z_{d,k}$ = vanishing locus of $\Lambda^{2k+2}\rho_{d,k}$

(Kokie's theorem)

<u>Lemma</u> $Z_{d,0}$ is the image of the Segre embedding $\mathbb{P}(V_d) \times \mathbb{P}^n \longrightarrow N_d(\mathbb{P}^n)$

$$([h], [a_0 \ldots a_n]) \longmapsto [a_0 h, \ldots, a_n h]$$

$$Bl_{\leq 0} N_d = Bl_{Z_{d,0}} N_d$$

$Bl_{\leq 1} N_d$: blow up proper transform of $Z_{d,1}$

- Define $Bl_{\leq k} N_d$ inductively:

Let $Z_{d,k}^{(k-1)}$ be the proper transform of $Z_{d,k}$ in $Bl_{\leq k-1} N_d \to N_d$

$$Bl_{\leq k} N_d = Bl_{Z_{d,k}^{(k-1)}}(Bl_{\leq k-1} N_d).$$

Exceptional divisors: $E_{d,0}^{(k-1)}, \dots, E_{d,k-1}^{(k-1)} \subset Bl_{\leq k-1} N_d$

$\Rightarrow E_{d,0}^{(k)} \dots E_{d,k-1}^{(k)}$ are their proper transforms

$E_{d,k}^{(k)}$ is the exceptional divisor of the k^{th} blowup.

Theorem (induction)

1. $Bl_{\leq k} N_d$ is smooth
2. $Bl_{\leq k} N_d$ is the closure of the graph of

$$\prod_{j=0}^{k} \psi_{d,j} : N_d \longrightarrow \prod_{j=0}^{k} \mathbb{P}\left(\mathrm{Hom}\left(\Lambda^{2j+2} W, \Lambda^{2j+2} V_{d+j}\right)\right)$$

$$[f_0, \dots f_n] \longmapsto \prod_j [\Lambda^{2j+2} \rho_{f_0,\dots f_n, j}]$$

3. $Z_{d,k+1}^{(k)} \simeq \mathbb{P}(V_{d-k-1}) \times M_{k+1}(\mathbb{P}^n)$

lower degree instance of desired compactification

Corollary $M_d(\mathbb{P}^n) = Bl_{\leq d-1} N_d$ is smooth.

Theorem (transversality)

1. $E_{d,0}^{(k)} \dots E_{d,k}^{(k)}$ is a divisor with normal crossings
2. The scheme theoretic intersection
$$Z_{d,k+1}^{(k)} \cap \bigcap E_{d,i_j}^{(k)} \simeq \mathbb{P}(V_{d-k-1}) \times \bigcap_{j=1}^{k} E_{k+1,i_j-1}^{(k)}$$

Theorem $M_d(\mathbb{P}^n)$ is a good compactification (satisfies 1, 2).

Hodge polynomial: $e_X(u,v)$ $\lambda = uv$

Theorem (recursive formula)
$$e_{M_d} = e_{N_d} + \sum_{k=0}^{d-1} e_{M_k} \frac{\lambda^{d-k+1}-1}{\lambda-1} \frac{\lambda^{n(k+1)}-\lambda}{\lambda-1}$$

Notation $\alpha = (\alpha_1, \dots \alpha_r) \in \mathbb{Z}_{>0}^r$

$|\alpha| = \sum_{j=1}^{n} \alpha_j$

$R_\alpha = \prod_{j=1}^{r} R_{\alpha_j}$ $R_0 = 1$

$\Rightarrow$ Closed formula $e_{M_0} = \sum_{0 \le |\alpha| \le d} R_\alpha \, e_{N-|\alpha|}$

Generalizations: try to replace $\mathbb{P}^n$ by another target X.

Inductive step looks very similar, but initial step $N_d(X)$ is not smooth (even for Grassmannian)
— hope that all of its singularities get resolved by the inductive process.

Conjecture: Same picture holds for any homogeneous space G/P.

Example: $X = Gr(r, \mathbb{C}^n)$
Naive compactification:

$$N_d(Gr(r, \mathbb{C}^n)) \hookrightarrow N_d(\mathbb{P}(\Lambda^r \mathbb{C}^n))$$ Kuznetsov

closure of projective embedding $\leadsto$ "quantum Grassmannian", which is singular... but can resolve by a Quot scheme, small resolution(?)