Note Title 3/31/2007

Daniel Huybrechts - Autoequivalences of Derived Categories of K3 surfaces

(with Macri & Stellari)

X K3 surface = compact complex surface, $K_X \simeq \mathcal{O}_X$, $H^1(X)=0$.

Intersection pairing on $H^2(X,\mathbb{Z}) \Rightarrow$
even unimodular lattice of rank 22
$= 2\cdot(-E_8) \oplus 3U$ (U = hyperbolic plane)

Signature $= (3,19)$

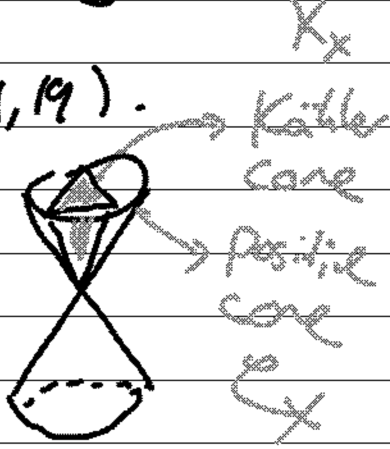
$$H^2(X,\mathbb{C}) = \underset{\mathbb{C}}{\underset{\shortparallel}{H^{2,0}}} \oplus \underset{20\text{ dim}}{H^{1,1}} \oplus \underset{\mathbb{C}}{\underset{\shortparallel}{H^{0,2}}}$$

$H^{1,1}(X,\mathbb{R}) \simeq \mathbb{R}^{20}$, signature $(1,19)$.

$\cup$

$\{\alpha \in H^{1,1}(X,\mathbb{R}) : \alpha^2 > 0\}^{o}$

two connected components



$f : X \longrightarrow X$ automorphism

$\Rightarrow f_* : H^2(X,\mathbb{Z}) \circlearrowleft$ Hodge isometry: compatible with intersection pairing & Hodge decomposition.

But also $f_*(K_X) = K_X \Rightarrow f_*(\mathcal{C}_X) = \mathcal{C}_X$: ... use existence of Kähler classes to show the two components are preserved.

$f_* \mathcal{C}_X = \mathcal{C}_X \iff f_*$ is <u>orientation preserving</u>:

pick $F \subset H^2(X,\mathbb{R})$ three-dimensional so that $(,)|_F$ is positive definite.

Pick an orientation of F, σ

$F, \ f_* F \subset H^2(X,\mathbb{R})$
$\sigma \quad f_*\sigma$

Orthogonal projection gives an isomorphism $F \longrightarrow f_* F$.
"Orientation preserving" means this projection identifies the orientations σ, $f_*\sigma$.

Theorem (Donaldson)
If $f : X \to X$ is a diffeomorphism
$\Rightarrow f_*$ is orientation preserving.

So $\eta : \mathrm{Diff}\, X \to \mathrm{Aut}\, H^2(X, \mathbb{Z})$
has image $= O_+(H^2(X, \mathbb{Z}))$
group of orientation preserving isometries.
$\supset$: uses global Torelli theorem (Borcea)
$\subset$: Donaldson's theorem.

Kernel not understood - e.g. is it connected?
related to Weil's strong form of global Torelli for K3, still unknown.

Derived categories

$\mathrm{Coh}\, X$ = abelian category of coherent sheaves
$D^b(X) = D^b(\mathrm{Coh}\, X)$ bounded derived category.

Consider $\phi : D^b(X) \xrightarrow{\sim} D^b(X)$.
always given by $\mathcal{E} \in D^b(X \times Y)$
$X \xleftarrow{p} \quad \xrightarrow{q} Y$
$F \mapsto q_*(p^* F \otimes \mathcal{E})$

Mukai: To any such ϕ can associate a Hodge isometry $\phi^H : H^*(X,\mathbb{Z}) \circlearrowleft$

... for K3 surfaces can do this integrally

higher dim only get rational statements.

(note! ϕ^H knows what to do with nonalgebraic classes, not just algebraic ones!)

$$\phi^H : \alpha \longmapsto q_*\left(p^*\alpha \cdot ch(\mathcal{E}) \cdot \sqrt{td(X \times X)}\right)$$

.... isometry for Mukai pairing, not quite the intersection pairing: change signs on H^0, H^4.

Put weight 2 Hodge structure on full $H^*(X)$

eg $\tilde{H}^{1,1}(X) = H^{1,1}(X) \oplus H^0(X) \oplus H^4(X)$

$H^{2,0}, H^{0,2}$ don't change. . Signature $= (4,20)$

$$\Rightarrow \quad \rho : \operatorname{Aut} D^b(X) \longrightarrow O(\tilde{H}(X,\mathbb{Z}))$$

group of all Hodge isometries

$$\phi \longmapsto \phi^H$$

Q: What are $\operatorname{Im}\rho$ & $\operatorname{Ker}\rho$?

[Bridgeland: conjecture for $\operatorname{Ker}\rho$ as a certain fundamental group]

[Torelli: automorphisms of X give all Hodge isometries which preserve Kähler cone ... don't include reflections in (-2) classes!]

Orlov, Hosono-Lian-Oguiso-Yau, Ploog:

$$O_+(\widetilde{H}(X,\mathbb{Z})) \subset \operatorname{Im}\rho.$$

... isometry preserving the four positive directions

<u>Theorem!</u> (in progress!) $\operatorname{Im}\rho = O_+(\widetilde{H}(X,\mathbb{Z}))$
i.e. every ϕ^H is orientation preserving

((Conjecture of Szendrői: explains this is the <u>mirror</u> of Donaldson's theorem.))

<u>Examples</u> i) $f: X \xrightarrow{\sim} X \implies$

$f_*: D(X) \xrightarrow{\sim} D(X)$. $\mathcal{E} = \mathcal{O}_{\Gamma_f}$

Γ_f = graph of f.

ii) shift $[1]$. $\mathcal{E} = \mathcal{O}_\Delta[1]$. $\phi_H = -\mathrm{id}$

iii) $L \in \mathrm{Pic}\, X$, $L \otimes \_$. $\phi^H = ch(L) \cdot \_$

iv) Sometimes X is a moduli space of stable sheaves on X itself [Mukai]
$\mathcal{E}$ = universal family over $X \times X$

$\phi = \phi_{\mathcal{E}}$. ϕ^H complicated

v) Spherical twists (Kontsevich; Seidel-Thomas)
$T_E : D^b X \circlearrowleft$ with $E \in D^b(X)$ a spherical object

ie $\mathrm{Ext}^\bullet(E,E) = H^\bullet(S^2, \mathbb{C})$

$$T_E(F) = \mathrm{Cone}\left(\mathrm{Hom}^\bullet(E,F) \otimes E \xrightarrow{\mathrm{eval}} F\right)$$

e.g. any line bundle $L \in \mathrm{Pic}\, X$ is spherical
(but T_L is <u>not</u> $\otimes L$).

e.g. $\mathbb{P}^1 \cong C \subset X$ (-2) curve $\Rightarrow$
$\mathcal{O}_C(i)$ are spherical.

E spherical $\Rightarrow$ its Mukai vector

$v(E) = ch(E)\sqrt{td_X} \in \tilde{H}(X, \mathbb{Z})$

is a -2 vector.

$T_E{}^H$ is just reflection in hyperplane orthogonal to E.

So if $E = \mathcal{O}_C(1) \Rightarrow$ reflection $s_{[C]}$ in -2 class.

[L line bundle $\Rightarrow$
$(1 + c_1 L + \frac{c_1^2 L}{2} + 1)$]

$\mathcal{E} = \mathrm{Cone}(E \boxtimes E^\vee \xrightarrow{tr} \mathcal{O}_\Delta)$

Can prove any of the autoequivalences of types i - iv are orientation preserving.

Complicated functors: come from spherical twists.

Q What happens if there are no spherical objects?

... never happens: even trivial line bundle acts nontrivially, so even on analytic K3s ...

Theorem 2 If (X, α) is a generic twisted K3 surface $\Rightarrow$ $D^b(X, \alpha)$ does not contain spherical objects,

$$\& \ \mathrm{Im}\,\rho = O_+(\tilde{H}(X, \mathbb{Z})) \quad (\text{i-iv generate im}\rho)$$

$$\mathrm{Ker}\,\rho = [2]$$

$\alpha \in H^2(X, \mathcal{O}_X^*)_{\text{torsion}}$ Brauer class.

An α twisted sheaf is a collection of sheaves on open coverings satisfying α-twisted cocycle condition.

$$D^b(X, \alpha) = D^b(\mathrm{Coh}(X, \alpha))$$

Sketch of proof
- no spherical objects,
- no rigid objects $(\mathrm{Ext}^1(-,-) = 0)$
- any $x \in X$ gives a semi-rigid / elliptic object:

$$\mathrm{Ext}^*(-,-) = H^*(S^1 \times S^1, \mathbb{C})$$

Under any autoequivalence semi-rigids go to semi-rigid.

(Mukai) $*$ $\sum \dim \operatorname{Ext}^1(\mathcal{H}^i(F), \mathcal{H}^i(F)) \leq \dim \operatorname{Ext}^1(F,F)$
$= 2$

for $F = \Phi(k(x))$ semi-rigid

At most one such extension $\neq 0$
The others are rigid, but there are none such $\Rightarrow$ so
F is a sheaf in deg 0 up to shift.

$\Rightarrow$ the object $\mathcal{E}$ on $X \times X$ is actually a sheaf (up to shift), in fact a universal family stable sheaves. So we've reduced to an understood situation. ∎

Theorem 3 If $\operatorname{Pic} X = 0$ then $\mathcal{O}_X$ is the only spherical object up to shift &

$\operatorname{Aut} D^b(X) = \operatorname{Aut} X \oplus \mathbb{Z} \cdot [1] \oplus \mathbb{Z} \cdot T_{\mathcal{O}_X}$

<u>Sketch of proof</u> $\phi : D^b(X) \longrightarrow D^b(X)$.

Enough to show $\exists\, n, m$ s.t.
$T^n_{\mathcal{O}_X}\, \phi\, [m] : k(x) \longrightarrow k(y) \quad \forall x$

take points to points up to shift & twist $\Rightarrow$ then get automorphism of X.

Construct t-structure on $D^b(X)$ with heart $\mathcal{A}$ s.t. point sheaves are only semi rigid minimal sheaves (no nontrivial subsheaves).

$\mathcal{A}' = \phi(\mathcal{A})$

enough to show $\exists\, n, m \quad \forall y \in X \quad T^n k(y)[m]$ is in $\mathcal{A}'$ & <u>minimal</u>.

– use inequality $*$ above to show all but one cohomologies of $k(y)$ are spherical – but we know that $\mathcal{O}_X$ is the only spherical one. Using $T_{\mathcal{O}_X}$ then make $k(y)$ into something of smaller length $\rightsquigarrow$ land in $\mathcal{A}'$.

— really need to put stability conditions & resulting Harder–Narasimhan filtrations to reduce to minimal sheaves. ... ∎

Apply this to Theorem 1:

$\phi_{\mathcal{E}} : D^b(X) \circlearrowleft$ Deform this (by deforming $\mathcal{E}$) to autoequivalence of $D^b(X_t) \circlearrowleft$ in some family. If X_t generic

$\Rightarrow \phi^H_{\mathcal{E}_t}$ preserves orientation.

But $\phi^H_{\mathcal{E}_t}$ <u>constant</u> in t

— problem! must deal with convergence issues of deformations!

- instead work with rigid analytic space of formal scheme $\mathfrak{X} \longrightarrow \mathrm{Spf}(\mathbb{C}[[t]])$

$\Rightarrow$ derived category $D^b(\mathfrak{X}) / D^b(\mathfrak{X})_{t\text{-torsion}}$

- mod out by sheaves supported on some
  multiple of the general fiber

  $\Rightarrow$ get model for D^b of the rigid analytic
space : get $\mathbb{C}((A))$ - linear category
& prove Theorem 3 in this context.