János Kollár - Quotients of Calabi-Yau Varieties

Note Title 3/31/2007

with M. Larsen

Ex 1. $E_1 = (y^2 = x^3 - 1)$

$\tau(x,y) \longrightarrow (x,-y)$

$E_1/\tau = \mathbb{P}^1$ $\quad E_1 \times E_1/(\tau,\tau)$ K3 surface

2. $E_2 = (y^3 = x^3 - 1)$

$\sigma : (x,y) \longrightarrow (x, \varepsilon y)$, $\varepsilon^3 = 1$

$E_2/\sigma = \mathbb{P}^1$ $\quad E_2 \times E_2/(\sigma,\sigma)$ $\quad E_2 \times E_2/(\sigma,\sigma^2)$

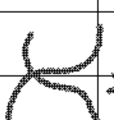
Let's write these down:

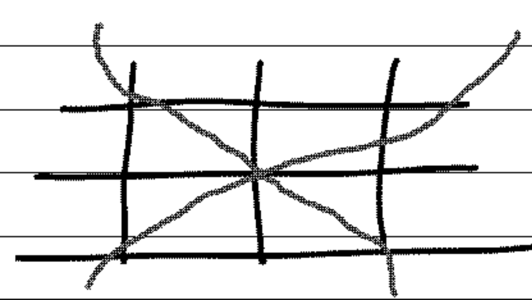
σ,σ $\quad y_1^3 = x_1^3 - 1$, $\quad y_2^3 = x_2^3 - 1$ $\Rightarrow$ invariants $u_1 = y_1 y_2^2$

equation $u_1^3 = (x_1^3 - 1)(x_2^3 - 1)^2$

σ,σ^2 : $u_2 = y_1 y_2$, $\quad u_2^3 = (x_1^3 - 1)(x_2^3 - 1) \Rightarrow$ K3

σ,σ case: Take triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along some curves:

 = singular curve of type (2,2) on $\mathbb{P}^1 \times \mathbb{P}^1$, a rational curve



Claim: preimage of this curve C, C',
has only two actual ramification point
(possible ramification at intersection with ###
but multiplicities cancel
- while in σ, σ^2 case get lots of
ramification points!)

find a $\mathbb{P}^2$ of rational curves containing
after counting number of such C
$\Rightarrow$ this is a rational surface
So some quotients K3, some rational!

3. $F_3 = (y^6 = x(x-1)^2(x+1)^3)$
$\omega : (x,y) \longrightarrow (x, \varepsilon y)$ $\quad \varepsilon^6 = 1$

<u>Claim</u> $F_3^n / (\omega, \omega)$ is rationally connected
$\Longleftrightarrow n \leq 5$

Problem X CY variety, smooth
(just need $K_X \equiv 0$ numerically)

$G \subset \mathrm{Aut}\, X$ finite subgroup
What kind of variety is X/G?

3 cases:
1. X/G is CY (with canonical singularities)
2. otherwise X/G is uniruled
3. stronger: X/G is rationally connected

Aim is to decide among these cases using:
$x \in X \rightsquigarrow G_x = \mathrm{Stab}_x$
$\rho_x : G_x \circlearrowright T_x X$

- read off above cases from stabilizer representations.
- will completely distinguish 1 & 2 &
almost always from 3.

Reid-Tai criterion $G \subset GL_n$ finite

$g \in GL_n \rightsquigarrow$ can diagonalize $g \sim \begin{pmatrix} e^{2\pi i a_1} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i a_n} \end{pmatrix}$

$\mathrm{age}(g) = \sum a_i$ (Reid) $\qquad 0 \le a_i < 1$

Def. $g \neq 1$ satisfies the Reid-Tai criterion if
$\text{age}(g) \geq 1$.

Theorem 1 X smooth projective CY, $G \subset \text{Aut}\, X$ finite
$\Rightarrow$ X/G is uniruled iff
$\exists x \in X$, $\exists g \in G_x$ which fails Reid-Tai.

$K_X = 0$

$$\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle \pi} & & \\ X/G & \xleftarrow{g} & Y \end{array}$$ resolution of singularities

$K_Y = g^*(\pi_* K_X) + E$ where E is supported on $\text{Ex}(g)$ exceptional divisor $\cup$ Branch (π).

(Here $K_X = 0$.)

Reid-Tai says precisely where
E is effective : $E \geq 0 \Longleftrightarrow$ RT holds

Pf of Theorem 1 : Miyaoka-Mori criterion:
Z variety covered by $\{C_t\}$ family of curves & $C_t \cdot K_Z < 0 \Longrightarrow$
Z uniruled.

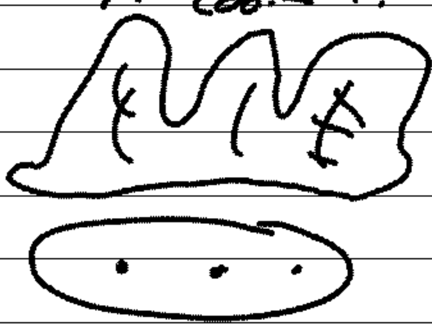
If π ramifies in codim 1 - branch divisor appears in E with negative multiplicity
may take C_t to be hyperplane sections,
No problem.

Interesting case is when no ramification in codim 1.

Y

$\downarrow$

Y/G



$E = \sum a_i E_i$

& $\exists i_0$ st $a_{i_0} < 0$

Try 1: find family C_t
s.t. $C_t \cdot E_{i_0} > 0$, $C_t \cap E_i = \emptyset$ for $i \neq i_0$.

Don't know how to produce them. ...
but this is overkill

Try 2: Enough to find C_t, s.t.
$C_t \cdot E_{i_0} >> 1$, $C_t \cdot E_i < M$ $i \neq i_0$

Some fixed M.

In dimension 2 everything is easy: $\forall E_i^2 < 0 \Rightarrow$

independent extremal rays in cone of curves.

– can find divisor with 0 intersection with E_i's & large intersection with E_{i_0}. Perturbation will be ample, some power will give us desired family of curves

In general try to cut down to surfaces.

- .... need some tricks but basically this reduction works. Lazarsfeld: this follows from description of nef cone in Demailly...

∎

$$\begin{tikzcd} X \arrow[d] \arrow[dd, dotted, bend left] \\ X/G \arrow[d, dashed, "G"] \\ Z \end{tikzcd}$$

Now assume X/G uniruled

$\Rightarrow$ MRC fibration: almost all rational curves lie in fibres & fibers rationally connected, & Kodaira dim $Z \geq 0$.

$X \dashrightarrow Z$ rational map from CY to something with Kod ≥ 0..

Can happen: e.g. $X_1 \times X_2 \longrightarrow X_1$ projection for X_1, X_2 CY.

<u>Theorem 2</u> X CY, $g: X \dashrightarrow Z$

rational map with $\kappa(Z) \geq 0$

$\Rightarrow$ can find a finite cover $\tilde{X} \to X$

which decomposes as $\tilde{Z} \times \tilde{F}$ with $\tilde{Z}, \tilde{F}$ CY

$$\begin{array}{ccc} \tilde{Z} \times \tilde{F} = \tilde{X} & \xrightarrow[\text{étale}]{} & X \\ {\scriptstyle \text{proj}} \downarrow & & \vdots \\ \tilde{Z} & \underset{\substack{\text{generically} \\ \text{finite}}}{\dashrightarrow} & Z \end{array}$$

(étale for X smooth)

So up to finite cover any such g is a projection on a factor.

If X is simply connected & not a product (e.g a CY 3 fold with $\pi_1 = 0$ is always irreducible.. so this won't happen)

$G \subset Gl_n$ subgroup which fails RT

ie $\exists\ 1 \neq g \in G \quad \text{age}(G) < 1$.

– easy to verify algorithmically for given G.

Tricky case: $\mathbb{Z}/m \subset Gl_3$ open for some time if it on fail RT

for $\mathbb{Z}/m \subset Gl_4$ still open to describe all such satisfying RT)

Reductions ① reduce to case where
$\langle GgG^{-1} \rangle = G$ conjugacy class of g generates G

② Age is additive $\Rightarrow$ can reduce to case where representation is irreducible.

③ Look at image $G \to PGl_n$:
if RT fails for $G \to Gl_n$, enlarge G throwing in m^{th} roots of unity
- if age $g < 1$ age $g \cdot \varepsilon < 1$ ε root of unity close to 1 ...
$\Rightarrow$ construct rep into PGl_n where RT fails.

Easy examples where RT fails:
reflection groups!

Theorem 3 If RT fails, $\langle G g G^{-1} \rangle = G$
+ have irreducible representation
$\Rightarrow$ either 1. G is projectively equivalent to a reflection gp.

2. Case $|G_{29}| = 7680$, $G_{29} \subset GL_4$
reflection group —
but have another projective rep in GL_4
which is not equivalent to a reflection rep

3. Possibly finitely many other exceptions

So for a typical group the quotient will be CY ... reflection groups are the boring case where we have branching in codimension 1 $\Rightarrow$ get uniruled quotient.

Theorem 4 $X = A$ simple abelian variety, $G \subset \operatorname{Aut} A$ finite $\Rightarrow A/G$ is a singular CY.

Idea of Thm 3 $G \subset U(n)$

age measures distance to eigenvalue on a circle,
i.e. not the natural metric on S^1.

Write $b_i = \min\{a_i, 1-a_i\}$ for $e^{2\pi i a_i}$

Metric on $U(n)$: $d(g,e) = \sqrt{\sum b_i^2}$

Our G is generated by $G \cap B_1(e)$ small neighborhood of identity (gen. by GgG^{-1})

Claim 1 If G is irreducible, $\exists h \in G$ s.t. $d(h,e) \geq \sqrt{n}$. :

Orthogonality of characters $\Rightarrow \sum_{g \in G} tr(g) = 0$

$\Rightarrow \exists h \in G$ with $\mathrm{Re}\, tr(h) \leq 0$

.... so need eigenvalues with negative real part $\rightsquigarrow$ simple estimate.

Covering # of G : how many elements

of our conjugacy class do we need to multiply to get all of G —

$$\min_c G = \underbrace{GgG^{-1} \cdots GgG^{-1}}_{c \text{ times}}.$$

So if RT fails $\Rightarrow c(G) \geq \frac{1}{2}\sqrt{n}$.

Known: If G is a simple group,
$c(G) \leq C \log |G|$ C some constant

<u>Fact</u>: $A_{n+1} \subset GL_n$ is by far the biggest simple group –

$|A_{n+1}| \sim n^n$.

So if understand reps of groups of Lie type, find that for any other simple group

$|G| \leq n^{C \log n}$

$\Rightarrow c(G) \leq c(\log n)^2$

$\rightsquigarrow$ reduce to reflection groups which are described using symmetric group.... ☐

So for quotients of Calabi-Yau
this proves the conjecture that
negative Kodaira dimension $\Rightarrow$ uniruled!