Rob Lazarsfeld - Syzygies of Multiplier Ideals

Note Title 3/31/2007

X smooth variety / $\mathbb{C}$
$\dim X = d$
$\mathfrak{b} \subseteq \mathcal{O}_X$ ideal sheaf.

Multiplier ideals: given $c > 0 \implies$
construct new ideal
$\mathcal{J}(\mathfrak{b}^c) \subseteq \mathcal{O}_X$ measures
singularities of functions $f \in \mathfrak{b}$

worse singularities $\longleftrightarrow$ "deeper" $\mathcal{J}(\mathfrak{b}^c)$

<u>Definition</u> Choose generators $f_1, \ldots, f_r \in \mathfrak{b}$

$$\mathcal{J}(\mathfrak{b}^c) \underset{\text{locally}}{=} \left\{ h : \frac{|h|^2}{(|f_1|^2 + \cdots + |f_r|^2)^c} \text{ locally integrable} \right\}$$

... away from zero locus e.g. $h = 1$ works
get everything.

Near zeros, denominator has singularities
$\rightarrow$ need h to vanish to keep this integrable.

<u>Definition'</u> Let $\mu : X' \to X$ be a log resolution of $\mathfrak{b}$

$\mathfrak{b}\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ $\quad$ F cartier divisor

$$\mathcal{J}(\mathfrak{b}^c) = \mu_* \mathcal{O}_{X'}(K_{X'/X} - [cF])$$

... functions vanishing to degree c upstairs up to correction.

<u>Ex</u> $\quad \mathfrak{b} = (x_1^{e_1}, \dots, x_d^{e_d}) \in \mathbb{C}[x_1, \dots x_d]$

monomial ideal

$$\mathcal{J}(\mathfrak{b}^c) = \left\langle x_1^{m_1} \cdots x_d^{m_d} : \sum \frac{m_i + 1}{e_i} > c \right\rangle$$

As we increase the e's our functions are getting more singular $\Longleftrightarrow$ have to increase m's to keep inequality ie ideals are getting deeper.

Kawamata-Viehweg-Nadel vanishing theorem for multiplier ideals $\Rightarrow$ many applications & special properties.

Global applications

- Deformation invariance of plurigenera (Siu, later Kawamata)
- Boundedness for pluricanonical mappings (Tsuji, Hacon-McKernan, Takayama)

- Minimal model program!

Local applications

- uniform & effective results in commutative algebra (Ein - L - Smith)

ex. $Z \subseteq X$ irred subvariety of codimension e, $\mathcal{I} \subseteq \mathcal{O}_X$ ideal of Z.

Consider $f \in \mathcal{O}_X$ s.t. $\operatorname{ord}_x(f) \geq m \cdot e$ at general point $x \in Z$.

$\Longrightarrow$ $f \in \mathcal{I}^m$!

-- trivial at smooth points, hard at singular points.

i.e. $\mathcal{I}^{[me]} \subset \mathcal{I}^m$ symbolic power $\subset$ usual power.

Key point : $\mathcal{J}(\mathfrak{a}^{c_1}\mathfrak{b}^{c_2}) \subseteq \mathcal{J}(\mathfrak{a}^{c_1}) \cdot \mathcal{J}(\mathfrak{b}^{c_2})$

subadditivity of multiplier ideals.

Connections with D-modules & char. p commutative algebra

<u>Vague Question</u> Which ideals are multiplier ideals?

ie which ideals can locally be realized this way

e.g. Given $\mathfrak{a}$, is $\mathfrak{a} = \mathcal{J}(\mathfrak{b}^c)$ for some $\mathfrak{b}, c$?

<u>No!</u> .... have local properties

<u>Integral Closure</u> $\mathfrak{a} \subset \mathcal{O}_X$ ideal

Say $\mathfrak{a}$ is integrally closed if (equivalently)

(i) Take local generators $f_1, \dots f_r \in \mathfrak{a}$ and say $g \in \mathcal{O}_X$ is function s.t.

$|g(z)| \leq C \sum |f_i(z)|$ (ie vanishes at least like f's)

$\Rightarrow g \in \mathfrak{a}$.

(ii) $\exists$ proper birational map $\nu : X^+ \longrightarrow X$ (X^+ normal) & effective Cartier divisor F on X^+ s.t. $\mathfrak{a} = \nu_* \mathcal{O}_{X^+}(-F)$.

... can determine membership in $\mathfrak{a}$ by orders of vanishing on a divisor

Intuition ideal $\mathfrak{a}$ <-----> linear series (take linear combinations of generators)

$\mathfrak{a}$ integrally closed <-----> complete linear series

Corollary Multiplier ideals are always integrally closed.

Other local properties: none known up to now!

Ex (Mustaţă et al) Every integrally closed monomial ideal is a multiplier ideal.

Dim $d=2$: [Favre-Jonsson, Lipman-Watanabe] Every integrally closed ideal is locally a multiplier ideal.

..... sequence of blowups at points will smooth a point on a surface. nef cone of the resolution is a simplicial polyhedral cone, very simple structure, can control.

Kyungyong Lee : The ideal of a suitable union of lines in $\mathbb{C}^3$ is not a multiplier ideal.

Local Syzygies of multiplier ideals

Fix $x \in X^d$, work in local ring $(\mathcal{O}, m)$ of X at x

<u>Theorem</u> (Lazarsfeld-Lee) Let $J = J(\lambda^c)_x$ be (the germ at x of) any multiplier ideal

If $p \geq 1$ no minimal p^{th} syzygy of J can vanish modulo m^{d+1-p}.

<u>Explanation of statement</u>

- Case $p=1$: choose minimal generators $h_1, \dots h_s \in J$

Consider functions $g_1, \dots g_s \in \mathfrak{m}$ giving a minimal syzygy, ie $\sum g_i h_i = 0$.

Then $\mathrm{ord}_x(g_i) \leq d-i$ for some i

... not all coefficients vanish to high order.

In general consider a minimal resolution

$$\cdots \to \mathcal{O}^{b_2} \xrightarrow{u_2} \mathcal{O}^{b_1} \xrightarrow{u_1} \mathcal{O}^{b_0} \to J \to 0$$

Columns of u_1 are syzygies

" " u_2 are " between syzygies

$$\mathrm{Syz}_p(J) = \mathrm{Im}(u_p) \subseteq \mathcal{O}^{b_{p-1}}$$

Theorem says no minimal generator of Syz_p lies in $\mathfrak{m}^{d+1-p}\mathcal{O}^{b_{p-1}}$

[Forced by integral closure in dim 2!]

Rmk Cannot exist any restrictions on order of vanishing of generators of multiplier ideals

.... since $\mathfrak{m}^d = J(\mathfrak{m}^{d+n-1})$ is a multiplier ideal $\forall d$.

Example If $d \geq 3$, choose $m = d-1 \geq 2$ general functions $h_1, \dots h_{d-1} \in \mathfrak{m}^d$.

$I = (h_1, \dots h_{d-1})$ full singular curve

... but reduced, hence integrally closed.

But violating conditions of thm!

syzygies $\underbrace{h_2}_{g_1} h_1 - \underbrace{h_1}_{g_2} h_2 = 0$ (Koszul)

vanish to order d, so not a multiplier ideals

– so general ideals are not multiplier ideals.

Idea of proof : local version of Castelnuovo-Mumford regularity.

Skoda complexes Write $\mathfrak{m} = (z_1, \dots z_d) \subseteq \mathcal{O}$

have $\mathfrak{m}^l \cdot J(\mathfrak{m}^k \mathfrak{b}^c) \subset J(\mathfrak{m}^{k+l} \mathfrak{b}^c)$

$\Rightarrow$ can define Koszul-type complexes. Skoda

$$\to \mathcal{O}^{\binom{d}{2}} J(\mathfrak{m}^{d-2} \mathfrak{b}^c) \to \mathcal{O}^d \otimes J(\mathfrak{m}^{d-1} \mathfrak{b}^c) \xrightarrow{[z_i]} J(\mathfrak{m}^d \mathfrak{b}^c) \to 0$$

Subcomplex of Koszul complex, keeping track of orders of vanishing.

<u>Theorem</u> (Ein-Lazarsfeld...) $Skod_\bullet$ is exact.
($\Rightarrow$ Skoda's theorem $\mathcal{J}(\mathfrak{m}^d \mathfrak{b}^c) = \mathfrak{m}\mathcal{J}(\mathfrak{m}^{d-1}\mathfrak{b}^c)$)

– <u>effective</u> version of Koszul.

Complements & Variants

I. Singular versions (w. Lee, Smith)
ex. Assume X^d Gorenstein with rational singularities $\rightarrow$ no minimal p-syzygy of $\mathcal{J}(\mathfrak{b}^c)$ vanishes mod $\mathfrak{m}^{2d-p}$.

II. Analytic setting: $\mathbb{C}^d \supseteq \Omega = \text{ball} \ni 0$
ϕ plurisubharmonic on Ω
$\mathcal{J}(\phi) = \left\{ h : \frac{|h|^2}{e^{2\phi}} \text{ is locally } L^1 \right\}$

(D. Kim) $\psi = \log\left(\sqrt{\sum |z_i|^2}\right)$ analog of $\mathfrak{m}$

Skoda complex:

$$\cdots \to \mathcal{O}^d \otimes J((d-1)\varphi + \phi) \xrightarrow{\begin{pmatrix} z_1 \\ z_i \end{pmatrix}} J(d\varphi + \phi) \to 0$$

Theorem (Danokin): This is exact, ⟹ some syzygy conditions hold.

III Positive characteristic

(Hara, Yoshida, Takagi) $X = \mathbb{A}^d$ $k = \bar{k}$ char $p > 0$

$\mathfrak{b} \subseteq k[X]$

Def (Blickle–Mustata–Smith) The test ideal

$$\tau(\mathfrak{b}^c) = \left\{ \begin{array}{l} \text{unique minimal ideal } J \text{ s.t.} \\ \mathfrak{b}^{\lceil cp^e \rceil} \subseteq J^{[p^e]} \quad \text{for } e \gg 0 \end{array} \right\}.$$

$\lceil \ \rceil$ = round up.

$J^{[p^e]}$: Frobenius powers of the ideal.

For fixed e, prove existence of minimal such J_e, & then show it stabilizes for large e.

Ex $\mathfrak{b} = (x^3, y^2)$

$\tau(\mathfrak{b}^{5/6}) = (x, y)$.

Yoga: $\tau(\mathfrak{b}^c)$ analog of multiplier ideal
& if $\mathfrak{b}$ is a reduction from char. 0
$\Rightarrow \tau(\mathfrak{b}^c)$ is reduction of $\mathcal{J}(\mathfrak{b}^c)$
for sufficiently large p.

- $\tau(\mathfrak{a}^{c_1} \mathfrak{b}^{c_2}) \subseteq \tau(\mathfrak{a}^{c_1}) \tau(\mathfrak{b}^{c_2})$
- satisfy Skoda $\tau(\mathfrak{m}^d \mathfrak{b}^c) = \mathfrak{m}\, \tau(\mathfrak{m}^{d-1} \mathfrak{b}^c)$

Yoshida: any ideal is a test ideal!