Luis Alvarez-Consul : Moduli of coherent sheaves

Note Title 4/13/2007

from moduli of Kronecker modules

(w/ Alastair King, Invent. Math.)

1. Goals X projective scheme $/ k = \bar{k}$

$\mathcal{O}$ structure sheaf

$\mathrm{Coh}\, X$ category of coherent sheaves

$\mathcal{O}(1)$ ample line bundle

$\Rightarrow \mathrm{Coh}\, X \longrightarrow \mathrm{Coh}\, X \quad E \mapsto E(n) = E \otimes \mathcal{O}(1)^{\otimes n}$

For $n >> 0$ $\dim H^0(E(n)) = P(E, n)$

given by Hilbert polynomial.

$\mathrm{Coh}\, X \rightsquigarrow M_X(P) =$ moduli scheme of semistable sheaves E w/ fixed $P(E) = P$.

Goals: try to shed some light on

- construction of $M_X(P)$
- projectivity

} previously known

- natural homogeneous coordinates on $M_X(P)$ ... theta functions.

2. An adjunction
$$\begin{array}{c} \mathrm{Coh}\, X \\ \Phi^\vee \uparrow \downarrow \Phi \\ \mathrm{mod}(A) \end{array}$$

Q Why are $M_X(P)$ projective

A1. ∵ constructed by GIT

Want a more natural reason in terms of functors

$\Phi : \mathrm{Coh}\, X \longrightarrow \mathrm{mod}\, A :$

$E \longmapsto \mathrm{Hom}_X(T, E)$:

Choose integers $n_0 < n_1$

$T = \mathcal{O}(-n_0) \oplus \mathcal{O}(-n_1)$ $(A, \mathcal{O}_X)$-bimodule

for assoc. algebra $A = \begin{pmatrix} k & H \\ 0 & k \end{pmatrix}$

$H = H^0(\mathcal{O}(n_1 - n_0))$.

— If $H^0(\mathcal{O}_X) = k$ take $A = \mathrm{End}_X T$

A-modules $V \iff V = V_0 \oplus V_1$

$\alpha : V_0 \otimes H \longrightarrow V_1$

$\Phi(E) = V \longleftrightarrow V_0 = H^0(E(n_0)),\ V_1 = H^0(E(n_1))$

These are H-Kronecker modules = representations of
$$0 \xrightarrow{H} 1$$
H worth of arrows from 0 to 1.

$\Phi = \mathrm{Hom}_X(T, -)$ has left adjoint $\Phi^\vee = - \otimes_A T$.

3. Embedding & projectivity of $M_X(P)$:

Given a dimension vector $v = (v_0, v_1)$ $\Rightarrow$ projective moduli space
$$M_H(v) = \begin{array}{c}\text{semistable}\\ H\text{-Kronecker modules}\\ \text{of dim vector } v\end{array}$$
Projective (constructions by Drezet, King)

<u>Theorem 1</u> For $n_1 >> n_0 >> 0$ Φ induces a closed embedding $\varphi : M_X(P) \hookrightarrow M_H(v)$
$$E \mapsto [\Phi(E)]$$
$v = (P(n_0), P(n_1))$

<u>Rmk</u> 1) Applying this to points we get
$$X \hookrightarrow \mathbb{P}H = M_H(1,1) \qquad x \mapsto \Phi(\mathcal{O}_x)$$

(ie this is our given projective embedding)

2.) Reinterpretation of Simpson's construction of $M_X(P)$

3.) φ is a scheme-theoretic closed embedding in char $=0$, only know set theoretically for char $p>0$ on strictly semistable locus.

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4. Embedding regular sheaves

Def (Castelnuovo-Mumford) E is n-regular if $H^i(E(n-i))=0 \quad \forall i>0$

Theorem 2 If $\mathcal{O}(-n_0)$ is n_1-regular $\Rightarrow$

$$\Phi : \left\{\begin{matrix} n_0\text{-reg} \\ \text{sheaves} \end{matrix}\right\} \xhookrightarrow[\text{faithful}]{\text{fully}} \operatorname{mod}(A)$$

ie $\Phi^\vee \Phi(E) \simeq E$.

Boundedness Theorem (Maruyama [char 0], Langer)

Given P, $\exists$ $n \gg 0$ s.t all s-s sheaves E of Hilbert polynomial P are n regular.

5. Preservation of semistability

To get from Thm 2 to Thm 1

Def An A-module is semistable if

$\forall V' \subset V$ $\quad \dfrac{\dim V_0'}{\dim V_1'} \leq \dfrac{\dim V_0}{\dim V_1}$ ("slope of V")

E is pure if $\forall E' \subset E$, $\dim \operatorname{Supp} E' = \dim \operatorname{Supp} E$ $[= \deg P(E)]$

Def (Simpson) E is semistable if it is pure

& $\forall E' \subset E$ $\quad \dfrac{P(E',n)}{r(E')} \leq \dfrac{P(E,n)}{r(E)}$ $\quad n \gg 0$.

($r(E)$ - leading coefficient of $P(E)$ ---
on nice varieties this is rank(E)).

<u>Lemma</u> E ss $\iff$ $\forall E' \subset E$

$\Phi(E') \subset \Phi(E)$ doesn't destabilize $\Phi(E)$

for $n_1 \gg n_0 \gg 0$ depending on E'.

i.e $\dfrac{P(E', n_0)}{P(E', n_1)} \leq \dfrac{P(E, n_0)}{P(E, n_1)}$ $n_1 \gg n_0 \gg 0$

<u>Theorem 3</u> Given P, for $n_1 \gg n_0 \gg 0$ (only depending on P)

1. any E of Hilbert polynomial P is ss $\iff$ E n_0-reg & pure, & $\Phi(E)$ is semistable.

2. $\forall E$ ss with $P(E) = P$ we have bijections of Jordan-Holder filtrations of E & $\Phi(E)$

Theorem 3.1 & adjunction $\Phi, \Phi^\vee$ for families $\Rightarrow$ locally closed $M_X(P) \hookrightarrow M_H(P)$.
Langton's theorem $\Rightarrow$ $M_X(P)$ proper.

6. Theta functions

[Schofield et al for quivers]:

natural homogeneous coordinates on $M_H(v)$ well understood

<u>Thm</u> [Schofield - van den Bergh; Derksen - Weyman]

Let $P_i = \Phi(\mathcal{O}(-n_i))$ (ie the indecomposable projectives) $\Rightarrow$ an A-module V is semistable

$\Leftrightarrow \exists\, \gamma: P_1^{k_1} \longrightarrow P_0^{k_0}$ s.t.

$\mathrm{Hom}_A(\gamma, P): V_0^{k_0} \xrightarrow{\sim} V_1^{k_1}$ invertible

i.e $\theta_\gamma(V) \neq 0$ where $\theta_\gamma(V) = \det \mathrm{Hom}_A(\gamma, P)$

— Schofield's <u>determinantal</u> semi-invariant.

Let $\dim V_i = v_i$

$G = GL(V_0) \times GL(V_1) \curvearrowright R = \mathrm{Hom}(V_0 \otimes H, V_1)$

R/G = isom classes of A-modules of dim v.

[King]: GIT: $M_H(v) = \mathrm{Proj}\, S^\bullet(v)$

$$S^\bullet(V) = \left\{ \begin{array}{l} G\text{-invariant polynomial functions} \\ \theta : R \longrightarrow (\det V_0)^{k_0} \otimes (\det V_1)^{k_1} \end{array} \right\}$$

with $\frac{k_0}{k_1} = \frac{v_1}{v_0}$

Schofield et al: $S^\bullet(V)$ spanned by the θ_γ's.

Theorem $\exists\ \gamma_0, \dots \gamma_N : P_1^{\oplus k_1} \longrightarrow P_0^{\oplus k_0}$ defining natural homogeneous coordinates on moduli, ie

$\Theta_\gamma : M_H(P) \longrightarrow \mathbb{P}^N$ is an embedding.

Transfer these results to $M_X(P)$ using $\varphi : M_X(P) \hookrightarrow M_H(\,)$

$$\theta_\gamma \big|_{M_X(P)} = \theta_\delta \quad \text{with } \delta = \underline{d}^\vee(\gamma)$$

ie $\delta : \mathcal{O}(-n_1)^{\oplus k_1} \longrightarrow \mathcal{O}(-n_0)^{\oplus k_0}$

Theorem 4 Given P, for $n_1 \gg n_0 \gg 0$ $\forall E$ with $P(E) = P$, E is semistable $\iff$ E is n₀-reg & pure & $\theta_\delta(E) = \det \mathrm{Hom}(\delta, E) \neq 0$ for some $\delta : \mathcal{O}(-n_1)^{k_1} \longrightarrow \mathcal{O}(-n_0)^{k_0}$.