САША БЕЙЛИНСОН (08.09.9

X – smooth / $\mathbb{C}$

$\mathcal{D}_X$ – ring of diff. operators

$\mathcal{M}^l_{\mathcal{D}}(X) = \mathcal{M}^l(X)$ – category of left $\mathcal{D}$-modules

example (i) $X = \operatorname{Spec} A$, $D_A = \Gamma(X, \mathcal{D}_X)$

$\Gamma : \mathcal{M}^l_{\mathcal{D}}(X) \xrightarrow{\sim} D_A$-mod.

ii) Coherent $\mathcal{D}$-module := loc. fin. gen. $\mathcal{D}$-module
= loc. fin. presented $\mathcal{D}$-mod.
Smooth $\mathcal{D}$-module := coherent as $\mathcal{O}_X$-mod
= locally free of finite rank $\mathcal{O}_X$-mod.

iii) $\mathcal{M}^l_{\mathcal{D}}(X) \underset{\otimes \mathcal{D}}{\rightleftarrows} \mathcal{M}_{\mathcal{O}}(X)$ quasicoh. sheaves.

iv) $\Gamma^{\nabla} : \mathcal{M}^l_{\mathcal{D}}(X) \longrightarrow \mathrm{Vect}$

$\Gamma^{\nabla}(M) = \operatorname{Hom}(\mathcal{O}_X, M)$

$\mathcal{M}^l_{\mathcal{D}}(X)$ is an abelian tensor category

$M_1 \underset{\mathcal{O}_X}{\otimes} M_2$ $\qquad$ $\mathcal{O}_X$ is a unit

i) $X = \operatorname{Spec} A$: $\mathcal{M}^l(X) = D_A$-mod.

ii) $\otimes$ does not preserve coherence

iii) Lack of duals

Ex) M has dual iff M is smooth

$\mathcal{O}_n = \mathbb{C}[[t_1, \dots, t_n]]$

$\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathcal{O}_n)$ is the group of $\mathbb{C}$-points of an affine proalg. group $= \mathrm{Aut}^\circ(\mathcal{O}_n)$

$\mathrm{Lie}\,\mathrm{Aut}^\circ(\mathcal{O}_n) = \underbrace{\mathrm{Der}^\circ(\mathcal{O}_n)}_{\text{preserving max. ideal}} \subset \mathrm{Der}(\mathcal{O}_n)$

$\underline{\mathrm{Aut}(\mathcal{O}_n) = \text{group ind-scheme}}$

$\mathrm{Aut}(\mathcal{O}_n) = (\mathrm{Der}(\mathcal{O}_n), \mathrm{Aut}^\circ(\mathcal{O}_n))$ — Хариш-Чандра pair $(\mathfrak{g}, K)$ $\mathrm{Lie}\,K \hookrightarrow \mathfrak{g}$ action of K on $\mathfrak{g}$

($\mathrm{Aut}(\mathcal{O}_n)(R) = \mathrm{Aut}_{\text{top. } \mathbb{C}\text{-alg}} R[[t_1, \dots, t_n]]$
it is represented by ind-scheme)

$\mathrm{Aut}^\circ(\mathcal{O}_n)\text{-mod} \underset{\mathrm{Ind}}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{Aut}\,\mathcal{O}_n\text{-mod}$ — Х.-Ч.-modules

$\mathrm{Ind}(V) = U(\mathrm{Der}(\mathcal{O}_n)) \underset{U(\mathrm{Der}^\circ(\mathcal{O}_n))}{\otimes} V$

X, $\dim X = n$

$X^\sim$ $\ni (x, \lambda_x) = (\mathrm{Spec}\,\mathcal{O}_n \to X)$ $m \mapsto x$ with nonvan. diff.
$x \in X$, λ_x formal coord. system at x.

$\pi : X^\sim \to X$ $\qquad$ $\mathrm{Aut}^\circ(\mathcal{O}_n)$-action on $X^\sim$

$X^\sim$ is a principal $\mathrm{Aut}^\circ(\mathcal{O}_n)$-bundle over X.

$$\mathrm{Aut}^{\circ}(\mathcal{O}_n)\text{-mod} \underset{\Delta^{\sim}}{\overset{\Gamma^{\sim}}{\rightleftarrows}} \mathcal{M}^{\ell}_{\mathcal{O}}(X)$$

$\Delta^{\sim}$ is left adj. to $\Gamma^{\sim}$

$\Gamma^{\sim}(N) := \Gamma(X^{\sim}, \pi^{*}N)$

$\Delta^{\sim}(V) := V_{X^{\sim}} = \left(V \otimes_{\mathbb{C}} \pi_{*}\mathcal{O}_{X^{\sim}}\right)^{\mathrm{Aut}^{\circ}(\mathcal{O}_n)}$

$\Delta^{\sim}$ is a tensor functor

<u>Claim</u> The $\mathrm{Aut}^{\circ}(\mathcal{O}_n)$-action extend canon. to an $\mathrm{Aut}(\mathcal{O}_n)$

<u>Important</u>: $\mathrm{Der}\,\mathcal{O}_n$ - action on $X^{\sim}$ is formally simply trans.

<u>Remark</u> $(\mathfrak{g}, K)$

X $\dim X = \dim(\mathfrak{g}/\mathrm{Lie}\,K)$

<u>Definition</u> A $(\mathfrak{g}, K)$-str. on X is a fibration $X^{\wedge} \to X$ together with $(\mathfrak{g}, K)$-action on $X^{\wedge}$ s.t. $X^{\wedge}$ is a princ. K-bundle over X and $\mathfrak{g}$ acts simply trans.

$$\mathrm{Aut}(\mathcal{O}_n)\text{-mod} \underset{\Delta^{\sim}}{\overset{\Gamma^{\sim}}{\rightleftarrows}} \mathcal{M}^{\ell}_{\mathcal{D}}(X)$$

those $\mathcal{D}$-modules which appear this way are called "Natural".

A. Beilinson - Vertex Operator Algebras Fall '95

(Lecture 3) X smooth alg variety, dim n. $(\mathfrak{g}, K)$ H-Ch. pair. $\dim(\mathfrak{g}/\mathfrak{k}) = n$ ($\mathfrak{g}$ affine group scheme - not nec. f.dim).

$(\mathfrak{g}, K)$ structure on X: $\pi : \tilde{X} \to X$ fibration, with $(\mathfrak{g}, K)$ action on $\tilde{X}$, $\tilde{X}$ K-torsor $/X$, $\mathfrak{g}$ acts simply transitively on X. [K simply transitively on fibers.]

Ex. i) $\mathfrak{g} = \mathrm{Lie}\, G$, $G \supset K$: take $X = K \backslash G$, $\tilde{X} = G$. "integrable example"

ii) $(\mathfrak{g}, K) := \mathrm{Aut}\, \mathcal{O}_n := (\mathrm{Der}\, \mathcal{O}_n, \mathrm{Aut}^\circ \mathcal{O}_n)$ - canonical $(\mathfrak{g}, K)$ structure on any smooth X.

To any $(\mathfrak{g}, K)$ structure on X assign sheaf of Lie algebras $\tilde{\Theta}$ on X [acting on X] - i.e. morphism $\tilde{\Theta} \xrightarrow{\sigma} \Theta_X$.

$\tilde{\Theta}(U) \subset \Theta_{\pi^{-1}(U)}$ - vector fields inv. wrt $(\mathfrak{g}, K)$ action - commute with $\mathfrak{g}$ action. inf. symmetries of the $(\mathfrak{g}, K)$ structure

* i) : $\tilde{\Theta} = \mathfrak{g}$ via right translations

ii) : $\tilde{\Theta} = \Theta_X$.

$\Longrightarrow$ $\mathcal{D}$-modules on X from H-Ch $(\mathfrak{g}, K)$ modules:

$$\begin{tikzcd} K\mathrm{mod} \arrow[r, shift left, "\tilde{\Delta}"] & \mathcal{M}_{\mathcal{O}}(X) \arrow[l, shift left, "\tilde{\Gamma}"] \\ (\mathfrak{g}, K)\mathrm{mod} \arrow[u, shift left, "o"] \arrow[r, shift left, "\tilde{\Delta}"] & \mathcal{M}^{l}_{\mathcal{D}}(X) \arrow[l, shift left, "\tilde{\Gamma}"] \arrow[u, shift left, "o"] \end{tikzcd}$$

with Ind going up from the top row to the bottom row on both sides.

$\tilde{\Gamma}$: pull back $\mathcal{O}$-mod to $\tilde{X}$, take global sections

$\tilde{\Delta}$: twist by our K-torsor

Comment V a $(\mathfrak{g}, K)$-mod: Action of $\mathcal{D}_X$ on $\tilde{\Delta} V$ is given by:

* $\Theta_X = \tilde{\Delta}(\mathfrak{g}/\mathfrak{k})$ (normal bundle to fibers upstairs is $\mathfrak{g}/\mathfrak{k}$, depends on twist as above..)

$= (\pi_* \mathcal{O}_{\tilde{X}} \otimes \mathfrak{g}/\mathfrak{k})^K$ .. which:

* $\pi_* \mathcal{O}_{\tilde{X}} \otimes \mathfrak{g}$ acts on $\pi_* \mathcal{O}_{\tilde{X}} \otimes V$ $\supset \tilde{\Delta}(V) = (\cdot)^K$

$\rightarrow$ get $(\pi_* \mathcal{O}_{\tilde{X}} \otimes \mathfrak{g}/\mathfrak{k}) \otimes \tilde{\Delta}(V) \to (\pi_* \mathcal{O}_{\tilde{X}} \otimes V) = \tilde{\Delta}(V)$

and same for K-invariants.

o - forgetful functor. $\tilde{\Delta} o = o \tilde{\Delta}$, $\tilde{\Delta}\, \mathrm{Ind} = \mathrm{Ind}\, \tilde{\Delta}$ etc.

$\tilde{\Delta}$ is/are tensor functors

The $\mathcal{D}$-modules we get, $\tilde{\Delta}(V)$, are equipped with action of $\tilde{\Theta}$ - two separate Θ actions, as part of $\mathcal{D}$ and as part of symmetries of the structure.

$\mathcal{D}_X$-algebra := associative ~~commutative~~ unital algebra in our tensor category $\mathcal{M}^{l}_{\mathcal{D}}(X)$ - a $\mathcal{D}$-mod A, $A \otimes_{\mathcal{O}_X} A \to A$, $\mathcal{O}_X \to A$: $\mathcal{O}$-algebra with flat connection, horizontal unit etc.

Category of $\mathcal{D}_X$-algebras: $\mathrm{Comm}_{\mathcal{D}}(X) \overset{o}{\dashrightarrow}$, versus $\mathrm{Comm}_{\mathcal{O}}(X)$
- tensor categories (the tensor product is a categorical coproduct: $A \to A\otimes B \leftarrow B$, $\searrow \downarrow \swarrow$ C) o commutes with $\otimes$ (tensor in $\mathrm{Comm}_{\mathcal{D}}$ is $\otimes_{\mathcal{O}_X}$)

$\boxed{\text{Jets}}$ Lemma i) o admits a left adjoint $\mathcal{J}: \mathrm{Comm}_{\mathcal{O}}(X) \to \mathrm{Comm}_{\mathcal{D}}(X)$
ii) $\mathcal{J}$ commutes with $\otimes$

Proof i) R an $\mathcal{O}_X$-algebra: $\mathrm{Hom}_{\mathcal{D}\text{-alg}}(\mathcal{J}R, A) = \mathrm{Hom}_{\mathcal{O}\text{-alg}}(R, oA)$
So take $\mathcal{J}R$ as follows: need universal morphism $R \to o\mathcal{J}R$.
$\mathcal{J}R = \mathrm{Sym}(\mathcal{D}_X \otimes_{\mathcal{O}_X} R) / [\mathcal{D}_X\text{-ideal gen. by } 1_R - 1,\ 1\otimes r_1 r_2 - (1\otimes r_1)(1\otimes r_2)]$
* ii prove using coproduct property

$\mathcal{J}R$ is the "jet algebra" for R.:
Prop B, R $\mathcal{O}_X$-algebras, then $\mathrm{Hom}_{\mathcal{O}_X\text{-alg}}(o\mathcal{J}R, B) = \varprojlim \mathrm{Hom}_{\mathcal{O}_X^{(n)}\text{-alg}}(\mathcal{O}_{X^{(n)}} \otimes_{\mathcal{O}} R, B \otimes_{\mathcal{O}} \mathcal{O}_{X^{(n)}})$

Here $\mathcal{O}_X^{(n)}$ is $\mathcal{O}(n^{th}$ inf neigh of $X) = \mathcal{O}_{X\times X}/I^{n+1}$, I diagonal ideal.

** Sketch of proof: RHS $\subset \varprojlim \mathrm{Hom}_{\mathcal{O}_X^{(n)}\text{-mod}}(\ ) = \mathrm{Hom}_{\mathcal{O}_X}(R, \varprojlim_n B \otimes \mathcal{O}_{X^{(n)}})$

$\varprojlim B \otimes \mathcal{O}_{X^{(n)}} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{D}_X, B)$: $b \otimes f \otimes g \mapsto (\partial \mapsto bf\partial(g))$
above $= \mathrm{Hom}_{\mathcal{O}\text{-mod}}(\mathcal{D}_X \otimes R, B) \supset \mathrm{Hom}_{\mathcal{D}_X\text{-alg}}(\mathcal{J}R, B)$: map determined by action on generators. $\square$

Cor. $\mathbb{C}$-points of $o\mathcal{J}R$ are the same as pairs (x, γ), $x \in X$, γ a section of $\mathrm{Spec}\, R/X$ on the formal neighborhood of x.
Thus $\mathrm{Spec}\, \mathcal{J}R$ is the space of infinite jets of sections of $\mathrm{Spec}\, R$.

- general principle: $\overset{Y}{\underset{X}{\downarrow}}$ fibration, infinite jets of sections have canonical flat connection, as above construction.

Globalize: $\mathcal{D}_X$-scheme := X-scheme with a flat connection along X.
(of which above is affine case.)

Claim the $\mathcal{J}$-construction is compatible with Zariski or étale localization, hence passes to $\mathcal{D}_X$-schemes: $\mathcal{D}_X\text{-sch} \underset{\mathcal{J}}{\overset{o}{\rightleftarrows}} \mathcal{O}_X\text{-sch}$

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$Sch_D(X) \underset{\mathcal{J}}{\overset{o}{\rightleftarrows}} Sch(X)$

* Exer. $(\mathfrak{g}, K)$ HC pair Commutative unital algebras in category of $(\mathfrak{g},K)$ mods:
$Comm(\mathfrak{g}, K) \underset{\mathcal{J}}{\overset{o}{\rightleftarrows}} Comm(K)$: define adjoint $\mathcal{J}$, show
Δ, Γ commute with $o, \mathcal{J}$.

Modules over $\mathcal{D}_X$ algebras : A a $\mathcal{D}_X$-algebra $\Rightarrow$ sheaf of rings
$A[\mathcal{D}_X]$: sheaf of algebras on X with alg embeddings $A \to A[\mathcal{D}_X]$
(diff ops with coeffs in A), + Lie algebra embedding $\Theta_X \to A[\mathcal{D}_X]$
with obvious compatibilities - $A[\mathcal{D}_X]$ generated by A, Θ_X with only
these relations. $(\mathcal{O} \subset A, \Theta)$. Also set $\mathcal{D}_X \hookrightarrow A[\mathcal{D}_X]$.

Exer $A[\mathcal{D}_X] = A \otimes_{\mathcal{O}} \mathcal{D}_X$ as A-$\mathcal{D}_X$ bimodule.
. A a $\mathcal{D}_X$-algebra allows one to multiply these A-valued differs.

Claim A-modules = sheaves of $A[\mathcal{D}_X]$-modules q.c. as $\mathcal{O}_X$-modules.
$\rightsquigarrow$ define qc modules on $\mathcal{D}$-schemes, projectivity
(not very good to globalize, in inf dim'l ex. - want f.g. projective)

Lemma fin gen projective $A[\mathcal{D}_X]$-modules are "local" objects.
(i.e. projective on some (fin?) covering $\Rightarrow$ projective - not true w/o f.g.).

Sheaf of (Kähler) differentials $A \to \Omega_A$: given any A-module M
a differential $A \xrightarrow{d} M$ is a morphism satisfying Leibniz $d(ab) = a\,db + b\,da$
i) Universal differential $A \xrightarrow{d} \Omega_A \xrightarrow{\exists !} M$

ii) $I \subset A \otimes A \overset{\text{product}}{\twoheadrightarrow} A$ diagonal, $\Omega_A = I/I^2$, A-module.
$d: A \to \Omega_A$, $da = a \otimes 1 - 1 \otimes a \mod I^2$.

Smooth $\mathcal{D}_X$-schemes : A $\mathcal{D}_X$-algebra.
Def A is formally smooth if for any $\mathcal{D}_X$-alg $C \supset I$ ($\mathcal{D}$ ideal) s.t. $I^2 = 0$,
any morphism $A \to C/I$ lifts to a morphism $A \to C$.

Ex. $A = \mathrm{Sym}\, M$. Then A is formally smooth iff M is projective $\mathcal{D}$-mod.
: if M is proj, $A \to C/I$ is mor of $\mathcal{D}$-mod $M \to C/I \Rightarrow$
liftable, conversely (exercise) - lets you check projectivity

Assume X affine - huge supply of proj $\mathcal{D}$-mods (e.g. free).
A any $\mathcal{D}_X$-alg, can be written as quotient of formally smooth
$J \hookrightarrow B \twoheadrightarrow A$

We may write $\Omega_B \to \Omega_A$

$$0 \to \Omega^{(2)}A \to J/J^2 \to \Omega_B \otimes A \to \Omega_A \to 0 \quad \text{exact.}$$

$\quad\quad u \mapsto du \qquad = \Omega_B / J\Omega_B$

$\Omega^{(2)}A$ depends on A only for B formally smooth.

* Show (using J) formally smooth $\mathcal{O}_X$-alg is formally smooth $\mathcal{D}_X$-alg.

Then $\Omega^{(2)}A$ is just $\Omega^{(2)}A$ as $\mathcal{O}$-module.

Criterion. A is formally smooth iff i) $\Omega^{(2)}_A = 0$, ii) Ω_A is projective (as $A[\mathcal{D}_X]$-module)

(A formally sm. i) obvious, ii) take B = polynomial algebra ...

$J/J^2 \to B/J^2 \rightleftarrows A \to 0$ for A fm smooth, get

Ω_A as direct summand of $\Omega_B \otimes A \Rightarrow$ proj.

Conversely Ω_A proj $\Rightarrow$ splitting $\Rightarrow$ splitting $B/J^2 \leftarrow A$

$$\begin{array}{ccc} B/J^2 & \to & A \to 0 \\ & & \downarrow \\ C & \to & C/I \end{array}$$

etc.)

A is fm sm. as $\mathcal{D}_X$-alg iff it is f.sm. as $\mathcal{O}_X$-alg and Ω_A is projective.

Exerc. A is f sm as $\mathcal{O}_X$ alg iff it is f sm. as $\mathbb{C}$-algebra.

Def. A is smooth if it is f sm and finitely generated (as $\mathcal{D}_X$-algebra – quotient of symm algebra of f.g. free $\mathcal{D}$-module.)

Claim smoothness is local $\Rightarrow$ notion of a smooth $\mathcal{D}_X$-sch.

Exerc. J sends f.sm. $\mathcal{O}_X$-alg $\Rightarrow$ f sm $\mathcal{D}_X$ alg and smooth schemes $\Rightarrow$ sm. $\mathcal{D}_X$ sch.

Question Is any smooth $\mathcal{D}_X$-alg is finitely presented? (as $\mathcal{D}_X$-alg)

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Ex. of fin gen not fin pres $\mathcal{D}_X$-alg: $\mathrm{Sym}(\mathcal{D}_X)/\mathrm{Sym}^{\geq 2}$: $\mathrm{Sym}^{\geq 2}$ is not finitely generated ..

Pitfall: Assume we have Y smooth $\mathcal{D}_X$-scheme. Are there coords on Y?

Def – A coord system on Y is a morph of $\mathcal{D}_X$-sch $Y \xrightarrow{f} \mathrm{Spec}(\mathrm{Sym}\,\mathcal{D}_X{}^n) \simeq \mathcal{I}$

s.t. $df : f^*\Omega_{\mathcal{I}} \to \Omega_Y$ is an isomorphism.

[always have local coordinates on a smooth scheme.]

Not true for smooth $\mathcal{D}_X$-schemes.. e.g. $\begin{smallmatrix} y \\ \downarrow \\ X \end{smallmatrix}$ smooth $\begin{smallmatrix} JY \\ \downarrow \\ X \end{smallmatrix}$ jets, smooth $\mathcal{D}_X$ scheme – these do have coords – but not true for arbitrary $\mathcal{D}$-sch even at generic pnt.

Reason Assume $Y = \operatorname{Spec} A$, A integral. $A[\mathcal{D}_X]$ has few invertible elements: $A[\mathcal{D}_X]^* = A^*$ – invertible differential operator must have deg 0 (say $\dim X = 1$, top symbol is function, degrees of symbols add...).

Now assume A is smooth and $\Omega_A \simeq A[\mathcal{D}_X]$ free one generator ($n = \dim$). $A \subset A[\mathcal{D}]$ corresponds to some $\mathcal{L} \subset \Omega_A$, well defined/unique up to right mult of invertible operator $\Rightarrow A^*$ which fixes A!

so $\mathcal{L} \subset \Omega_A$ is canonical. – line subbundle in Ω_A, field of hyperplanes.. is it integrable?

Integrability: $d(\mathcal{L}) \subset \Lambda^2 \Omega_A$ must sit in $\mathcal{L} \wedge \Omega_A$..

If we had coord system this must be integrable – $\mathcal{L}$ generated by exact form. Usually this doesn't hold (canonical contact structure...)

Example/Exercise $v + u'v' = 0$, $X = \operatorname{Spec} \mathbb{C}[x]$. $B = \mathbb{C}[x, u, u', \ldots, v, v', \ldots, v'^{-1}]$
– gives $\mathcal{D}_X$-scheme. Take $\mathcal{D}_X$ ideal $I \subset B$ gen by $\varphi = v + u'v'$,
$A = B/I$, so the DE is satisfied & v' invertible.
Claim A is smooth and $\mathcal{L}_A$ is not integrable.

$d\varphi = dv + d(u'v') = dv + u'dv' + du'v = dv + u'\partial_x dv + v'\partial_x du$.

Horizontal sections (classical analog of space of conformal blocks)

Comm Assoc $\mathbb{C}$-alg Com $\longrightarrow$ $\mathrm{Comm}_{\mathcal{D}}(X)$

$\mathbb{C}/R \longmapsto R \otimes \mathcal{O}_X$

Obvious right adjoint: $A \in \mathrm{Comm}_{\mathcal{D}}(X) \longmapsto \Gamma^\nabla(X, A)$ global horizontal sections

By Leibniz this is subring

Left adjoint: space of horizontal sections. Need "maximal constant quotient" of A! projective limit $A \longmapsto \varprojlim \Gamma^\nabla(A/I)$ over A/I constant $\mathcal{D}_X$-algebra.

call this $H_\nabla(X, A) = H_\nabla(A)$

"constant algebra" – image of $R \longmapsto R \otimes \mathcal{O}_X$.

Is there a smallest ideal that will do the trick?

Proposition If X is compact then any A has a smallest ideal I s.t. A/I is const.

[Will return to proof]

What is "space of sections"? R-points of sections $\to$ morphisms $X \times \operatorname{Spec} R \to \operatorname{Spec} A \times \operatorname{Spec} R$

So we want such morphisms $A \to R \otimes \mathcal{O}_X$ which are $\mathcal{D}$-morphisms, i.e. horizontal.

Via functor of jets, this contains usual notion of sections (by adjunction).

Right D-modules $\mathcal{M}_D^l(X)$ vs. $\mathcal{M}_D^r(X)$

a. L left, M right, then $L \otimes_{\mathcal{O}} M$ is naturally a right D-module: τ vector field
$(l \otimes m)\tau = -\tau(l) \otimes m + l \otimes m\tau$

b. Canonical right D-mod $\omega_X = \Omega_X^{\dim X}$ – $\nu\tau = -\mathcal{L}ie_\tau(\nu) = d(\tau \lrcorner \nu)$

c. $\mathcal{M}_D^l(X) \to \mathcal{M}_D^r(X)$, $L \mapsto L \otimes \omega_X = L^r$ is an equivalence of categories.

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$D^b\mathcal{M}_D^l(X) \xrightarrow{\sim} D^b\mathcal{M}_D^r(X)$

$\overset{\omega}{M} \longmapsto M \otimes \overset{\omega}{\omega}_X[\dim X]$ – should shift naturally

– the object here is the dualizing sheaf $\omega_X[\dim X]$.

Thus $D^b\mathcal{M}(X)$ has two natural cores (abelian subcategories)

$\mathcal{M}^l \nearrow \nwarrow \mathcal{M}^r$, the left core $\mathcal{M}_D^l$ coinciding with $\mathcal{M}^r[\dim X]$.

Pull-back functors $f: Y \to X$, $M \in \mathcal{M}_D^l(X) \Rightarrow$

f^*M as D-module: pullback of bundle with connection is bundle with connection.

$\Rightarrow$ right exact functor $f^*: \mathcal{M}_D^l(X) \to \mathcal{M}_D^l(Y)$, left derived is

$Lf^*: D^b\mathcal{M}_D^l(X) \to D^b\mathcal{M}_D(Y)$, which we denote by f^+

Exer.
$$\begin{array}{ccc}
D^b\mathcal{M}_{\mathcal{O}}(X) & \xrightarrow{Lf^*} & D^b\mathcal{M}_{\mathcal{O}}(Y) \\
\uparrow o^l & & \uparrow o^l \\
D^b\mathcal{M}_D(X) & \xrightarrow{f^+} & D^b\mathcal{M}_D(Y) \\
\downarrow o^r & & \downarrow o^r \\
D^b\mathcal{M}_{\mathcal{O}}(X) & \xrightarrow{Rf^!} & D^b\mathcal{M}_{\mathcal{O}}(Y)
\end{array}$$
– forgetting left D-mod structure

– forgetting right D-mod structure

DeRham cohomology sheaves: $M \in \mathcal{M}^r \Rightarrow$ sheaf of vector spaces (Z. or Ét.)

$h(M) = M \otimes_{D_X} \mathcal{O}_X = M_{\Theta_X}$ coinvariants wrt vector fields, $M \ni m \mapsto \int m$.

Projection $\int : M \twoheadrightarrow h(M)$

DeRham complex: $L \in \mathcal{M}^l$, $DR(L)^{\cdot} = \Omega_X^{\cdot} \otimes_{\mathcal{O}_X} L$, integrable connection gives differential.

$M \in \mathcal{M}^r$, $DR(M)^{\cdot} = M \otimes_{\mathcal{O}_X} \Lambda^{-\cdot}\Theta_X$ (in negative degrees).

$M = L \otimes \omega_X$ these get identified (contract ω with $\Lambda^{i}\Theta_X \to \Omega_X^{n-i}$.)

So canonical functor $D^b\mathcal{M}(X) \xrightarrow{DR} D^b Sh(X)$

Index 0 $\to M\otimes \Theta_X \to M$, $m\otimes \tau \mapsto m\tau$ so H_0 (cokernel) is precisely $h(M)$ = coinvariants.

left: $DR^l = R(L \mapsto L^\nabla = Hom_D(O_X, L)) = R\,Hom_{M^l}(O_X, L)$

right: $DR(M) = M \overset{L}{\otimes} O_X$.

$\boxed{\text{Induction}}$ $M^r_D(X) \underset{\mathrm{Ind}^r = m}{\overset{or}{\rightleftarrows}} M_O(X)$

Ψ Ψ

$\tilde F = F \underset{O}{\otimes} D_X \longleftarrow F$

Lemma i. $h(\tilde F) = F$ $[\tilde F \otimes_{D_X} O_X = F \otimes_{O_X} D_X \otimes_{D_X} O_X = F.]$

ii. $Hom(\tilde F_1, \tilde F_2) \xrightarrow{h} Hom_{\mathbb{C}}(F_1, F_2)$ is injective, with image the differential operators $Diff(F_1, F_2)$.

(Prove by reducing to local, reduce to F free by taking inductive limits and exactness in first argument..)

Example Introduce category $Diff(X) \supset M_O(X)$, same objects but morphisms are differential operators - Lemma ii states

$Diff(X) \xrightarrow{\text{fully faithful}}$

$\tilde M_O(X) \xrightarrow{\sim} \tilde M^r_D(X)$

Example $M \in M^r(X)$. $DR(M)$ is a complex in $Diff(X)$ (morphs = differentials not O-linear). So may consider $\widetilde{DR}(M)$ as complex of D-modules:

$\widetilde{DR}(M) = (\cdots \to M \otimes_{O_X} \Theta_X \otimes_O D_X \to M \otimes_{O_X} D_X)$.

$\downarrow$ $m\otimes \partial$

M $\mapsto m\partial$

Claim: this morphism $\widetilde{DR}(M) \to M$ is a quasi-isomorphism i.e. $\widetilde{DR}$ gives canonical left resolution of a right D-module M. (Spencer resolution)

Exercise Define $D^b Diff(X)$ (D^b of nonabelian) via quisms of complexes... then $D^b Diff(X) \underset{DR}{\overset{\widetilde{}}{\rightleftarrows}} D^b M^r_D(X)$ are mutually inverse equivalences of categories.

$\boxed{\text{Functoriality}}$: Direct image (somehow better for right, pullback for left).

Case a) Open embedding $j: U \hookrightarrow X$, $j_* N_U = j_* N_U$ as ~~sheaves~~ sheaves. "meromorphic continuation".

b. Closed embedding $i: Y \hookrightarrow X$ closed smooth subscheme, $O_Y = O_X / I$

$i^!: M^r_D(X) \longrightarrow M^r_D(Y)$, $i^! M = \{m \in M \mid Im = 0\} = Hom_{O_X}(O_Y, M)$

To make this a D_Y-mod, extend vector field arbitrarily to X, action (from right) independent of extension ($\underline{mI}\,\Theta_X$ ..)
Pushforward is now $i_*: \mathcal{M}_D^r(Y) \to \mathcal{M}_D^r(X)$ left adjoint to $i^!$.

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$M \in \mathcal{M}_D^r(Y)$: $i_* N = N \otimes_{D_Y} i^* D_X$
where D_X is a left D-mod on X, pull back to Y, $i^* D = \mathcal{O}_Y \otimes_{\mathcal{O}_X} D_X$
$= D_X / I\, D_X$ $\quad (\mathcal{O}_Y = \mathcal{O}_X / I)$
Alternatively define $D_{Y/X} \hookrightarrow D_X$ as $\{\partial : \partial(I) \subset I\}$
(two sided ideal) $\quad D_{Y/X} \twoheadrightarrow D_Y$ projection by action on Y.
So set $i_* N = N \otimes_{D_{Y/X}} D_X$ and <u>show it's the same</u>.

$i^!$ is left exact and i_* is in fact exact ...

[What is $\hom(i_* N, M)$ – maps sends N to subsheaf of section killed by ideal
$D_{Y/X} \twoheadrightarrow D_Y$ so set precisely D_Y morphisms ......]

Locally $X = Y \times Z$ in étale topology. (Y smooth) $Y = Y \times \bullet$, $\bullet \in Z$.
D-mod on $\bullet$ is vector space. $(\bullet \hookrightarrow Z)_* \mathbb{C} := \delta_\bullet$,
$i_* N = N \boxtimes \delta_\bullet$

<u>Lemma (Kashiwara)</u> $\quad \mathcal{M}_D^r(Y) \xrightarrow{i_*} \mathcal{M}_D^r(X)_Y$ is equiv. of categories,
<u>with inverse</u> $i^! / \mathcal{M}_D^r(X)_Y$.
<u>Proof</u>. Need to show natural transforms $i_* \leftrightarrow i^!$ are equivalences:
$i_* i^! M \xrightarrow{\sim} M$, $\quad N \xrightarrow{\sim} i^! i_* N$
Case Y point – algebraic version of Stone-von Neumann ($\mathbb{Z} \ni pt\ \bullet$)
support at 0 means mult by x is locally nilpotent $\Rightarrow$
rep of Heisenberg algebra loc nilp, $\Rightarrow \oplus$ of Fock $= \delta_\bullet$,
add parameters $\blacksquare$

This functor is compatible with DR: $\quad DR\, i_* N \longleftrightarrow i_\cdot DR\, N$ (sheaf theoretical push forward (ext. by 0))
is a quism – in particular $h(i_* N) = i_\cdot h(N)$
Also compatible with induction for F $\mathcal{O}_Y$-mod : $i_* \tilde{F} = \widetilde{i_\cdot F}$
(follow from adjunction/universality ..)

Open vs. closed : $Y \xhookrightarrow{i} X \xhookleftarrow{j} U$ complement
$i_* R i^! M \longrightarrow M \longrightarrow R j_* j^* M$ is an exact triangle

<u>Formal definition of general push forward</u> $f: Y \to X$

$f_*: D^b \mathcal{M}_D^r(Y) \to D^b \mathcal{M}_D^r(X)$

N, $f_* N = Rf_\cdot(N \otimes^L_{\mathcal{D}_Y} f^* \mathcal{D}_X)$ ← pullback of $\mathcal{D}_X$ as left $\mathcal{D}_X$-mod.

Alternatively decompose f as $Y \xrightarrow{\Gamma_f} Y \times X \xrightarrow{p} X$ - compatible with compositions.

Γ_f closed, $(\Gamma f)_* N$ exact, take p_* for projection: take de Rham complex fibrewise, differs along Y $\mathcal{O}_X \otimes \mathcal{D}_Y$: [$f_*$ as $\mathcal{O}$-modules]

$p_* (\Gamma_f)_* N = R p_\cdot DR_Y (\Gamma_{f*} N)$ on which $\mathcal{D}_X$ acts along X in product, commuting with $\mathcal{D}_Y \otimes \mathcal{O}_X$ (?)

When X = point, then $f_* N = R\Gamma(Y, DR(N))$ - de Rham cohomology.

Natural morphism $\int: Rf_\cdot N \to f_* N$ since N is subsheaf of $(N \otimes^L f^* \mathcal{D}_X)$ generalizing (local version) of our previous $\int$.

$f_* N$ doesn't have expression as single derived functor - neither left nor right exact, comp of Rf_* and $\otimes^L$.

Theorem (key property of $*$) $f: Y \to X$ proper, then $D^b \mathcal{M}_D^\bullet(Y) \underset{f^\dagger}{\overset{f_*}{\rightleftarrows}} D^b \mathcal{M}_D(X)$

Then f_* is left adjoint to $f^\dagger$. [Borel]

<u>Particular case</u> X compact $\xrightarrow{\dim n} \bullet$ pt. $M \in \mathcal{M}_D^r(X)$.

$H^n_{DR}(X, M) = H^n(X, DR(M))$.

<u>Lemma</u> There is a canonical morphism of $\mathcal{D}$-mods $M \to H^n_{DR}(X, M) \otimes \omega_X$ (constant $\mathcal{D}$-mod) identifying the RHS with the maximal constant quotient of M.

– theorem tells us how to compute $\mathrm{Hom}(M, \pi^* V)$ – it is $\mathrm{Hom}(\text{De Rham } M, V)$ by adjunction:

$x \in X \supset U_x = X \setminus \{x\}$. $M_x = M^1 / m_x M$ fiber at x

Fibrewise $M_x \twoheadrightarrow H^n_{DR}(X, M)$: constructed from our canonical triangle $\bullet \xrightarrow{i_x} X \hookleftarrow U_x$

$(i_x)_* \underbrace{R i_x^! M}_{} \to M \to R(j_x)_* j_x^* M$

M_x indep n: look at formal neigh where our connection trivializes M as M_x

So get map $i_{x*} M_x[-n] \to M$, surjective since U_x is open, no top coh.

The kernel can be computed via

$$H^{n-1}(U_x, M) \xrightarrow{Res} M_x \to H^n_{DR}(X, M) \to 0$$

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Assume $\dim X = 1$, X compact. $L \in \mathcal{M}_{\mathcal{D}}^{\ell}(X)$
What is the maximal constant quotient of L ($\oplus$ of copies of $\mathcal{O}$ ..)

Prop–construction $H_{DR}^{top}(X, L)$ ($= H^2_{DR}(X, L)$ in our case):
There is a canonical morphism of left $\mathcal{D}$-mod $L \twoheadrightarrow H^2_{DR}(X, L) \otimes \mathcal{O}_X$ ..

Proof May assume L has no torsion as $\mathcal{O}$-mod – can quotient out by torsion without changing the question. Take $x \in X$, $j : U = X \setminus \{x\} \hookrightarrow X$
Consider the fibre $L_x = L/m_x L = i_x^! [\mathrm{coker}(L \hookrightarrow j_* j^* L) \otimes \omega_x]$
– $h = i_x^! : \mathcal{M}_{\mathcal{D}}^r(X)_x \to \mathrm{Vect}$ .. $= h[\ \ "\ \ ]$
Why is this? identification is multiplication by $\frac{dt}{t}$:
take element of L_x, extend locally, tensor with $\frac{dt}{t}$ –
different extensions differ by $L \otimes \omega_x$ (regular forms)
– i.e. residue map.

$$0 \to L \to j_* j^* L \to L_x \otimes \delta_x \to 0$$

$\Rightarrow$ long exact: top dR of $j_* j^* L$ vanishes, as for $L_x \otimes \delta_x$ .. so get

$$H^1(U, L) \to L_x \to H^2_{DR}(X, L) \to 0$$

U is affine so may write quotient $\Gamma(U, L \otimes \omega_x) \twoheadrightarrow H^1(U, L)$, $\xrightarrow{Res_x} L_x$

Restrict L to formal neigh of x, ∇ trivializes, i.e.
$L_x^\wedge = \varprojlim L/m_x^n L \xrightarrow{\sim} \varprojlim L_x \otimes (\mathcal{O}/m^n)$
The map above is just Res_x.
$\Rightarrow H^2_{DR}(X, L)$ is cokernel of residue map.
Do this in families over X & look at diagonal in $X \times X$...]
so the above becomes stalk of the map we were to construct.
Maximality of quotient: say $L \to V \otimes \mathcal{O}_X$, this is to factor through H^2_{DR}: apply H^2 to it:
$H^2_{DR}(X, L) \to V \otimes H^2_{DR}(X, \mathcal{O}_X) = V$, which is the map we wanted .....

Cor. application we used before: A a $\mathcal{D}_X$-algebra, $\dim X = 1$, X compact
then A has maximal constant quotient $H_\nabla(X, A)$:
$R \subset A \twoheadrightarrow H^2_{DR}(X, A) \otimes \mathcal{O}_X$. R (kernel) is it ideal...
so maximal quotient is $A/\mathcal{D}_X$-ideal gen by R.
$H_\nabla(X, A) = A/AR$. The spec of this is spec of horiz sections...
$A_x \xleftarrow{Res} \Gamma(X \setminus \{x\}, A \otimes \omega_x)$, $A_x / A_x Res(\ ) = H_\nabla(X, A)_x$

In quantized situation this will be conformal blocks, standard coinvariants definition.

New pseudotensor structure on $\mathcal{M}_D^r(X)$

$\psi\otimes$ categories: Example. $\mathcal{M}, \otimes$ tensor category (symmetric monoidal – strictly comm & assoc.) Given $\{M_i\}_{i\in I}$ I finite nonempty $\Rightarrow$ $\otimes_I M_i$, need not demand I ordered etc.

Polylinear operations $P_I(\{M_i\}, L) = \mathrm{Hom}(\otimes_I M_i, L)$

composition: $J \twoheadrightarrow I$, family $\{K_j\}_J$

$\psi_i \in P_{J_i}(\{K_j\}, M_i)$, $\varphi \in P_I(\{M_i\}, L)$

$\varphi(\psi_i) = \varphi \circ (\otimes_I \psi_i) \in P_J(\{K_j\}, L)$ associative.

Definition $\mathcal{M}$ category, a $\psi\otimes$ structure on $\mathcal{M}$ is a law $\{M_i\}, L \mapsto P_I(\{M_i\}, L)$. More generally just assume $\mathcal{M}$ is a set, get law of composition P: $\mathrm{Hom}(M, L) = P_{\cdot}(\{M\}, L)$, assume we have an identity in here.

"Difference from usual tensor category:

Assume functor $P_I(\{M_i\}_{i\in I}, -)$ is representable – by "$\otimes_I M_i$")

Composition law gives canonical morphism $\otimes_J K_j \xrightarrow{\epsilon_\pi} \otimes_I(\otimes_{J_i} K_j)$

Now we have

Lemma A tensor category is a representable $\psi\otimes$ cat such that all ϵ_π's are isomorphism

Exam2 $\psi\otimes$ category with single object: I finite nonempty $\mapsto$ P_I composition operations $\Longleftrightarrow$ operad:

$n \geq 1 \mapsto P_n$ composition law, + S_n action. Set $P_n = P_{\{1,2,\dots n\}}$

S_n acts thanks to the composition

This will be tensor if all $P_I = M$, commutative monoid, compositions are products.

Obvious notion of $\psi\otimes$ functor $\to$ sub $\psi\otimes$ cat etc.

Examples of subcat: full subcategories, take any collection of objects and their operations e.g. any object $\Rightarrow$ operad.

$\psi\otimes$ on $\mathcal{M}_D^r(X)$ which is in fact (poly)linear: $P_I^*(\{M_i\}, L) := \mathrm{Hom}(\boxtimes_I M_i, \Delta_*^{(I)} L)$ – purely local concentrated on diagonal, form sheaf on X.)

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$*$-structure on $\mathcal{M}_D^r(X)$, $P_I^*(\{M_i\}, L) = \mathrm{Hom}(\boxtimes M_i, \Delta_*^{(I)} L)$

$J \twoheadrightarrow I$, $\varphi_i \in P_{J_i}^*(\{K_j\}, M_i)$, $\psi \in P_I^*(\{M_i\}, L)$

$$\Rightarrow \boxtimes \varphi_i : \boxtimes_J K_j \longrightarrow \boxtimes_I \Delta_*^{(J_i)} M_i \;. \quad X^I \xrightarrow{\Delta^{(J/I)}} X^J$$

$$\Delta_*^{(J/I)}(\boxtimes M_i)$$
$$\downarrow \Delta_*^{(J/I)}(\psi)$$
$$\Delta_*^{(J/I)} \Delta_*^{(I)} L = \Delta_*^{(J)} L$$

check this is associative etc.

Def We say a $\psi\otimes$ structure on an abelian category M is abelian if P_I are left exact. (generalize usual exactness of Hom, & right exactness of $\otimes$.)

On the full subcategory $\mathrm{Diff}(X) \hookrightarrow \mathcal{M}_D^r(X)$ (induced D-mod):
$M_i = \widetilde{F_i} = F_i \otimes_{\mathcal{O}} D_X$, $L = \widetilde{G}$, then
$P_I^*(\{M_i\}, L) = \mathrm{Hom}(\widetilde{(\boxtimes F_i)}, \widetilde{(\Delta_* G)})$. since $\boxtimes$, Δ_* commute with $\sim$:
so $= \mathrm{Diff}(\boxtimes F_i, \Delta_* G) =$ polydifferential operator from $\otimes F_i \to G$.

We have $h: \mathcal{M}_D^r(X) \to Sh(X)$, a tensor category with usual tensor of sheaves.
Claim h is a $\psi\otimes$ functor $\mathcal{M}_D^r(X)^* \to Sh(X)$.
Pf. $P_I^*(\{M_i\}, L) \longrightarrow \mathrm{Hom}(\otimes h(M_i), h(L))$:
lift section (local) of $\otimes h(M_i)$ to $\otimes$ sections of M_i, take ext product, & Δ_* commutes with h: i.e. natural maps
$\boxtimes h(M_i) \to h(\boxtimes M_i)$, $h(\Delta_*^I L) = \Delta_* h(L)$, compatible with compositions .... ∎

Assume $\mathcal{O}$ is an $(\otimes)$ operad. We have notion of an algebra/$\mathcal{O}$:
Def An $\mathcal{O}$-algebra := vector space V with $\mathcal{O}$-action, i.e. maps
$\mathcal{O}_n \to \mathrm{Hom}(V^{\otimes n}, V) =$ morphism $\mathcal{O} \to \mathrm{End}(V)$, endo operad of V.
M a $\mathbb{C}$-$\psi\otimes$ category, an $\mathcal{O}$ alg in M = $\psi\otimes$ functor from $\mathcal{O} \to M$, i.e. an object $V \in M$ with a morphism $\mathcal{O} \to$ the assoc. operad.

Ex 1 For us the most important is the Lie operad $\mathcal{L}ie$ - space $\mathcal{L}ie_n$ consists of all natural n-Lie operations, generated by $[,]$ in degree 1 freely with two relations - Jacobi & skew-symmetry.
Ex 2 $\mathcal{C}om$ - just $\mathbb{C}$ in each degree, only one way to n-compose.
Ex 3 $\mathcal{A}ss$ - gen by $\cdot$ in deg 2, no relations at all except associativity
Ex 4 $\mathcal{P}oiss$

Given $\mathcal{O}$ can define free $\mathcal{O}$-algebra on n gens (universal property), which "is" the space of $\mathcal{O}$.

A Lie algebra in M is $L \in M$ with $[\,] \in P_2^*(\{L,L\},L)$ + skew symmetry & Jacobi identity in $P_3^*(\{L,L,L\},L)$.

An L-module is $V \in M$ with $\cdot_V \in P_2^*(\{L,V\},V)$

— Show this gives an abelian category L-mod for M abelian $\neq 0$.

In particular we have Lie* algebras...

If L is a Lie*-algebra then $h(L)$ is a sheaf of Lie algebras on X,
V an L-module $\Rightarrow$ $h(V)$ is an $h(M)$ module.
h is not faithful, but not too far from it...

<u>EX</u> For induced modules, a Lie algebra in $Diff(X) = \mathcal{O}$-mod + Lie bracket given by a bidifferential operator,
This gives a huge supply — vect fields, diff ops, endos of vector bundle etc ...

<u>Ex2</u> Simplest example of noninduced Lie*-algebra : $\mathfrak{g}$ ~~simple~~ Lie algebra, $(,) \in (S^2\mathfrak{g})^{*\mathfrak{g}}$. Assume $\dim X = 1$, consider $\mathfrak{g} \otimes \mathcal{O}_X$, Lie algebra with $\mathcal{O}$-linear bracket, induce to get Lie*-alg $\mathfrak{g} \otimes D_X = \mathfrak{g} \otimes \mathcal{O}_X$
$(,)$ defines a central extension by means of ω_X:

$$0 \to \omega_X \to \mathfrak{g} \otimes D_X^\sim \to \mathfrak{g} \otimes D_X \to 0 \quad ;$$

usual K-M cocycle $\mathfrak{g} \otimes \mathcal{O}_X \times \mathfrak{g} \otimes \mathcal{O}_X \to \omega_X$

$a, b \longmapsto (a, db)$ — gives two cocycle & we pass from ω to deRham complex..

Induced version : bilinear $*$-operation on $\mathfrak{g} \otimes D_X \otimes \omega_X \otimes D_X \to \omega_X$

- get almost cocycle, but not skew-symmetric
- define $\mathfrak{g} \otimes D_X^\sim = \omega_X \oplus \mathfrak{g} \otimes D_X$

with lift of cocycle (a, db) — operation in $P_2^*(\{\mathfrak{g} \otimes D_X, \mathfrak{g} \otimes D_X\}, \omega_X \otimes D_X)$ push it forward.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \omega_X$

Explicitly, take the product operation $\mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X$, what is the corresp $*$-operation in $P_2^*(\{D_X, D_X\}, D_X)$:

map $D_X \boxtimes D_X \to \Delta_* D_X$ = module gen by δ with relation : being killed by diagonal

$\mathcal{O}_{X \times X}, \ni 1 \longmapsto \delta$

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Diff(X) A diffop between $\mathcal{O}$-mods $F_1 \xrightarrow{\partial} F_2$ is the same as a morphism between the induced modules $\tilde{F_1} \to \tilde{F_2}$:

$\tilde{F_1} = F_1 \otimes_{\mathcal{O}} \mathcal{D}_X = \mathrm{Diff}(\mathcal{O}_X, F_1)$ via $f \otimes \partial \mapsto (\varphi \mapsto \partial(\varphi) f)$

$\mathrm{Diff}(\mathcal{O}_X, F_1) \xrightarrow{\partial \circ} \mathrm{Diff}(\mathcal{O}_X, F_2)$: composition from left with ∂.

$\mathrm{Diff}(X)^* \to \mathcal{M}^r(X)^*$ is a $\otimes$ functor, . given

L a $\mathrm{Diff}(X)^*$-Lie algebra $[\cdot]: L \otimes_{\mathbb{C}} L \to L$ bidifferential

$\longmapsto$ $\tilde{[\,]} \in P_2^*(\{\tilde{L}, \tilde{L}\}, \tilde{L})$

i.e. $\tilde{[\,]}: \tilde{L} \boxtimes \tilde{L} \to \Delta_* \tilde{L}$

$$(L \otimes_{\mathcal{O}_X} D_X) \boxtimes (L \otimes_{\mathcal{O}_X} \mathcal{O}_X) \longrightarrow \Delta_*(L \otimes D_X)$$

$$(L \boxtimes L) \otimes_{\mathcal{O}_{X\times X}} \mathcal{D}_{X\times X} \longrightarrow L \otimes \mathcal{O}_X (I\mathcal{D}_{X\times X} \backslash \mathcal{D}_{X\times X})$$ I ideal of diagonal

generalizes substrate $\leftarrow$ $L \otimes_{\mathbb{C}} L \longrightarrow L \subset L \otimes_{\mathcal{O}_{X\times X}} \mathcal{D}_{X\times X}$

Examples (i) $[\,]$ is $\mathcal{O}$-linear :

$\tilde{[\,]}$ sends $L \otimes_{\mathbb{C}} L \subset (L \boxtimes L) \otimes \mathcal{D}_{X\times X}$ to $L \subset L \otimes_{\mathcal{O}_{X\times X}} \mathcal{D}_{X\times X}$

and coincides on it with $[\,]$.

e.g. K-M case $\mathfrak{g} \otimes \mathcal{O}$: bracket $(\mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{g}) \otimes \mathcal{D}_{X\times X} \to \mathfrak{g} \otimes_{\mathbb{C}} (I\mathcal{D}_{X\times X} \backslash \mathcal{D}_{X\times X})$

$I\mathcal{D}_{X\times X} \backslash \mathcal{D}_{X\times X}$ - diagonal operators $t_2 \to t_1$ with δ-functions in transverse directions

- constant in transverse (t_2) direction - $t_1^a \partial_{t_1}^b \partial_{t_2}^c$ . .

(ii) $L = \Theta_X$ with standard bracket, get $*$ bracket on $\tilde{\Theta}_X$

On $\Theta_X \otimes_{\mathbb{C}} \Theta_X$ the $*$ bracket is $[\tau_1, \tau_2] = [\tau_1,\tau_2] \otimes 1 + \tau_2 \otimes \tau_1^{(2)} - \tau_1 \otimes \tau_2^{(1)}$

in $\Theta_X \otimes (I\mathcal{D}_{X\times X} \backslash \mathcal{D}_{X\times X})$ — $\tau_1^{(2)}$ means τ_1 as diff op acting along the second variable

e.g. $[\partial, \partial] = \partial \otimes (\partial_y - \partial_x)$

How do we get this : $\varphi_1^{(1)} \otimes \varphi_2^{(2)} \mapsto [\varphi_1 \tau_1, \varphi_2 \tau_2]$ is the new differential operator $= \varphi_1 \varphi_2 [\tau_1, \tau_2] + \varphi_1 \tau_1)(\varphi_2) \tau_2 - (\varphi_2 \tau_2)(\varphi_1) \tau_1$

"Skew symmetry" : take $[\tilde{\tau_2}, \tau_1] = -\mathrm{inv} \cdot [\tau_1, \tilde{\tau_2}]$ where inv is the involution of $X \times X$ switching factors.

What are $*$-Lie ~~modules~~ ?

Recall $h: \mathcal{M}^r(X)^* \to Sh(X)$ $\otimes$ functor.

$(A \boxtimes B \to \Delta_* C) \in P_2^*(\{A, B\}, C)$.

Let's apply h transversely to the diagonal - say to A, along first

variable .. $\Delta \cap C$ transversely is δ-functions hence its h is $\mathbb{C}$:
get $h(A) \otimes_{\mathbb{C}} B \longrightarrow C$, i.e. $h(A) \to Hom_{\mathcal{D}}(B, C)$
$\xrightarrow{h} Hom(h(B), h(C))$ which is h applied to the whole P_2 operation — get more structure via this partial application of h.

Example L Lie* algebra, M an L-mod. Then the L-action on $M \in P_2^*(\{L, M\}, M)$, apply partial h yields
$h(L) \longrightarrow Hom_{\mathcal{D}}(M, M) = End_{\mathcal{D}}(M)$. $h(L)$ is a sheaf of plain Lie algebras $\longrightarrow$ $\boxed{a}$ this gives an action of the Lie sheaf $h(L)$ on M (follows from general functorialities to come.)
$\boxed{b}$ The functor Lmod $\longrightarrow$ $\{h(L)$-mod in $\mathcal{M}^r(X)\}$ is fully faithful.

L a Lie*-alg, M an Lmod $\Rightarrow$ the Lie alg $h(L)$ acts on M 10/4

The above property of h makes it an <u>augmentation functor</u> for the * ⊗ structure : write $h(M) = P_\emptyset^*(M)$

Now the $P_I(\{L_i\}, M)$ have composition $J \twoheadrightarrow I \Rightarrow P_I^* \otimes .. \otimes P_{J_i} \to P_J^*$

Now can extend this to $J \to I$ not surjective : for <u>empty</u> fibers, replace with $P_\emptyset$ — extra operations.

h right exact, hom left exact — can't <u>really</u> write $h(M)$ as a $Hom(-, M)$.

In the case $J \hookrightarrow I_{\{M_i\}}$: ~~$P_I \otimes$~~

$$P_I(\{M_i\}, L) \otimes \bigotimes_{i \notin J} h(M_i) \longrightarrow P_J(\{M_i\}, L)$$

i.e. $P_I^*(\{M_i\}, L) \longrightarrow Hom(\bigotimes_{I \setminus J} h(M_i), P_J^*(\{M_i\}, L))$.

<u>Exercise–Proposition</u> : If J is nonempty then this map is injective

<u>Example</u> L is a Lie*-algebra, $x \in X$, what are Lmod$_x$ - Lmods supported at x?

$\mathcal{M}(X)_x \xrightarrow{h} Vect$. So we have vector spaces w/ some extra structure.

$h(L)_x$ is a Lie algebra, + a topological vector space :

The stalk of $h(L)$ is always topological for any $\mathcal{D}$-mod L — consider all $\mathcal{D}$-submodules $L' \subset L$ s.t. L/L' is supported at x

$\Rightarrow$ $h(L')_x \to h(L)_x \longrightarrow h(L/L') \to 0$ exact.

The open sets are then the images of these maps $h(L')_x \to h(L)_x$.

Lemma If the L_i's are fin gen D-modules then any $*$-operation $\varphi \in P_J^*(\{L_i\}, N)$ induces a continuous operation
$h(\varphi)_x : \bigotimes_I h(L_i)_x \to h(N)_x$.

(take an open in $h(N)$, killed by sufficiently high powers of (x), hence so will $h(L_i)$ in inverse image...)

So $h(L)_x$ is a topological Lie algebra, and a module over it is a continuous $h(L)_x$ module (wrt discrete topology on the vector space: Consider δ-function sections on M (supp at x) $h(M) \subset M$: so action $L \boxtimes M \to \Delta_* M$ gives $L \boxtimes h(M) \to M$ ~~...~~ M along first variable). Then $h(L) \boxtimes h(M) \to h(M)$ (h along first variable) is continuous...

We can also go backwards: continuous map $h(L) \otimes h(M) \to h(M)$ - so factors (at a point) $h(L) \to h(L/L') \to h(M)$ - but $h(L/L')$ supp at point, may remove h by Kashiwara, get $L \otimes h(M) \to M$, $L \boxtimes M \to \Delta_* M$ just by lifting back up to D.

Example K-M $\widetilde{\mathfrak{g} \otimes \mathcal{O}_x} = \mathfrak{g} \otimes D_x$ Lie*, apply h get back $\mathfrak{g} \otimes \mathcal{O}_X$ - no longer just a plain sheaf: has a topology, x-adic top on $\mathfrak{g} \otimes \mathcal{O}_X \xrightarrow{\wedge} \mathfrak{g} \otimes \hat{\mathcal{O}_X}$, modules are vector spaces on which this completed K-M acts - i.e. every element is killed by $\mathfrak{g} \otimes m^N$ - i.e. category $\mathcal{O}$!

Now Combine D-algebras & Lie* structure

We need to study symmetries in geometry of D_x-alg, but dualizing doesn't work, need $*$-theory instead of group schemes...

Compatibility between $\otimes$ and P^*: Fix attention on $M^r(X)$, which now has $\otimes$ tensor structure & $*$ ~~$\psi\otimes$~~ structure:
$M_1 \otimes^! M_2 = (M_1 \underset{\mathcal{O}}{\otimes} \omega_X^{-1}) \underset{\mathcal{O}}{\otimes} (M_2 \underset{\mathcal{O}}{\otimes} \omega_X^{-1}) \underset{\mathcal{O}}{\otimes} \omega_X = (M_1 \otimes \omega_X^{-1}) \underset{\mathcal{O}}{\otimes} M_2$
- since we discussed $\otimes$ on left, and now we must shift to right..
(corresp to $\Delta^!(M_1 \boxtimes M_2)$ ... up to shift)

Compatibilities: $i_0 \in I$, $j_0 \in J$, families $\underset{L}{\{M_i\}}$, $\underset{N}{\{K_j\}}$
have canonical map $P_I^*(\{M_i\}, L) \otimes P_J^*(\{K_j\}, N) \to P^*_{I \vee J}(\{M_i, K_j, M_{i_0} \otimes^! K_{j_0}\}, L \otimes^! N)$

($I \underset{i_0 j_0}{\vee} J$ union glued at $i_0 \sim j_0$) $\Rightarrow M_i\ (i \neq i_0)\ K_j\ (j \neq j_0).\ M_{i_0} \otimes K_{j_0}$

What is the map? $\boxtimes M_i \to \Delta_*^{(I)} L$, $\boxtimes K_j \to \Delta_*^{(J)} N$,

$\boxtimes M_i \boxtimes \boxtimes K_j \longrightarrow \Delta_*^{(I)} L \boxtimes \Delta_*^{(J)} N$

Consider hyperplane $X^{I \vee J} \hookrightarrow X^{I \sqcup J}$

- pull back our module to this hyperplane – still get something sitting on a diagonal of X^-, which is just what's written above.

Case of one module: $M \boxtimes K \to L \boxtimes N \Rightarrow M \otimes^! K \to L \otimes^! N$.

So above operations are $*$-generalization of tensoring morphisms ...

More fancy formulation: note that $\otimes \psi$ is not self-dual: dual category has arrows going wrong ways. Let's make it self-dual by brute force – i.e. cat. with $\otimes \psi$ on it & on the dual – "compound structure" with some compatibility. E.g. $\otimes$ defines $\psi\otimes$ on dual ($\otimes$ self-dual) $\Rightarrow$ compound structure.

Silliest example: $\otimes$, $\otimes$ itself – can certainly compound $\otimes M_i \to L$, $\otimes K_j \to N$.

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$$\begin{tikzcd} & M^r(X) & \\ \otimes^! \arrow[ur] \arrow[rr, dash, "\text{pairing}"'] & & P^* \arrow[ul] \end{tikzcd}$$

$I \ni i_0$, $j_0 \in J \Rightarrow I \underset{i_0 = j_0}{\vee} J$

$\otimes_{IJ}^{i_0 = j_0} : P_I^* \otimes P_J^* \longrightarrow P^*_{I \underset{i_0=j_0}{\vee} J}$

Commutativity clear – switch I, J, i_0, j_0.

Associativity: I, J, K with i_0, j_0', j_0'', k_0

$I \underset{i_0 j_0'}{\vee} J \underset{j_0'' k_0}{\vee} K$, two compositions $(I\,J)\,K$; $\to I \vee J \vee K$ ...

Different order pullbacks commute...

$\Rightarrow$ Compound tensor category (classical structure?)

Matrix algebras – Assoc algebras wrt $*$, come from following model!

V, V' vector spaces, $\langle\,\rangle : V' \times V \longrightarrow \mathbb{C}$

$V \otimes V'$ is then an associative algebra:

$v_1 \otimes v_1' \cdot v_2 \otimes v_2' = \langle v_1', v_2 \rangle\, v_1 \otimes v_2'$ (coordinate free matrix mult.!)

It acts on V from left, or V' from right.

Now assume $V, V' \in M^r(X)$, $\langle\,\rangle \in P^*(\{V', V\}, \omega_X)$

Claim $V \otimes^! V'$ is a $*$-associative algebra that acts on V from left, V' from right.

Action on V: $id_V \otimes \langle\,\rangle$: $\{V', V\} \to \omega_X$, $V \xrightarrow{id} V$

$\Rightarrow \{V \otimes V', V\} \longrightarrow V$

V' take $\langle\,\rangle \otimes id_{V'}$

$$\begin{array}{l} V \longrightarrow V' \\ \{V', V\} \longrightarrow \omega_X \\ \{\} \longrightarrow V \end{array}$$

Product : $id_V \otimes \langle\rangle \otimes id_{V'}$

Endomorphism algebras : M, L in a $*\otimes$ category "$*$"

$M, L \Rightarrow \mathcal{H}om^*(M, L)$? Assume we have a pairing $\langle\rangle \in P^*(\{X, M\}, L)$.

~~Def~~ Then for any $I, \{K_i\}$ get map

$P_I^*(\{K_i\}, X) \longrightarrow P_{I+1}^*(\{K_i, M\}, L)$

Def $(X, \langle\rangle)$ is $\mathcal{H}om^*(M, L)$ if for any $I, \{K_i\}$

$P_I^*(\{K_i\}, X) \xrightarrow{\sim} P_{I+1}^*(\{K, M\}, L)$

This is inner hom in our tensor category.

Say I is a single object : $Hom(K, \mathcal{H}om(M, L)) = P_2^*(\{K, M\}, L)$

Usual $\otimes$-cat : $\mathcal{H}om(M, L)$ is an object X equipped with a $X \otimes M \to L$ s.t. $Hom(K, X) \to Hom(K \otimes M, L)$ is iso.

Certainly unique if exists..

- trying to represent functor F as $Hom(\ldots, X)$: identifying these is like giving a canonical element $\alpha \in F(X)$ corresp to id_X : $Hom(X \otimes M, L) \longleftarrow Hom(X, X)$

$\alpha \longleftarrow id_X$

For M coherent this exists.

Exer Define inner P_I^* .... & show one can compose inner Hom.

$P^*(\{\mathcal{H}om^*(M, L), \mathcal{H}om^*(L, N)\}, \mathcal{H}om^*(M, N))$

Then $End^*(M)$ is an ass*-algebra acting on M from left & on ...

Lemma In $\mathcal{M}^r(X)^*$, $\mathcal{H}om(M, L)$ exists if M is coherent.

Pf Build it locally : $\mathcal{H}om(M, L) =$ sheaf $\underline{Hom}(\mathcal{D}_X, \mathcal{H}om(M, L))$

$= P_2^*(\;) = \underline{Hom}(\mathcal{D}_X \boxtimes M, \Delta_* L)$

$= \underline{Hom}(M, \Delta_* L) = \underline{Hom}(M, L \otimes_{\mathcal{O}_X} D_X)$

say 2nd struct. on diagonal

If M is not coherent this last term will be infinite sum :

Hom(quasicoh, q.c.) not nec q.c., but Hom(coh, q.coh) is coh.

Example $M^\circ := \mathcal{H}om^*(M, \omega_X)$ usual duality for D-modules.

$D_X^\circ = \omega_X \otimes_{\mathcal{O}} D_X$ - inner $*$-duality.

Example $V, V', \langle\rangle \Rightarrow$ alg $V \otimes V' \to End^* V$

Lemma (exercise) If V is locally free D-mod of finite rank, $V' = V^\circ$, $\langle\rangle$ canonical then $V \otimes V^\circ \xrightarrow{\sim} End^* V$.

Now apply h. First assume $V = \tilde{F}$ is induced, F coherent $\mathcal{O}$-mod.
$h(\mathrm{End}\, V)$ sheaf of assoc algebras acting on $\tilde{F}$.
Claim $h(\mathrm{End}^* \tilde{F}) = \mathrm{Diff}(F,F) \xrightarrow{\sim} \mathrm{End}\, \tilde{F}$.

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L a Lie*-algebra, Lmod $\hookrightarrow h(L)$-mod in $M_D^r(X)$ - a tensor cat. as usual.
Claim L-mod is a tensor category, $h(L)$ is a tensor functor which is a fully faithful embedding as tensor subcategory.
Proof should define an L-action on $M \otimes N$ (M, N are Lmods):
$\cdot_M \in P_2^*(\{L, M\}, M)$
$\cdot_M \otimes id_N \in P_2^*(\{L, M \otimes N\}, M \otimes N)$ by compound structure.
Use Leibniz formula, interchange M, N:
Laction on $M \otimes N$ is $\cdot_{M \otimes N} = \cdot_M \otimes id_N + id_M \otimes \cdot_N$
or explicitly $[L \boxtimes N \to \Delta_* M]$ a tensor in second variable with N
$[\omega_X \boxtimes N \to \omega_X \boxtimes N]^{\otimes !} = \Delta_*(M \otimes N)$ . . . .
A $D_X$alg $A \otimes^! A \to A$ can define L-action on D_X alg using tensor of L-modules.

$\boxed{\text{Duality}}$ $\quad O_n : M_D^r(X)^{\circ}_{coh} \longrightarrow M_D^r(X)$ (covariant)
$M \mapsto M^{\circ} = \mathcal{H}om(M, \omega_X) = \underline{Hom}_{D_X}(M, \omega_X \otimes D_X)$

($\omega_X \otimes_{\mathcal{O}_X} D_X$ has two right D-mod structures, from D as right & from D as left $\otimes$ with ω_X).
Consider both sides of $\circ$ as $\psi\otimes$ cats, $M_D^r(X)$ via $\otimes^*$ and $M_D^r(X)^{\circ}_{coh}$ from usual $\otimes$.
Claim $\circ$ lifts canonically to a $\psi\otimes$ functor
$$M^r(X)^{\circ !}_{coh} \longrightarrow M_D^r(X)^*$$

(Note - tensor of two coh. D-mods is not coh e.g. $D_X \otimes D_X$ inf generated so $M^r(X)^{\circ !}_{coh}$ is really only $\psi\otimes$..)
- as follows - in $M^r(X)^{\circ !}_{coh}$ $\quad P_I(\{L_i\}, M) := Hom_{M^r(X)}(M, \otimes^! L_i)$
.
$$P_I^*(\{L_i^{\circ}\}, M^{\circ}) \xleftarrow{\text{maps}}$$

Pf Recall we have a pairing $\langle\,\rangle_{L_i} \in P_2^*(\{L_i, L_i^\circ\}, \omega_X)$
$\Rightarrow \otimes_I \langle\,\rangle_{L_i} \in P_I^*(\{\otimes_I^! L_i, L_i^\circ\}, \omega_X)$
- compound tensor of the $\langle\,\rangle_{L_i}$

If $\varphi \in \mathrm{Hom}_{M(X)}(M, \otimes^! L_i)$ can compose:
$(\otimes\langle\,\rangle_{L_i})(\varphi, id_{L_i^\circ}) \in P_I^*(\{M, L_i^\circ\}, \omega_X) \rightarrow$
$\rightarrow P_I^*(\{L_i^\circ\}, M^\circ)$ - inner hom — this is our φ_0 functor. ■

We have notion of D_X-schemes $\ni Y$, group D_X-schemes $\ni G$,
action $G \times Y \to Y$ of G on Y. What is infinitesimal action?
$CoLie(G)$ is well defined - cotangent fiber of G at 0, (I/I^2)
product on G yields cobracket $CoLie(G) \xrightarrow{\delta} CoLie(G) \otimes CoLie(G)$
Assume that $CoLie(G)$ is coherent (e.g. G locally of fin type as D_X-scheme).
Then $CoLie(G)$ is a coherent D-module $\Rightarrow$ our duality
(which is *not* the abstract inner duality of a $\otimes$ category) - turns
takes $! \to *$). Set $Lie(G) = CoLie(G)^\circ$ —
which is a Lie^* algebra!
This Lie^* alg will act on the sheaf of functions of the space G acts on..

e.g. $\begin{matrix} G \\ \downarrow \\ X \end{matrix}$ acts on $\begin{matrix} Y \\ \downarrow \\ X \end{matrix} \Rightarrow \begin{matrix} JG \\ \downarrow \\ X \end{matrix}$ acts on $\begin{matrix} JY \\ \downarrow \\ X \end{matrix}$

($\begin{matrix}\text{left} \\ \mathcal{O}\text{-mod}\end{matrix}$) $CoLie\, JG = J(CoLie\, G) = D_X \otimes_{\mathcal{O}_X} CoLie\, G$.
- more generally $J\, Sym\, V = Sym\,(D_X \otimes_{\mathcal{O}_X} V)$ by universality:
$\mathrm{Hom}_{D_X\text{-alg}}(J\,Sym\,V, A) = \mathrm{Hom}_{\mathcal{O}\text{-alg}}(Sym\,V, A) = \mathrm{Hom}_{\mathcal{O}_X\text{-mod}}(V, A)$
$= \mathrm{Hom}_{D\text{-mod}}(D_X \otimes V, A) = \mathrm{Hom}(Sym(D_X \otimes V), A)$.

$Lie\, JG = Lie\, G \otimes D_X \ldots$ when we have central charge can't write
our Lie-alg as dual to anything .. only in some derived cat...
These are the *real* examples..

Remark A comm D_X alg, can consider A-mod, a tensor category via $\otimes_A$:
$\otimes_A M_i = \otimes_{\mathcal{O}_X} M_i / J \otimes_{\mathcal{O}_X} M_i$, J being the ideal of $A^{\otimes I} \to A$

Claim A-mod is a compound tensor cat:
$P^*_{I,A}(\{L_i\}, M) \subset P_I^*(\{L_i\}, M) \ni \varphi$ s.t. $\varphi \otimes_{i_0} id_A \in P_I^*(\{L_i, L_{i_0} \otimes A\}, M \otimes A)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \qquad\qquad \downarrow$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \varphi(\cdot_{i_0}, id_{L_i}) \in P_I^*(\{L_i, L_{i_0} \otimes A\}, M)$

$\otimes^!$ - exercise

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Lie algebroids X alg. variety - algebroid is a sheaf L with:
a. L is a (q-c) $\mathcal{O}_X$ module   b. L is a sheaf of Lie algebras
c. $\sigma: L \to \mathrm{Der}\,\mathcal{O}_X$ (Lie alg homomorphism) & $\mathcal{O}_X$-modules
· $l_1, l_2 \in L$, $f \in \mathcal{O}_X$ then $[l_1, f l_2] = \sigma(l_1)(f)\, l_2 + f[l_1, l_2]$
L-module is an $\mathcal{O}_X$-module M with an L-action (as Lie algebra)
s.t. $l \in L$, $m \in M$, $f \in \mathcal{O}_X$, $l(fm) = \sigma(l)(f)m + (f\, l(m))$, $(fl)m = f(lm)$
Examples 1) sheaf of vector fields (with some finiteness conditions on X) - Θ_X. A Θ_X-module is precisely a D-module (left).
2. $\begin{smallmatrix} F \\ \downarrow \\ X \end{smallmatrix}$ a bundle - pairs $(\tau, \tilde{\tau})$ $\tau \in \Theta_X$, $\tilde{\tau}$ lifting on F
$\Rightarrow$ Lie algebroid. Can put extra conditions on $\tilde{\tau}$: if $\mathcal{F}$ is a G-bundle consider lifts $\tilde{\tau}$ commuting with G-action $\Rightarrow \mathcal{E}_{\mathcal{F}}$.
- A Lie algebroid is transitive if $\sigma: L \twoheadrightarrow \mathrm{Der}\,\mathcal{O}_X$.
So $\mathcal{E}_{\mathcal{F}}$ is transitive. $0 \to \mathfrak{g}_{\mathcal{F}} \to \mathcal{E}_{\mathcal{F}} \xrightarrow[\sigma]{} \Theta_X \to 0$
- In general $\ker \sigma$ is a Lie $\mathcal{O}_X$-algebra.
Connection — an $\mathcal{O}_X$-linear section of σ. It's integrable iff this section is a morphism of Lie algebras.
3. Let P be a Lie alg. acting on $X \Rightarrow \mathcal{O}_X \otimes P$ is a Lie algebroid on X. : get σ by $\mathcal{O}$-linearly extending action $P \to \mathrm{Der}\,\mathcal{O}_X$
Claim $\exists!$ algebroid structure on $\mathcal{O}_X \otimes P$ with $\sigma/_P$ = action of P.
Ex. $G \supset H$, G acts on $G/H = X \xleftarrow{\text{principal } G}$ H-bundle F
$\mathcal{O}_X \otimes \mathfrak{g}$ Lie algebroid $= \mathcal{E}_F$.
$\mathfrak{h} \subset \mathfrak{g} \to \Theta_X$ - $\mathfrak{h}$ usually not ideal in $\mathfrak{g}$ .. but we can "idealify" it : $\mathfrak{h}_X \subset \mathcal{O}_X \otimes \mathfrak{g}$ is in fact a sheaf of ideals, (with quotient Θ_X) !
This is Lie algebra version of groupoids :
Groupoid - device which gives you a space of orbits - equiv relation, $\begin{smallmatrix} Y \\ \downarrow \\ X \times X \end{smallmatrix}$ morphism (not nec. subvariety), which is symmetric, has lifting over diagonal $X \hookrightarrow X \times X$ (lifting to Y) + composition axiom
Playing with sets, this gives a category : objects points of X, arrows elements of Y over $x_1 \times x_2$ — same as category with all arrows isos. - morphisms here are elems of our category.
- so this is a groupoid on X. $X = pt \Rightarrow X$-groupoid is just a group.
G acts on $X \Rightarrow$ groupoid on X: $X \xleftarrow{\text{projection}} G \times X \xrightarrow{\text{action}} X$
Groupoid action on X is local - may restrict to any open set U. . .

How to go to Lie algebroid: take unit section of one of the projections & take normal bundle .. (image of diagonal)

Consider relative tangent bundle of projection P_1 — $\Theta_{Y/X}$

We have unit section $e: X \to Y$, our Lie algebroid is $e^* \Theta_{Y/X}$.

4) Poisson structure: $\{\}$ on X $\Rightarrow$ canonical Lie algebroid on Ω_X (cotangent sheaf), uniquely defined by this structure.

- the unique algebroid structure s.t. $\mathcal{O}_X \xrightarrow{d} \Omega'_X$ is a morphism of Lie algebras, & $\sigma(df)(g) = \{f, g\}$. $[df, dg] = d\{f, g\}$

- $\mathcal{O}_X$ as abstract Lie alg acts on $\mathcal{O}_X$, may take induced algebroid $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \twoheadrightarrow \Omega'$ $f \otimes g \mapsto f\, dg$.

Lemma Poisson structure on $X \Leftrightarrow$ Lie alg str. on Ω_X s.t. $d: \mathcal{O}_X \to \Omega'_X$ commutes with Ω_X action

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Compound Lie $\mathcal{A}$-algebroids: $\mathcal{A}$ is a $\mathcal{D}_X$-algebra, these are $\mathcal{D}_X$-mod $\mathcal{L}$ with the following structures: - $\mathcal{A}$-module
- Lie*-alg structure - σ action of $\mathcal{L}$ on $\mathcal{A}$ (Lie* alg on $\mathcal{D}_X$-alg action)

Compatibilities: - adjoint action of $\mathcal{L}$ on $\mathcal{L}$ is compatible with σ wrt $\mathcal{A}$-mod structure on $\mathcal{L}$ i.e. $\mathcal{A} \otimes^! \mathcal{L} \to \mathcal{L}$ is a morphism of $\mathcal{L}$-modules. — $\sigma \in P_2^*(\{\mathcal{L}, \mathcal{A}\}, \mathcal{A})$ is $\mathcal{A}$-linear wrt $\mathcal{L}$ variable

$\Rightarrow$ category $\mathcal{CL}(\mathcal{A})$ compound Lie $\mathcal{A}$-algebroids.

$\mathcal{L}$-modules are $\mathcal{D}_X$-mod M with structure of $\mathcal{A}$-module, $\mathcal{L}$-mod (as Lie* alg) and compatibility $\cdot_M \in P_2^*(\{\mathcal{L}, M\}, M)$ $\mathcal{L}$-linear and $\mathcal{A} \otimes M \to M$ compatible with action.

Ex. 1. Tangent algebroid $\Theta_{\mathcal{A}} := \mathcal{D}er^*(\mathcal{A}, \mathcal{A}) \subset \mathcal{H}om^*(\mathcal{A}, \mathcal{A})$
- polylinear maps in $P_I^*(\{\mathcal{L}, \mathcal{A}\}, \mathcal{A})$ which are derivations wrt. variable $\mathcal{A}$. (when corresp. functor is representable) — as $\mathcal{H}om_{\mathcal{A}}^*(\Omega_{\mathcal{A}}, \mathcal{A})$

If $\mathcal{A}$ is smooth then $\Theta_{\mathcal{A}}$ exists: then $\Omega_{\mathcal{A}}$ is a (loc. projective) free mod of finite rank over $\mathcal{A}[\mathcal{D}_X]$ - then $\mathcal{H}om_{\mathcal{A}}^*(\Omega_{\mathcal{A}}, \mathcal{A})$ is well defined...

Universality of tangent algebroid: every Lie algebroid $\mathcal{L} \in \mathcal{CL}(\mathcal{A})$ has $\exists!$ $\mathcal{L} \to \Theta_{\mathcal{A}}$, via action $\mathcal{L}$ on $\mathcal{A}$.

Tangent algebroid for jet algebras $\mathcal{A} = JB$: $\mathcal{A}[D_X] \otimes_B \Omega_B \xrightarrow{\sim} \Omega_{\mathcal{A}}$
(EXERCISE) . $\Theta_{\mathcal{A}} = \mathcal{H}om^*_{\mathcal{A}}(\Omega_{\mathcal{A}}, \mathcal{A}) =$
$Hom_{\substack{\mathcal{A}[D_X]\text{mod} \\ (\text{left})}}(\mathcal{A}[D_X] \otimes_B \Omega_B, \mathcal{A}[D_X])$
which is a right $\mathcal{A}[D_X]$-mod, $\mathcal{A}[D_X] = \Delta_X \mathcal{A}$:

$\Theta_{\mathcal{A}} = Hom(\Omega_{\mathcal{A}}, \mathcal{O}_{\mathcal{A}}) = P_2^*(\{\Omega^{\omega}_{\mathcal{A}}, D_X\}, \Delta \omega_X \mathcal{A}) = Hom(\omega_X \Omega_{\mathcal{A}} \otimes D_X, \omega_X \mathcal{A})$
But $\delta_X \omega_X \mathcal{A} = \omega_X \mathcal{A}[D_X]$ so may rewrite this as
$Hom_{\mathcal{A}[D_X]}(\Omega_{\mathcal{A}}, \mathcal{A}[D_X])$

$\Rightarrow \Theta_{\mathcal{A}} = Hom_{\mathcal{O}_B}(\Omega_B, \mathcal{A}[D_X]) = \Theta_B \otimes_B \mathcal{A}[D_X]$
Now $h(\Theta_{\mathcal{A}})$ should act on $\mathcal{A}$: $h(\Theta_{\mathcal{A}}) = \Theta_B \otimes_B \mathcal{A}$ acting in obvious way on $\mathcal{A}$ - inf. auts of B act on $\mathcal{A}$ = jets of B.

Now Assume $\dim X = 1$, $\mathcal{A}$ smooth
Def $\mathcal{L} \in \mathcal{E}\mathcal{L}(\mathcal{A})$ is an elliptic algebroid if
$\mathcal{L} \to \Theta_{\mathcal{A}}$ is injective, and the cokernel is a projective $\mathcal{A}$-mod of finite rank.

Compound Poisson (= Coisson) structure $\mathcal{A}$ D_X-algebra with $\{\} \in P_2^*(\{\mathcal{A}, \mathcal{A}\}, \mathcal{A})$ st. the adjoint action is an action of $\mathcal{A}$ (as Lie*-alg) on $\mathcal{A}$ (as D_X-alg).
$\mathcal{A}$ smooth have unique factorization $\mathcal{A} \to \Theta_{\mathcal{A}}$ through $\Omega_{\mathcal{A}}$, $\mathcal{A}$ linear
$\Rightarrow \Omega_{\mathcal{A}}$ canonically Lie*-$\mathcal{A}$ algebroid,
from which can recover Poisson on $\mathcal{A}$.
Def. A coisson is elliptic if $\Omega_{\mathcal{A}}$ is an elliptic algebroid.

A coisson structure is symplectic if $\Omega_{\mathcal{A}} \longrightarrow \Theta_{\mathcal{A}}$ is iso.
Elliptic is the true generalization of symplectic to infinite dimensions...
(symplectic not that interesting here.)

Ex 1. Kostant-Kirillov poisson structure on Sym $\mathfrak{g}$:
· If L is any Lie*-alg then Sym$^!L$ is a coisson algebra - namely the bracket $\{\}$ characterized by property $L \hookrightarrow \text{Sym}\, L$ is a morphism of Lie algebras.
2. Twisted version : $0 \to \omega_X \to \tilde{L} \to L \to 0$ central ext.
(in cat. of Lie*-alg) $\Rightarrow$ twisted symmetric algebras -
quotient of Sym $\tilde{L}$ by ideal gen by ω_X $(\mathbb{1} = 1)$ - commutative filtered algebra with assoc graded Sym $L \to \text{Sym}^{\sim} L$.

Ex. Heisenberg algebra $\{p(x), q(y)\} = \delta'(x-y)$:
extension $0 \to \omega_X \to \mathcal{H}eis \to p\mathcal{D}_X \oplus q\mathcal{D}_X \to 0$
Then ($Sym^\sim \mathcal{H}eis$) $sym^\sim(p\mathcal{D}_X \oplus q\mathcal{D}_X)$ is elliptic — this is $A = Sym(p\mathcal{D}_X \oplus q\mathcal{D}_X)$ with twisted bracket.
$\Omega \to \Theta$ ——
$A[\partial_x^n]dp \oplus A[\partial_x]dq \to A[\partial_x]\partial_p \oplus A[\partial_x]\partial_q$
via $dp \mapsto \partial_x \partial_q$, $dq \mapsto \pm \partial_x \partial_p$
with coker two copies of A (first-order terms)

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On a curve X we have the notion of elliptic coisson algebras A - take A smooth $\mathcal{D}_X$-alg with $\{\}$ $\Rightarrow$ Lie algebroid str. on $\Omega_A \xrightarrow{\sigma} \Theta_A$.
- elliptic: σ injective, coker σ is a proj. A-mod of finite rank.
Recall Ω_A is a projective $A[\underline{\mathcal{D}_X}]$-mod ($A$ smooth) - huge compared to A.

Note/Exercise - ω_X sitting in any $\mathcal{D}_X$-alg is always central:
$\omega_X \subset \tilde{L}$, pairing $\omega_X \boxtimes \tilde{L} \to \Delta_* \tilde{L}$. Fix a point in X (second variable) $\Rightarrow$ get morphism $\omega_X \otimes L_x \to \delta_x \otimes \tilde{L}_x$ - fibers
but there are no morphisms $\omega_X \to \delta_x$ other than zero
Furthermore ω_X will act trivially on any $\tilde{L}$-mod ($\omega_X \boxtimes P \to \Delta_* Q \ldots$)
— can't feel "level"/central charge classically - only quantized.
$Sym^\sim L$ - functions on hyperplane in dual to $\tilde{L}$....
No Kostant-Kirillov (untwisted) coissons are elliptic - degenerate at 0...
Untwisted - L loc. free $\mathcal{D}_X$ mod of finite rank. Now in general
$\Omega_{Sym L} = Sym L \otimes L$ via $ad\,l \leftrightarrow a \otimes l$.
$\Theta_{Sym L} \leftarrow Sym L \otimes L^\circ$ canonically - $l^\circ \in L^\circ$ gives morphism $Sym L \to \mathcal{D}_X$, use inner hom defn of $\Theta_{Sym L} = \mathcal{H}om^*_{Sym L}(\Omega_{Sym L}, Sym L)$
$= \mathcal{H}om^*(L, Sym L) \leftarrow \mathcal{H}om^*(L, \mathcal{O}_X)$ via $\mathcal{O}_X \to Sym L$
$\Rightarrow$ map $Sym L \otimes \mathcal{H}om^*(L, \mathcal{O}_X) \to \mathcal{H}om^*(L, A)$ which is ISO with the above freeness condition.

$$\begin{array}{lccc} A = Sym L & \Omega_A & \longrightarrow & \Theta_A = A \otimes L^\circ \supset L^\circ \\ & \cup & & \cup \\ & L & \xrightarrow{[,]} & L \otimes L^\circ = \mathcal{H}om(L, L) \end{array}$$

Image of the map starts with quadratics, in particular $L^\circ \to$ coker σ is injective : K-K bracket degenerates at 0 $\Rightarrow$ so not symplectic.

Assume this is elliptic - coker loc. free A-mod. . take it's quotient mod standard (aug) max ideal in A. Get a $\mathcal{D}$-mod, with $L^\circ \hookrightarrow \operatorname{coker}\sigma / I \operatorname{coker}\sigma$. But elliptic hypothesis's $\Rightarrow$ coker / I coker is loc. free $\mathcal{O}_X$-mod of fin. rank .. ■

Twisted case - twisted algebra $\mathrm{Sym}^\sim L$ for Heisenberg - usual symplectic bracket on hyperplane, nondegen! (fin dim) - via splitting of exact sequence.

Inf-dim - say we have $\mathcal{D}$-mod splitting $0 \to \omega_X \to L^\sim \rightleftarrows L \to 0$
- get K-K bracket with a correction term.

$$\begin{array}{ccc} \Omega_A & \longrightarrow & \Theta_A \\ \cup & & \cup \\ L & \xrightarrow{[\,]\oplus[\,]} & L^\circ \oplus L\otimes L^\circ \end{array}$$

since $[\,]_{L^\sim} = [\,]_L + c$
- $c \in P_2^*(\{L, L\}, \omega_X) = \mathrm{Hom}(L, L^\circ)$

Map no longer preserves grading, but filtered $(\mathrm{Sym}^{\geq i} L)\otimes L \longrightarrow \mathrm{Sym}^{\geq i} L \otimes L^\circ$
- successive graded quotients $\operatorname{gr}\sigma : \mathrm{Sym}^\bullet L \otimes L \to \mathrm{Sym}^\bullet L \otimes L^\circ$
$= \mathrm{id}_{\mathrm{Sym} L} \otimes c$ so if assoc graded is iso, original map is - so our coisson is symplectic/elliptic iff the diff op $c : L \longrightarrow L^\circ$ is iso or elliptic - here an elliptic diff op (map between $\mathcal{D}$-mod) is injective with coker loc free fin gen. $\mathcal{O}$-module. .

Ex $\mathfrak{g}, (\,) \Rightarrow$ K-M $0 \to \omega_X \to \mathfrak{g}\otimes \mathcal{D}_X^\sim \rightleftarrows \mathfrak{g}\otimes\mathcal{D}_X \to 0$, with cocycle $(\mathfrak{g}\otimes\mathcal{D}_X)\boxtimes(\mathfrak{g}\otimes\mathcal{D}_X) \to \Delta_* \omega_X = \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$
$(l_1 \otimes 1)\boxtimes(l_2\otimes 1) \longrightarrow (l_1, l_2)\, dx \otimes \partial_x$ $\quad dx \in \omega_X,\ \partial_x \in \mathcal{D}_X$ -
in local coordinates.

$$c : L \to L^\circ \quad \begin{tikzcd} \mathfrak{g}\otimes\mathcal{D}_X \arrow[rr] \arrow[dr, "\mathrm{id}\otimes\mathrm{canon}"'] & & \mathfrak{g}^*\otimes\omega_X \mathcal{D}_X \\ & \mathfrak{g}\otimes\omega_X\mathcal{D}_X \arrow[ur, "(,)\otimes\mathrm{id}_{\omega_X \mathcal{D}_X}"'] & \end{tikzcd}$$

canon. $\mathcal{D}_X \to \omega_X \mathcal{D}_X \quad 1 \longmapsto dx \otimes \partial_x$ - corresponds to diff op $d : \mathcal{O}_X \to \omega_X$ (these are induced)
coker canon is ω_X. If $(,)$ is nondegen, then coker $\sigma = \mathfrak{g}\otimes\omega_X \Rightarrow$ elliptic coisson! ■

Local Poisson algebras

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dim $X=1$, A any D_X-alg, $x\in X$. $A_x = A/m_x A$.

$j: X\setminus\{x\} \longrightarrow X$, $A_x = h(j_*j^*A/A)$ – vector space at point x. Spec A_x = space of horizontal sections of Spec A over Spec O_x (formal disc: $O_x \subset K_x \Leftrightarrow \mathbb{C}[[t]]\subset \mathbb{C}((t))$)

– any point of fiber may be extended via the connection to a horizontal section/a formal disc.

$A_{(x)} = \varprojlim A'_x$, A' varying over all D_X subalg $A'\subset A$ s.t. $A'|_{X\setminus\{x\}} = A|_{X\setminus\{x\}}$.

$\boxed{\text{Note } A'\subset A \Rightarrow A'_x \twoheadrightarrow A_x}$ – fiber at x is larger:

Consider $A'\to A\to A/A'\to 0$ – A/A' supp at point and infinitely divisible... tensor with $\otimes_{O} O/m_x$

$\Rightarrow A'_x \to A \to A/A'\otimes O/m_x = 0$.

(A/A' supp at point $\Rightarrow$ sum of δ-functions $\Rightarrow$ inf. divisible).

Claim: Spec $A_{(x)}$ = space of horiz sect of Spec A over Spec K_x

– first we have canonical projection $A_{(x)}\twoheadrightarrow A_x$, hence Spec $A_{(x)} \hookleftarrow$ Spec A_x restriction of horizontal sections.

– What is a section? horizontal O_x-morphism $A\to K_x$. Consider $O_x\subset K_x$ and the preimage $A'\to O_x$

– A' coincides with A outside x. Now get $A'_x\to\mathbb{C}$ "determining" $A'\to O_x$ – fiber at point.

Spec $A_{(x)} := \bigcup$ Spec A' (Spectrum of projective limit)...

Conversely localize $A'\to O_x$ to get $A\to K_x$ (agree when localized!) ■

Proposition Assume that A is f.gen as D_X algebra. Then any Poisson bracket on A yields a Poisson bracket on $A_{(x)}$.

Pf. Construction: $\forall$. $A_x \twoheadleftarrow h(j_*j^*A)$ canonically

$\Rightarrow$ morphism $A_{(x)} \xleftarrow{\nu} h(j_*j^*A)_x$ – dense image (surjective on any quotient - just A).

$h(j_*j^*A)$ has Lie bracket – $A_{(x)}$ is completion of this so bracket extends if it's continuous...

Continuity: for given A' and $f\in h(j_*j^*A)$, $\exists A''$ s.t. $\{A'',A'\}\subset A'$, $\{A'',f\}\subset A'$.

Check this.

This bracket is Poisson: $h(j_*j^*A)$ acts on j_*j^*A by derivations. Now take $f\in h(j_*j^*A)$ s.t. $ad_f: A'_x\to A'_x$ – must be derivation, so our action is direct limit of derivations. ■

Consider $I \subset A_{(x)} \twoheadrightarrow A_x$. Claim I involutive (closed under the Poisson bracket.) — $h(A)_x \to h(j_* j^* A)_x$ generates the ideal I .. for any A' we have $h(A) \to A'_x \to A_x \to 0$ which comes from h applied to $A \to j_* j^* A/A' \to (j_* j^* A)/A \to 0$.
$\Rightarrow$ kernel at each level is a quotient of $h(A)$ (h right exact)
$\Rightarrow$ I completion of $h(A)_x$ $\Longrightarrow$ involutive.

<u>Now</u> I/I^2 is an A_x module. I/I^2 is Lie, acts on $A_{(x)}$ ~~acts~~ by Poisson bracket : $\Rightarrow$ map $I/I^2 \to \Theta_A$
I acts on A_x $\Rightarrow$ <u>I/I^2 is a Lie A_x-algebroid</u> (Hamiltonian reduction).

[<u>General nonsense</u> - N an A-module, may form
$$h(N)^{\wedge}_x = \varprojlim_{N' \subset N \text{ A submod coinciding with } N \text{ outside } x} h(N/N')$$

This is an A_x-module : N/N' is an A-mod supp at point $\Leftrightarrow$ A_x module.

<u>Lemma</u> Assume N is a compound Lie A-algebroid $N \in \mathcal{CL}(A)$, and N is fin gen as A-module.
Then $h(\hat{N})_x$ is a Lie A_x-algebroid
- action on A_x comes from continuous action of $h(N)_x$ of which this is a completion.]

<u>Proposition</u> $I/I^2 = h(\Omega A)^{\wedge}_x$
<u>Pf</u> The map sends $(a\, db\, dx) \longmapsto a\, \nu(b\, dx)$ $\quad - b\,dx \in h(A)$.
<u>Prove this</u>. <u>Hint</u> Compute relative $\Omega_{A/A'}$.
<u>Answer</u> $\Omega_{A/A'} \cong A \otimes_{A'} A/A'$ $\quad a\,db \longleftarrow a \otimes b$ — using infinite divisibility of A/A'.

<u>The Global Space of Sections</u>
X - compact curve, A $\mathcal{D}_X$-algebra. $\Gamma^{\nabla}(X, \operatorname{Spec} A) = \operatorname{Spec} B$ space of horizontal sections, $B = H_{\nabla}(X, A)$ maximal constant quotient.
$x \in X$, $B = A_x / A_x \operatorname{Res}_x(H'(X \setminus \{x\}, A))$
via $\operatorname{Res}_x : H'(X \setminus \{x\}, A) \to A_x$
$S \subset X$ write $B = \bigotimes_{x \in S} A_x / J_S$, J_S is gen by the image of $\operatorname{Res}_S = \Sigma \operatorname{Res}_{x_i} : H'(X \setminus S, A) \to \bigoplus A_{x_i} \subset \bigotimes_{x \in X} A_x$

Consider $\mu_S : H'(X \setminus S, A) \to \bigoplus A_{(x)} \subset \hat{\bigotimes}_{x \in X} A_{(x)}$

$\Gamma^\nabla(X\setminus S, \operatorname{Spec} \mathcal{A}) = \operatorname{Spec} \hat{\otimes}_S \mathcal{A}_x / \mu_S(H'(X\setminus S, \mathcal{A}) \hat{\otimes}_S \mathcal{A}_x$

$\left(\begin{array}{ccc} \mathcal{A} & \to & j_* \mathcal{O} \\ \cup & & \cup \\ \mathcal{A}' & \longrightarrow & \mathcal{O} \end{array} \right.$; and pass to h.T. . .)

Suppose $\mathcal{A}$ is a coisson. $\hat{\otimes}_S \mathcal{A}_{(x)}$ will then be a topological Poisson algebra. Then μ_S will also be a morphism of Lie algebras – may be considered a Poisson action (Hamiltonian) of $H'(X\setminus S, \mathcal{A})$, a Lie algebra, on the Poisson algebra $\hat{\otimes}_S \mathcal{A}_{(x)}$

Rewrite it ~~as~~ $H'(X\setminus S, A) \to \otimes \mathcal{A}_{(x)}$ as a morphism $\prod \operatorname{Spec} \mathcal{A}_{(x)} \longrightarrow H'(\ )^*$, and then the space of horizontal sections of Spec $\mathcal{A}$ over $X\setminus S$ is the zero fiber of the moment map for this action.

($H'(X\setminus S, A) \to \otimes A_{(x)}$ ~~might~~ expand to Sym H' → ... Poisson)

Hamiltonian reduction – take invariants of this zero fiber..

Example $\mathfrak{g}$, (,) semisimple ⟹ K-M $\mathfrak{g}\otimes D_x^\sim$,

$\mathcal{A} = \operatorname{Sym}^\sim(\mathfrak{g}\otimes D_x)$. Spec $\mathcal{A}_{(x)}$ = the space of connections on the trivial G-bundle on the formal punctured disc at x. : as plain comm. algebra $\operatorname{Sym}^\sim = \operatorname{Sym}$ (extension splits),

Spec Sym ⟷ $\mathfrak{g}^* \otimes \omega_\emptyset$ – dual to $\mathfrak{g}\otimes K_x/\mathcal{O}_x$

$\partial + \mathfrak{g}\otimes\omega_\emptyset$ gives connections... $\omega_\emptyset$ – diffs on the disc

Horizontal sections will give space of connection on $X\setminus S$.. take Hamiltonian reduction .. [Spec A_x – connections on formal disc]

– isomorphism classes of (global) connections on curve – no reference to bundle.

Coisson reduction Standard situation $\mathfrak{g} \xrightarrow{\text{Lie}\ \text{Lie morph}} A^{\text{Poisson}}$,

Sym $\mathfrak{g} \to A$, take preim of 0 on spectra, reduce wrt $\mathfrak{g}$.

In other terms, $I = A$ (image of $\mathfrak{g}$) – involutive ideal

A/I doesn't inherit bracket .. take subspace invariant under I/I^2 – this acts on A/I – it is a Lie A/I-algebroid (has bracket etc) ⟹ $(A/I)^{I/I^2} = (A/I)^{\mathfrak{g}}$

in our situation – since I is gen by $\mathfrak{g}$.. in general only need the involutive ideal I, not $\mathfrak{g}$-action.

Coisson setting : $\mathcal{A}$ coisson $\mathcal{A} \supset \mathcal{I}$ involutive ideal

$\mathfrak{I}/\mathfrak{I}^2$ gets bracket, becomes $\in \mathcal{L}\langle A/\mathfrak{I}\rangle \Rightarrow$ invariants $(A/\mathfrak{I})^{\mathfrak{I}/\mathfrak{I}^2}$.. If $\mathfrak{I}$ is gen. by a Lie* subalg $\mathfrak{g}$ it's sufficient to take $\mathfrak{g}$-invariants
$\Rightarrow$ new coisson algebra.

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X compact curve, $\mathcal{A}$ $\mathcal{O}_X$-alg. Spec $H_\nabla(X,\mathcal{A})$ · horiz. sections of Spec $\mathcal{A}$.
$S \subset X$ finite then $H_\nabla(X,A) = \bigotimes_{x\in S} \mathcal{A}_x / \mathfrak{I}_S$ – ideal gen. by the image of Res : $H(X\setminus S, \mathcal{A}) \to \oplus \mathcal{A}_x \subset \otimes \mathcal{A}_x$
Now $H(X\setminus S, \mathcal{A})$ is too large ... $\Rightarrow$

Lemma Let $L \subset \mathcal{A}$ be a $\mathcal{D}$-mod that generates $\mathcal{A}$ as $\mathcal{D}_X$ alg.
Then $\mathfrak{I}_S$ is generated by the image of $H(X\setminus S, \mathcal{L})$

Proof suffices to use char. as maximal const. quotient ...
Case (i) $\mathcal{A} = \mathrm{Sym}\, \mathcal{L}$ .. then $H_\nabla(X,A) = \mathrm{Sym}$(max const quotient of $\mathcal{L}$) $= H^2(X,\mathcal{L})$ top deRham
(ii) General :

$$\begin{array}{ccc} \mathrm{Sym}\,\mathcal{L} & \twoheadrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ H_\nabla(X,\mathrm{Sym}\,\mathcal{L}) & \twoheadrightarrow & H_\nabla(X,\mathcal{A}) \end{array}$$

cocartesian .. $\to$ gen of kernel
Ideals must come from $\mathcal{L}$ ∎

Corollary Hor sec. of Spec $\mathcal{A}$ on $X\setminus S$ = Spec $H_\nabla(X\setminus S, \mathcal{A})$ = Spec $\varinjlim H_\nabla(X, \mathcal{A}')$ over $\mathcal{A}' \subset \mathcal{A}$ subalg, coincides with $\mathcal{A}$ outside S.
Thus $H_\nabla(X\setminus S, \mathcal{A}) = \hat{\bigotimes}_{x\in S} \mathcal{A}_{(x)} / \mathfrak{I}_S$, generated by Res from $H(X\setminus S, \mathcal{A})$
Then again it's sufficient to take ideal coming from Res $(H(X\setminus S, \mathcal{L}))$

Coisson case : $H(X\setminus S, \mathcal{L}) \hookrightarrow H(X\setminus S, \mathcal{A})$ Vect subspace of this Lie algebra.
May consider reduction wrt this subspace – action of generators
$\Rightarrow (\hat{\otimes} \mathcal{A}_{(x)} / \mathfrak{I}_S)^{H(X\setminus S,\mathcal{A})} = (\hat{\otimes} \mathcal{A}_{(x)} / \mathfrak{I}_S)^{H(X\setminus S,\mathcal{L})}$
– same reduction just from $\mathcal{L}$.

Example $\mathfrak{g}, \langle\,\rangle \Rightarrow \mathfrak{g}\otimes\mathcal{D}_X^{\sim}$, $\mathcal{A} = \mathrm{Sym}^{\sim}(\mathfrak{g}\otimes\mathcal{D}_X)$ Kostant-Kirillov coisson algebra. This is canonically split as algebra.
Spec $\mathcal{A}_x = \mathfrak{g}^* \otimes \omega_{\mathcal{O}_x}$. Spec $\mathcal{A}_{(x)} = \mathfrak{g}^* \otimes \omega_{K_x}$
Using $\langle\,\rangle$ $\mathfrak{g}^* \simeq \mathfrak{g}$, Spec $\mathcal{A}_x \simeq \mathfrak{g}\otimes\omega_{\mathcal{O}_x}$ = connections on trivial G-bundle on Spec $\mathcal{O}_x$, similarly for Spec $\mathcal{A}_{(x)}$.
$h(\mathfrak{g}\otimes\mathcal{D}_X) = \mathfrak{g}\otimes\mathcal{O}_X$, fiber at x – completed at $\mathcal{O}_x$.
– acts as algebra of gauge transformations on connections
= adjoint action of $h(\,)$ on —.
Thus for $\mathcal{A}_{(x)}$.

Global horizontal sections - global connections on trivial bundle
- maximal const quot. of $\mathcal{A}$ · gen by $\mathfrak{g} \otimes \mathcal{D}_X$ ..
$Hom(\mathfrak{g} \otimes \mathcal{D}_X, \omega_X) = Hom(\text{max const quot}, \mathbb{C})$
"$\mathfrak{g}^* \otimes$ forms - so spec $H_0 A$ is $\mathfrak{g}^* \otimes$ 1-forms...
Take Hamiltonian reduction - Lie algebra acts by gauge transformations ... as affine scheme quotient get just point: Group of gauge transforms gives isom classes of trivial G-bundles with connections -- but this space has no functions but constants ⟹ point spec. .. More cleverly should do stack quotient..

DS reduction extracts from this space a nice affine slice...

General nonsense on elliptic Coisson algebras

$\dim X = 1$, $\mathcal{A}$ $\mathcal{D}_X$-alg $L \in \mathcal{CL}(\mathcal{A})$ elliptic .. assume $\mathcal{A}$ smooth..
$0 \to L \to \Theta_A \to \phi_L \to 0$, ϕ_L proj $\mathcal{A}$-mod of finite rank
L, Θ proj $\mathcal{A}[\mathcal{D}_X]$ mods
Claim ϕ_L is a Lie algebra in the tensor cat. of $\mathcal{A}$-mods-

Ex. L comes (as $\mathcal{R}_A$) from an elliptic coisson algebra.
Kac-Moody case $A = Sym^2(\mathfrak{g} \otimes \mathcal{D}_X)$, spec $\mathcal{A}_X$ connections on triv G
- get sheaf of Lie algebras on this, locally free family, whose fiber at any connection is some Lie algebra..
— flat endomorphism of your bundle (endos preserving connection) - twisted version of $\mathfrak{g}$..

Definition of this bracket - Preliminaries
DeRham-Chevalley complex of a Lie algebroid:
Usual situation: $\mathcal{A}$ comm alg, L a Lie $\mathcal{A}$-algebroid.
- A constant, L just a Lie alg this is a dg-algebra - L cochains.
- A smooth, L vector fields ⟹ deRham complex of A.
- General $C^i(L, A) = Hom_A(\Lambda^i_A L, A)$ - can replace A by any L-mod M. This is a complex - using Chevalley differential for Lie alg. cohomology

$\boxed{11/1/95}$

$\mathcal{A}$ $\mathcal{D}_X$-alg, smooth. $\mathcal{L} \xrightarrow{\sigma} \Theta_{\mathcal{A}}$ Lie algebroid over $\mathcal{A}$
proj. f.g. $\mathcal{A}[\mathcal{D}_X]$-modules

Claim consider the dual $\mathcal{L}^\circ$ (duality even reflexive for f.g. proj..).
$0 \leftarrow \phi_{\mathcal{L}} \leftarrow \mathcal{L}^\circ \xleftarrow{\sigma^*} \Omega_{\mathcal{A}}$ – then the cokernel $\phi_{\mathcal{L}}$
is a Lie coalgebra in the tensor category of $\mathcal{A}[\mathcal{D}]$-modules
e.g. if $\sigma = 0$, $\mathcal{L}$ is Lie alg. over $\mathcal{A}$, then $\mathcal{L}^\circ$ becomes
Lie coalg ∴ Duality : $\mathcal{M}^{f.g.}_{\mathcal{A}}(X) \to \mathcal{M}^{*}_{\mathcal{A}}(X)$ – inner hom to $\mathcal{A}\otimes\omega$..
This is an equivalence of the categories if we consider f.g.
projective $\mathcal{A}[\mathcal{D}_X]$-modules.

In particular, in the elliptic situation : $(\text{coker } \sigma)^* = \phi_{\mathcal{L}} \Rightarrow$
coker σ is a Lie algebra in the tensor cat. of $\mathcal{A}[\mathcal{D}_X]$-modules.
Elliptic case – coker σ is f.g. $\underline{\mathcal{O}\text{-mod}}$ – so no
morphisms ~~from~~ to $\mathcal{D}$ – so σ^* is also injective
$0 \to A \to B \to V \to 0$
free $\mathcal{D}_X$-mod of finite rank, V free $\mathcal{O}_X$-mod of
finite rank – ~~plane~~ vector bundle with connection.
$0 \leftarrow V^* \leftarrow A^\circ \leftarrow B^\circ \leftarrow 0$ V^* dual v.b. with dual connection
$(\ )^\circ = \mathcal{H}om(-, \mathcal{D}_X)$ $V^* = \mathcal{H}om(-, \mathcal{O})$ – naive duality

Chevalley–deRham complex for a Lie algebroid – common gen. of Chevalley & deRham.
In compound setting –
$\mathcal{L} \in \mathcal{CL}(\mathcal{A})$ – get $\boxed{\text{global CdR complex}}$ $C^*_{\mathcal{A}}(\mathcal{L}, M)$
(M $\mathcal{L}$-mod). Terms $P^{*\,\text{skewsym}}_{\mathcal{A},n}(\{L\}_{1,\dots,n}, M)$

[$C^*_{\mathcal{A}}(\mathcal{L}, \mathcal{A})$ is a dg algebra.] Differential is usual formula
$C^*_{\mathcal{A}}(\mathcal{L}, M) \subset C^*(\mathcal{L}, M)$ – forget $\mathcal{A}$-structure, $\mathcal{L}$
plain Lie* algebra – define this to be a subcomplex.
similar to embedding Lie alg complex into all vector fields.
$C^*_{\mathcal{A}}(\mathcal{L}, A)$ is a dg. commutative algebra – but differential
not A-linear. → like usual deRham complex.

$\boxed{\text{Local CdR complex}}$ $\mathcal{C}^*_{\mathcal{A}}(\mathcal{L}, M)$ – replace polylinears P..
by inner homs : terms $\mathcal{P}^{*\,\text{skew}}_{\mathcal{A},n}(\{L\}_{1\dots n}, M)$
Exists when $\mathcal{L}$ is a proj f. rank $\mathcal{A}[\mathcal{D}_X]$-mod.

From duality, $\mathcal{P}_n^{*\,sh}(\{L\},M) = (\Lambda^n_A L^\circ)\otimes_A M$

$A = M$: terms are $\Lambda^n_A L^\circ = \mathcal{C}^n_A(L, A)$ dg comm alg.

Claim $\mathcal{C}^*_A(L, A)$ is a comm. dg-algebra in the tensor cat. of $~~A~~[\mathcal{D}_X]$-modules (diff. not A-linear.)

$L \xrightarrow{\sigma} \Theta_A$ gives a canonical morphism of dg-algebras $DR(A) \to \mathcal{C}_A(L, A)$. DR - deRham complex.

$0 \leftarrow \phi_L \leftarrow L^\circ \leftarrow \Omega_A$ just first term of this.

Quotient is $\mathcal{C}_A(L, A) / \mathrm{Ideal}(\text{im of } \Omega_A) = \Lambda^\bullet_A \phi_L$

* - so latter is dg-alg, in deg 0 coincides with A.

* Diff is zero in deg 0 - so A linear differential

* $\Rightarrow$ dg alg$/A$ on $\Lambda^\bullet_A \phi_L \iff$ coLie structure on ϕ_L
- bracket comes from degree one differential

The Quantum Picture

Quantize coisson algebras... - purely in terms of the $\mathcal{D}_X$ scheme itself - space of functions solving some differential equations ....

Chiral operations X is a curve. $\mathcal{M}_D^r(X)$

$P^I_{ch}(\{L_i\}, M) := \mathrm{Hom}(j_*^{(I)} j^{(I)*} \boxtimes L_i, \Delta_*^{(I)}, M)$

$j^{(I)} : U^{(I)} \hookrightarrow X^I$ complement of all diagonals - config space.

$- j_* j^* \boxtimes L_i = j_*^I \mathcal{O}_{U^{(I)}} \otimes_{\mathcal{O}_{X^{(I)}}} \boxtimes L_i$ - all poles along diagonal..

Compositions : $J \twoheadrightarrow I$ $\quad \varphi \in P_I^{ch}$

$J_i \to i$

$\varphi_i \in P^{ch}_{J_i}(\{K_j\}, L_i)$

$\varphi_j (x_a - x_b)^{-1} \otimes \boxtimes (j_*^{(J_i)} \boxtimes_{J_i} K_j) \xrightarrow{\boxtimes \varphi_i} \boxtimes_I \Delta_*^{(J_i)} L_i$

where $a, b \in J$ project to different elts of I. $\quad = \Delta_*^{(J/I)}(\boxtimes L_i)$

$\otimes \mathcal{O}(\;)^{-1}$

So may compose to M. $\quad \longleftarrow \quad \Delta_*^{(J/I)}(j_*^{(I)} j^{(I)*} \boxtimes L_i)$

Definition A chiral algebra is a Lie algebra in the $\otimes$ cat $\mathcal{M}^{ch} = (\mathcal{M}_D^r(X))^{ch}$