[?]a Beilinson

Chiral algebras

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X - curve, $\mathcal{M}^{!}_{\mathcal{D}}(X)$, $\mathcal{M}^{ch}_{\mathcal{D}}(X)$

Proposition $P^{ch}(\omega_X) \cong \mathcal{L}ie$

the operad generated by ω_X in $\mathcal{M}^{ch}_{\mathcal{D}}(X)$

the free Lie operad

Proof - Construction:

$\omega_X^{\boxtimes I} \cong \omega_{X^I}$ but this isomorphism is not canonical.

To make it canonical we need to introduce the 1-dimensional vector space $\lambda_I := \det(\mathbb{C}^I)$.

Then there is a canonical isomorphism

$$\omega_X^{\boxtimes I} \cong \omega_{X^I} \otimes \lambda_I$$

Indeed, the problem with the isomorphism $\omega_X^{\boxtimes I} \cong \omega_{X^I}$ comes from the fact that we would like to send

$$dt_1 \boxtimes dt_2 \boxtimes \cdots \boxtimes dt_n \quad \text{to} \quad dt_1 \wedge \cdots \wedge dt_n$$

which means that we have to choose an order on I. This choice makes the isomorphism non-canonical.

Instead we choose a basis $e_1, \dots, e_n \in \mathbb{C}^I$ and we send

$$dt_1 \boxtimes \cdots \boxtimes dt_n \longmapsto dt_1 \wedge \cdots \wedge dt_n \wedge e_1 \wedge \cdots \wedge e_n$$

Consider now $|I| = 2$

$$p^{ch}(\omega_X)_I = \mathrm{Hom}(j_*j^*\omega_X^{\otimes I}, \Delta_*\omega_X)$$
$$\| $$
$$j_*j^*(\lambda_I\, \omega_{X^I})$$

But we have an exact sequence

$$0 \to \omega_{X^I} \to j_*j^*\omega_{X^I} \to \Delta_*\omega_X \to 0$$

$$\Rightarrow \qquad H^! \Delta^! \omega_{X^I} = \omega_X$$

$$\Rightarrow \quad p^{ch}(\omega_X)_I = \mathrm{Hom}(j_*j^*(\lambda_I\, \omega_{X^I}), \Delta_*\omega_X) = \lambda_I = \mathcal{L}ie_I$$

Claim This identification extends uniquely to an isomorphism

$$\mathcal{L}ie \xrightarrow{\sim} p^{ch}(\omega_X)$$

Existence: Cousin resolution

$$0 \to \omega_{X^I} \to C^0 \to C^1 \to \cdots$$

To describe $C^\bullet$ denote by $Q(I)$ the set of all quotient sets of I and for a $T \in Q(I)$ consider

$$\Delta^{(I/T)} : X^T \hookrightarrow X^I$$

given by $I \twoheadrightarrow T$. Then

$$C^i = \bigoplus_{\substack{T \in Q(I) \\ |T| = |I| - i}} \Delta_*^{(I/T)} j_*^{(T)} \omega_{U^{(T)}}$$

The differentials d^0 are matrices whose entries are residue maps.

where $j^{(T)} : U^{(T)} \hookrightarrow X^{(T)}$

We have $\mathcal{L}ie_2 \longrightarrow P^{ch}(\omega_X)_2$ and we want to show that it induces a morphism

$$\mathcal{L}ie \longrightarrow P^{ch}(\omega_X).$$

Consider the free operad $\widetilde{\mathcal{L}ie}$ generated by a single skew-symmetric binary relation.

Since $\widetilde{\mathcal{L}ie}$ is freely generated by $\mathcal{L}ie_2$ $\Rightarrow$ we get

$$\widetilde{\mathcal{L}ie} \longrightarrow P^{ch}(\omega_X).$$

To show that this factors through $\mathcal{L}ie$ i.e. that we have

$$\begin{tikzcd} \widetilde{\mathcal{L}ie} \arrow[r] \arrow[d, two heads] & P^{ch}(\omega_X) \\ \mathcal{L}ie \arrow[ur] & \end{tikzcd}$$

We need to show that the Jacobi $\in \widetilde{\mathcal{L}ie}$ goes to 0 in $P^{ch}(\omega_X)$.

But $C^0 = j_*^{(I)} j^{(I)*} \omega_{X^I}$, $C^2 = \Delta_*^{(I)} \omega_X$

and $d^1 \circ d^0$ is precisely the image of the Jacobi (exercise).

$\Rightarrow$ o.k.

isomorphism Special $\mathcal{D}$-module on X^I is a $\mathcal{D}$-module M that has an increasing finite filtration $W_\bullet$ s.t.

gr_i^W is a direct sum of finitely many copies of $\Delta_*^{(I/T)} \omega_{X^T}$ $T \in Q(I)$, $|T| = |I| - i$

Properties of special $\mathcal{D}$-modules

1. If M - special $\Rightarrow$ W is uniquely defined

2. The category $\mathcal{M}^{sp}(X^I)$ of all special $\mathcal{D}$-modules on X^I is abelian subcategory of $\mathcal{M}(X^I)$ closed under subquotients

3. Any morphism is strictly compatible with the W-filtration, i.e. gr^W is an exact functor (because the simple objects in $\mathcal{M}^{sp}(X^I)$ are irreducible $\Rightarrow$ there are no non-trivial hom's between them.)

Lemma $\Delta_*^{(I/T)} j_*^{(T)} \omega_{U(I)}$ is special

Proof: It suffices to consider $j_*^{(I)} \omega_{U(I)}$. Induction by $|I|$. $\square$

Lemma $\Delta_*^{(I/T)} j_*^{(T)} \omega_{U(T)}$ are injective objects in $\mathcal{M}^{sp}(X^I)$

Proof: T, $|T| = i$

N / W_{-i-1} operates on N in an exact way i.e.

$\mathrm{Hom}(N, \Delta_*^{(I/T)} \ldots) = \mathrm{Hom}(N/W_{-i-1}, \ldots)$

Next check the statement for $T = I$ i.e.,
for $j_*^{(I)} \omega_{U(I)}$.

...

$\square$

Last step: Since $C^\bullet$ has terms that are special $\mathcal{D}$-modules $\Rightarrow$ consider

$$\mathrm{Hom}(\underset{\substack{\uparrow \\ \text{special} \\ \mathcal{D}\text{-modules}}}{C^\bullet}, \underset{\substack{\uparrow \\ \text{injective special} \\ \mathcal{D}\text{-modules}}}{\Delta_*^{(I)} \omega_X})$$

$\Rightarrow$ get

$$0 \leftarrow \lambda_I \otimes p^{ch}(\omega_X)_I \leftarrow \bigoplus_{|T| = |I|-1} \lambda_T \otimes p^{ch}(\omega_X)_T \leftarrow \cdots$$

and this is an exact sequence

Using induction on $|I|$ and this sequence we see that

$$\mathcal{L}ie \longrightarrow p^{ch}(\omega_X)$$

is an isomorphism.

$\square$

$$p^{ch}(\omega_X)_I := \mathrm{Hom}(j_*^{(I)} \omega_{U(I)}, \Delta_*^{(I)} \omega_X)$$

$\wr\!\downarrow$ Riemann-Hilbert

$$\mathrm{Hom}(Rj_*^{(I)} \mathbb{C}_{U(I)}[|I|], \Delta_\bullet^{(I)} \mathbb{C}_X[1])$$

$\longleftarrow$ perverse sheaves

$\Rightarrow$

$$p^{ch}(\omega_X)_I = (\Delta^{(I)^{-1}} R^{|I|\setminus 2} j_*^{(I)} \mathbb{C}_{U(I)}, \mathbb{C}_X) =$$

$$= H_{|I|-1}(\text{configuration space of } I \text{ distinct pts in } \mathbb{C})$$

Simplest example — ω_X ..

<u>Proposition</u> The operad $P^{ch}(\omega_X) = \mathcal{L}ie$ — the Lie Lie operad.

This ω_X is canonically a chiral algebra — there's a canonical morphism from the operad $\mathcal{L}ie$ to $End(\omega_X)$!

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Two important $\otimes$ functors:

$$\mathcal{M}^{!Lie}(X) \xrightarrow{\alpha} \mathcal{M}^{ch}(X) \xrightarrow{\beta} \mathcal{M}^{*}(X)$$

On plain D-mods these are just identity...

Consider $\mathcal{M}^!(X)$ as $\otimes$, tensor it $\mathcal{M}^{!Lie}(X) := \mathcal{M}^!(X) \otimes \mathcal{L}ie$ (Lie operad) via $P_I^{!Lie}(\{L_i\}, M) = P_I^!(\{L_i\}, M) \otimes \mathcal{L}ie_I$

$= Hom(\otimes^!_I L_i, M) \otimes \mathcal{L}ie_I$

$\beta : P_I^{ch}(\{L_i\}, M) \to P_I^*(\{L_i\}, M)$ comes from obvious morphism

$\boxtimes L_i \longrightarrow j_*^{(I)} j^{(I)*} \boxtimes L_i$

α : Comes from $\mathcal{L}ie \xrightarrow{\sim} P^{ch}(\omega_X)$

There is a canonical morphism $\alpha_I : \mathcal{L}ie_I \to P_I^{ch}(\{L_i\}, \otimes^! L_i)$

for any collection L_i ..

so $Hom(\otimes^!_I L_i, M) \otimes \mathcal{L}ie_I \longrightarrow P_I^{ch}(\{L_i\}, M)$

$f \otimes \ell \longmapsto \Delta_*^{(I)}(f)\, \alpha_I(\ell)$

Define α_I : consider $(\Delta_*^{(I)} \omega_X) \otimes^! (\boxtimes L_i) = (\Delta_*^{(I)} \omega_X) \otimes_{\mathcal{O}_X} (\boxtimes L_i \omega_X^{-1}) \otimes \lambda_I$

(since $\omega_X^{\otimes I} = \omega_{X^I} \otimes \lambda_I$)

$= \Delta_*^{(I)}(\otimes^! L_i) \otimes \lambda_I$

simple extension by pairs!

Now $j_*^{(I)} j^{(I)*}(\boxtimes L_i) = (j_*^{(I)} j^{(I)*} \omega_{X^I}) \otimes^! (\boxtimes L_i)$

$= (j_* j^* \omega_X^{\boxtimes I}) \otimes^! (\boxtimes L_i) \otimes \lambda_I$.

So Lie operations $\leftrightarrow$ chiral ops. on ω, tensor with $\boxtimes L_i$ get chiral operations on $\{L_i\}$ ...

$\alpha_I(\ell) = \alpha(\ell) \otimes id_{\boxtimes L_i}$, $\alpha : \mathcal{L}ie \xrightarrow{\sim} P^{ch}(\omega_X)$

takes $j_* j^* \boxtimes L_i \longrightarrow \Delta_*^{I}(\otimes^! L_i)$.

α is faithful!

The composition $\beta_I \gamma_I = 0$ on I-operations, $|I| \geq 2$

$|I| = 2$ this is almost exact:

$$(\text{if } L_i \text{ torsion free} \xrightarrow{0})\ L_1 \boxtimes L_2 \longrightarrow j_* j^* L_1 \boxtimes L_2 \longrightarrow \Delta_*(L_1 \dot{\otimes} L_2) \longrightarrow 0$$

$$\xleftarrow[\text{Chiral}]{Res} \Delta_*(M) \quad \text{!Lie-operation}$$

– get left exact $0 \to P_S^{!Lie}(\{L_i\}, M) \to P^{ch}(\{L_i\}, M) \to P_I^*(\{L_i\}, M)$

which is onto if L_i are locally free – can extend any morphism in above exact sequence...

Comm Assoc algebras $\quad \mathrm{Com}(M^!) \xhookrightarrow{\text{fully faithful}} \mathcal{L}ie(M^{ch})$

$\xrightarrow{\sim} \mathcal{L}ie(M^{!Lie}) \nearrow_\alpha$ since α is faithful

see Lemma

$\mathrm{Com} \to \mathcal{O}_p^!(A)$

$\mathrm{Lie}^{\otimes} \to \mathcal{O}_p^{!Lie}(A)$ —— so this is a Lie algebra.

$\| \quad \nearrow$

Lie

<u>Lemma</u> Let M be any $\psi\otimes$ category. Then

$\mathrm{Com}(M) \xrightarrow{\sim} \mathcal{L}ie(M^{Lie})$ – operations are M-ops $\otimes$ Lie ops

<u>Proof</u> Binary operations are same, except transposition of coords. acts by -1 : $P_2^{M} = P_2^{M^{Lie}} \otimes \lambda_2$: so comm. binary op in $\mathrm{Com}^M \iff$ skew-com op in M^{Lie}

via $\bullet \iff [\,,\,]$: and associativity for $\bullet \iff$ Jacobi for $[\,,\,]$:

$\mathcal{L}ie_3 = \mathbb{C}[1,[2,3]] \oplus \mathbb{C}[3,[1,2]] \oplus \mathbb{C}[2,[3,1]] \,/\,$ diagonal

Now $[\;] = \bullet \otimes [1,2]$

$a\cdot(b\cdot c) \otimes [1,[2,3]] + \text{cyclic} = 0$ is Jacobi

$\quad \hat{P}_3^M \otimes \mathcal{L}ie$

$a\cdot(b\cdot c) \otimes [1,[2,3]]$

$+\, c\cdot(a\cdot b) \otimes [3,[1,2]] \quad \in P_3^M \otimes$ Diagonal

$+\, b\cdot(c\cdot a) \otimes [2,[3,1]]$

$\iff a\cdot(b\cdot c) = c\cdot(a\cdot b) = b\cdot(c\cdot a)$ in order to be on diagonal!

but we know $\bullet$ is commutative $\Rightarrow$ this implies associativity. ∎

So we now have $\mathrm{Com}(M^!) \hookrightarrow \mathcal{L}ie(M^{ch})$ is fully faithful – any morphism in the M^{ch} compatible with Lie operations corresponds to something compatible with com operations.

$\boxed{\text{Def}}$ $A \in \mathcal{L}ie(M^{ch})$. A unit in A is a morphism of ch. algebras $\omega_X \xrightarrow{i} A$

s.t. $P^{ch}(\{\omega_X, A\}, A) \xleftarrow{\sim} \mathrm{Hom}(A, A)$

$\qquad [\;]_A(i, id_A) \longleftarrow\!\!| \; Id$

M a $\mathcal{D}_X$-mod $\Rightarrow$ canonical operation $1 \in P_2^{ch}(\{\omega_X, M\}, M)$

$$0 \to \omega_X \boxtimes M \hookrightarrow j_*j^*\omega_X \boxtimes M \to \Delta_* M$$
$$\searrow \Delta_* M \nearrow 1 \qquad \to 0$$

(note $\omega_X \otimes^! M = M$ of $\mathcal{D}_X$-mod)

<u>Def</u> $A \in \mathrm{Lie}\, \mathcal{M}^{ch}(X)$. A unit in A is "1": $\omega_X \to A$ s.t.
[$\mu = \mu_A \in P_2^{ch}(\{A,A\}, A)$ the bracket] : $\mu(\text{"1"}, id_A) = 1_A \in P_2^{ch}(\{\omega_X, A\}, A)$

Ex. Show that if a unit exists, it is unique
It also follows that "1" is a morphism of chiral Lie algebras...
<u>Def</u>. A chiral algebra := a unital Lie algebra in $\mathcal{M}^{ch}$. $\Rightarrow \mathcal{C}h(X)$.

<u>Ex</u> We have functors forgetting, adding unit $\mathrm{Lie}(\mathcal{M}^{ch}(X)) \leftrightarrows \mathcal{C}h(X)$
— show these are adjoint..

Recall $\alpha: \mathcal{C}om(\mathcal{M}^!(X)) \to \mathrm{Lie}\, \mathcal{M}^{ch}(X)$, $\mathcal{D}_X$ algebras: <u>unitals</u> $\in \mathcal{C}om(\mathcal{M}^!(X))$
$\Rightarrow \alpha: \mathcal{D}_X\text{-algebras} \to \mathcal{C}h(X)$ <u>fully faithful</u>.
$\beta: \mathcal{M}^{ch}(X) \to \mathcal{M}^*(X)$
$\Rightarrow \beta: \mathrm{Lie}\,\mathcal{M}^{ch}(X) \to \mathrm{Lie}^*(X)$
$\beta: \mathcal{C}h(X) \nearrow$ — any chiral algebra is automatically
a Lie^*-algebra : $\mu: j_*j^* A \boxtimes A \to \Delta_* A$.
$$\uparrow \qquad \nearrow_{[,]}$$
$$A \boxtimes A$$

Thus $\mu \Rightarrow [,]$ bracket, but μ really plays the role of associative product.
<u>Def</u> A chiral alg. A is commutative iff $[\,]_A = 0$.
<u>Prop</u> The functor α is an equivalence between the category $\mathcal{CC}h(X)$ of commutative chiral algebras and the category of $\mathcal{D}_X$-algebras.

D. (de-)Quantization

Classical limit - suppose $A_\hbar$ is a family of chiral algebras.
Assume that A_0 is a commutative $\mathcal{D}_X$-algebra.
Then A_0 is automatically coisson :
$$\mu: j_*j^* A_\hbar \boxtimes A_\hbar \to \Delta_* A_\hbar$$
$$\uparrow \qquad \nearrow_{\hbar^{-1}[,]}$$
$$A_\hbar \boxtimes A_\hbar$$
— well defined since $[,]$ is divisible by $\hbar$!
$\Rightarrow \hbar^{-1}[,]$ is a Lie^* bracket on A_0

<u>Exercise</u>-<u>Claim</u> This makes A_0 coisson

O.P.E.

So $A_\hbar$ is a "quantization" of A_0

Operator Product Expansions

Notation Assume Y is a smooth manifold. Denote $\mathcal{D}_Y$-mod the category of arbitrary sheaves of left $\mathcal{D}_Y$ modules .. $\mathcal{M}^!_{\mathcal{D}}(Y) \subset \mathcal{D}_Y$-mod.

Assume we have a closed embedding $i: Z \hookrightarrow Y$ of smooth varieties

$\Rightarrow$ $i^*: \mathcal{D}_Y\text{-mod} \to \mathcal{D}_Z\text{ mod}$. $i^*F = \mathcal{O}_Z \otimes_{\mathcal{O}_Y} F = F/\mathcal{J}F$.

($\mathcal{J}$ the ideal of i)

Non-quasi-coh $\mathcal{D}$ modules arise first from:

$\boxed{\text{Lemma}}$ (i) i^* admits a right adjoint $\hat{i}_*: \mathcal{D}_Z\text{-mod} \to \mathcal{D}_Y\text{-mod}$

This functor is exact, fully faithful. $F \in \mathcal{D}_Y$ mod sits in $\hat{i}_*(\mathcal{D}_Z\text{-mod})$ iff $\exists F \xrightarrow{\sim} \varprojlim F/\mathcal{J}^n F$ isomorphism.

$\hat{i}_x(\mathbb{C}) = \hat{\mathcal{O}}_x := \varprojlim \mathcal{O}_x/m_x^n$ – formal completion of $\mathcal{O}$ at x – typical non-qc $\mathcal{D}_Y$-module.

(ii) For $G \in \mathcal{D}_Z$-mod, set $i_*G := (i_*\omega_Z)\,\omega_Y^{-1} \otimes_{\mathcal{O}_Y} \hat{i}_* G$

– δ-functions $\otimes$ formal series = δ-functions .... right $\mathcal{D}_Y$-mod supp'd on Z

This extends i_* on $\mathcal{M}^!_{\mathcal{D}}$ ..

Then $i_*: \mathcal{D}_Z\text{-mod} \to \mathcal{D}_Y\text{-mod}$ is exact, fully faithful & satisfies Kashiwara's lemma. – i.e. $F \in \mathcal{D}_Y$-mod is in $i_*\mathcal{D}_Z$-mod iff any local section is killed by some $\mathcal{J}^n$.

Assume Z is a divisor. Then define $\tilde{i}_*: \mathcal{D}_Z\text{-mod} \to \mathcal{D}_Y\text{-mod}$

$\tilde{i}_* G = j_*\mathcal{O}_{Y\setminus Z} \otimes_{\mathcal{O}_Y} \hat{i}_* G$

Lemma We have a canonical exact sequence

$$0 \to \hat{i}_* G \to \tilde{i}_* G \to i_* G \to 0$$

(Note $i_*\omega_Z\,\omega_Y^{-1} = j_*\mathcal{O}_{Y\setminus Z}/\mathcal{O}_Y$)

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$$0 \to \mathbb{C}[[t]] \to \mathbb{C}((t)) \to \text{"}\delta\text{-functions"} \to 0$$

$Z \xrightarrow{i} Y$

Lemma $F \in \mathcal{D}_Y$ mod, $G \in \mathcal{D}_Z$ mod

$$\mathrm{Hom}(F, \tilde{i}_* G) \longleftarrow \mathrm{Hom}(F \otimes_{\mathcal{O}_Y} j_*\mathcal{O}_{Y\setminus Z}, \tilde{i}_* G) \longrightarrow \mathrm{Hom}(F \otimes_{\mathcal{O}_Y} j_*\mathcal{O}_{Y\setminus Z}, \hat{i}_* G)$$

are both isomorphisms

Pf – first arrow clear since $\tilde{i}_* G$ is already a $j_*\mathcal{O}_{Y\setminus Z}$-module

second arrow: $F' \,\cancel{} = F \otimes j_*\mathcal{O}_{Y\setminus Z}$ has t invertible..

look at maps $F' \xrightarrow{\varphi} \hat{i}_* G = \varprojlim \hat{i}_* G / t^n \hat{i}_* G$

$\xrightarrow{\varphi_0} \hat{i}_* G$

But $t\cdot$ is invertible on F' .. $\tilde{\varphi} = \varprojlim \varphi_n$, $\varphi_n(f) = \cancel{}\; t^n \varphi_0(t^{-n} f)$

- $\tilde\varphi(f) = t^n \tilde\varphi(t^{-n} f)$ $\forall n$, now reduce mod t^n - same as reduction of $\tilde\varphi(t^{-n}f)$ mod all power series ..
So $\tilde\varphi = \lim \varphi_n = \lim t^n \varphi_0(t^{-n} f)$ . . . ■

Assume $G \in \mathcal{D}_X$ mod. $\Delta^{(I)} : X \hookrightarrow X^I$.
$\tilde\Delta_*^{(I)} G := \hat\Delta_*^{(I)} G \otimes_{\mathcal{O}_{X^I}} j_*^{(I)} \mathcal{O}_{U^{(I)}}$ - poles at arbitrary pairwise diagonals

<u>OPE operations</u>: $Q(\{F_i\}, G) := \mathrm{Hom}(\boxtimes F_i, \tilde\Delta_*^{(I)} G)$
Problem - can't compose arbitrarily, don't give category

Binary OPE operations: $F_1 \boxtimes F_2 \to \tilde\Delta_*^{2} G$, here the diagonal is a divisor : $j_*j^* F_1 \boxtimes F_2 \to \Delta_* G$, by lemma above
this gives an equivalence with lower arrow...
Assume F, G quasi-coh, pass to right D-modules $\Rightarrow$ precisely chiral operations : $Q_I(\{F_i\}, G) = P_I^{ch}(\{F_i \omega_X\}, G\omega_X) \otimes \lambda_I$
- λ_I comes in identifying $\omega_X \boxtimes \omega_X$ with ω_{X^2}.
- for $|I| \leq 2$
Commutative chiral algebras $\Longleftrightarrow$ comm assoc $\mathcal{D}_X$-alg
chiral algebra $\Longleftrightarrow$ " " OPE alg (<u>whatever this means!</u>)

<u>Composition of OPE operations</u>
$J \twoheadrightarrow I$, $X \xhookrightarrow{\Delta^{(I)}} X^I \xhookrightarrow{\Delta^{(J/I)}} X^J$
G a left D-mod on X, $\tilde\Delta_*^{(J,I)} G := \tilde\Delta_*^{(J/I)} \hat\Delta_*^{(I)} G$
where $\tilde\Delta_*^{(J/I)} : \mathcal{D}_{X^I}\text{-mod} \to \mathcal{D}_{X^J}\text{ mod}$

$\tilde\Delta_*^{(J/I)}(R) = \hat\Delta_*^{(J/I)} R \otimes j_*^{(J/I)} \mathcal{O}_{U^{(J/I)}}$ → complement to all pairwise diagonals containing X^I in X^J.

Now $\hat\Delta_*^{(J/I)} \hat\Delta_*^{(I)} G = \hat\Delta_*^{(J)} G$, no problem.
But adding poles....
on $\tilde\Delta^{(J,I)} G$ <u>any</u> equation of a diagonal is invertible
- iterated Laurent series vs. true Laurent $\tilde\Delta_*^{(J)} G \hookrightarrow \tilde\Delta_*^{(J,I)} G$.

e.g. $\{1,2,3\} \xrightarrow{3 \mapsto 2} \{1,2\}$ $\hat\Delta_*^{(J)} \mathcal{O}_X = \mathbb{C}[t_4][[t_1 - t_2, t_2 - t_3]]$
t_4 coord on X

$$\tilde\Delta_*^{(J)} \mathcal{O}_X = \hat\Delta_*^{(J)} \mathcal{O}_X\, (t_1 - t_2)^{-1} (t_2 - t_3)^{-1} (t_1 - t_3)^{-1}$$

Now $\tilde{\Delta}_X^{(I)} \mathcal{O} \rightsquigarrow \mathbb{C}[t_1][[t_2-t_1]](t_1-t_2)^{-1}$
$= \mathbb{C}[t_1]((t_1-t_2))$

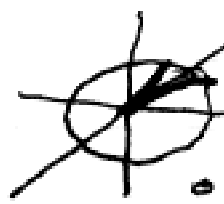
$\hat{\Delta}_X^{(J \to I)} \mathcal{O} \Rightarrow \mathbb{C}[t_1]((t_1-t_3))((t_2-t_3))$

$\mathbb{C}[t_1][[t_1-t_3, t_2-t_3]](t_1-t_2)^{-1}(t_2-t_3)^{-1}(t_1-t_3)^{-1}$

But our iterated Laurent does depend on projection $\{1,2,3\} \twoheadrightarrow \{1,2\}$

- the iterated can be written as functions on $0 < |t_2-t_3| << |t_1-t_2| << 1$
- but this depends on first joining 2,3 and then 1, (2,3).
- Taylor series sit in all of these regardless of order.

 diagonal - look at functions on small disc with poles on the diagonals

Laurent ———  defined in small punctured cone near our first diagonal

Problem with composition - lies in $\tilde{\Delta}_X^{(J,I)} G$, not $\tilde{\Delta}_X^{(J)} G$ — does depend on order of composition...

Then talk of operations which compose well - lie in $\tilde{\Delta}_X^{(J)} G$

— notion of associativity

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$J \overset{\psi_i}{\twoheadrightarrow} I_\varphi$ $\quad \varphi, \psi_i$ compose nicely if $\varphi(\psi_i) \in \hat{\Delta}_X^{(J,I)}(G)$ is actually in $\hat{\Delta}_X^{(J)}(G)$.

Def A comm, assoc OPE algebra is a left $\mathcal{D}$-mod A on X together with a binary OPE operation $\bullet \in \mathcal{O}_2(\{A,A\}, A)$ st. $\bullet$ is commutative and $\bullet(id_A, \bullet)$ compose nicely & associativity holds for it. (S_3 symmetry)

- $\varphi_1 \bullet \varphi_2 \bullet \varphi_3 \in \tilde{\Delta}_X^{(3)} A$ if compose nicely
  $\varphi_1 \bullet (\varphi_2 \bullet \varphi_3) \in A((t_2-t_3))((t_1-t_2)) \supset \tilde{\Delta}_X^{(3)} A$

Prop Chiral algebras $=$ comm. assoc OPE algebras

Pf $(B, \mu) \longmapsto (B\omega_X^{-1}, \bullet_\mu)$ $\quad$ μ Lie 2 $\Leftrightarrow$ $\bullet$ in OPE

- twist by λ then comm assoc.

Jacobi $\leftrightarrow$ associativity : Jacobi term $\mu(\mathrm{id}_B, \mu) \in P_3^{ch}(\{B\}, B)$

$|J|=3$, $J \twoheadrightarrow I$ $|I|=2$

– $\tilde{\Delta}_x^{(J/I)} A = \tilde{\Delta}_*^{(J/I)} \tilde{\Delta}_* A$

<u>Claim</u> $\mathrm{Hom}(A^{\otimes J}, \tilde{\Delta}_*^{(J,I)} A)^{\otimes J} \hookrightarrow P_J^{ch}(\{B\}, B)$

which sends $\bullet(\mathrm{id}_A, \bullet) \longmapsto \mu(\mathrm{id}_B, \mu)$:

This arrow is defined as follows:

$\tilde{\Delta}_x^{(J/I)}$, $\hat{\Delta}_*^{(I)}$ both embeddings of divisors, so have projection

$\hat{\Delta}_x^{(J/I)} \tilde{\Delta}_*^{(I)} A \longrightarrow \Delta_x^{(J/I)} \Delta_*^{I} A \longrightarrow \Delta_*^{(J/I)} \Delta_*^{(I)} A = \Delta_*^{(J)} A$!

– so we have a canonical projection

$\tilde{\Delta}_*^{(J,I)}(A) \longrightarrow \Delta_*^{(J)} A$

So $\mathrm{Hom}(A^{\otimes J}, \tilde{\Delta}_*^{(J,I)} A) = \mathrm{Hom}(j_*^{(J)} j^{(J)*} A^{\otimes J}, \tilde{\Delta}_*^{(J,I)} A) \longrightarrow$

$\longrightarrow \mathrm{Hom}(j_*)^* A^{\otimes J}, \Delta_*^{(J)} A)$

$A = B\omega_x^{-1}$. $\leadsto$ this last arrow is injective :

$A((t_2-t_3))((t_1-t_3)) \longrightarrow$ polar part + poles for both variables

((cousin projection)) $= \Delta_*^{(J)} A$

– push monomials to polar parts

by adding many poles $\Rightarrow$ injective.

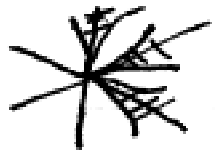
λ comes from problem of ordering of indices :

$\mu: \mathcal{L}ie_I \longrightarrow P_I^{ch}(B)$ $\leadsto$ $\mu \in P_I^{ch}(B) \otimes \lambda_I$ . . .

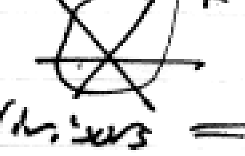
Why Jacobi $\Leftrightarrow$ assoc?

we need Jacobi $\Rightarrow$ • ~~???~~ composed . . . <u>Cousin theorem</u> :

compare

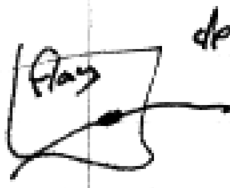


– need to extend these . . . analytic continuation



Cousin theorem (standard form) :

f function on disc with poles on these divisors $\Rightarrow$ may define singularity (polar part)

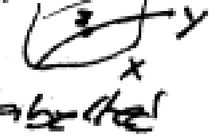


– first find principal part along the (generic point of) divisor –

localized function / regular ones : zero term $\rightarrow$ 1st term of complex.

Second step – consider polar part of result at our point $\Rightarrow$

δ-function at that point . . . with top-forms



$i_y \subset x$

$\omega_x \rightarrow$ $i_* \omega_y \rightarrow i_* \omega_z \rightarrow \ldots$ terms of Cousin complex labelled by subschemes, in a flag . . locally exact.

$$0 \to \omega_X \to C^0\omega_X \longrightarrow C^1\omega_X \longrightarrow C^2\omega_X \to \dots$$

$C^0\omega_X$ = mero forms; $C^1\omega_X = \bigoplus_{i:Y\to X \text{ divisor}} i_* C^0\omega_Y$; $C^2\omega_X = \bigoplus_{i:Y\to X \text{ codim } 2} i_* C^0\omega_Y$

Assume this stops.. does section of $C^1\omega_X$ (principal part) come from $C^0\omega_X$? locally this is equivalent to going to 0 in C^2 - differential (sum of princ. parts on codim 2) vanishes..

Now let's solve similar problem for iterated Laurent series rather than principal part: take our data, divide by arbitrary large powers $\left(\frac{1}{(t_1-t_2)(t_2-t_3)(t_1-t_3)}\right)^N$, take principal parts, should add up to zero...

i.e. $\forall$ - take principal part of Y... but same condition should be satisfied after dividing by arbitrary powers of diagonal - stronger condition

<u>Prop</u> Chals$_A$ = com assoc OPE algebras $A\omega_X^{-1}$ 11/17

$\lambda_2 \in P_2^{ch}(A) = O_2(A\omega_X^{-1})$

$\mu = [\cdot,\cdot] \longmapsto \cdot$

– Assume μ is skew com $\Leftrightarrow$ $\cdot$ comm. Then <u>Jacobi $\Longleftrightarrow$ assoc</u>:

J, $|J|=3$. X^J stratified by diagonals. Cousin: (δ-fns along diagonal)

$$C^\bullet = \{0 \to \omega_X^J \to j_*^{(I)}\omega_{U(J)} \longrightarrow \bigoplus_{\substack{J\twoheadrightarrow I\\ |I|=2}} \Delta_*^{(J/I)}\omega_{U(I)} \to \Delta_*^{(J)}\omega_X \to 0\}$$

Consider $\mathrm{Hom}_{O_X^J}(C^\bullet, \Delta_*^{(J)}A)$:

$$0 \leftarrow \Delta_*^{(J)}(A\omega_X^{-1}) \leftarrow T \leftarrow \bigoplus_I \hat\Delta_*^{(J/I)}\hat\Delta_*(A\omega_X^{-1}) \leftarrow \hat\Delta_*^{(J)}(A\omega_X^{-1}) \leftarrow 0$$

pushforward for left Dmods!

Why is this: divide ~~each~~ $\Delta_*^{(J)}A$ by various terms:

Dual to $\Delta_*^{(J)}\omega_X$: diff forms on diagonal extended by δ-function

— dual to formal local rings at that point by residues

— $\mathrm{Hom}(\hat O^*, \Delta_* A)$

Again: $\mathrm{Hom}(\Delta_*^{(J)}\omega_X, \Delta_*^{(J)}A) \xrightarrow{\varphi} \Delta_*^{(J)}(A\omega_X^{-1})$

define map φ is the same as a map $\cdot$ ~~$\mathrm{Hom}() \Rightarrow A\omega^{-1}$~~

left hand hom is same as maps $\mathrm{Hom}() \to A\omega_X^{-1}$

$\hat{\Delta}_*$ is defined by adjunction – so same as a map
$\mathrm{Hom}(\ )\otimes \mathcal{O}_x \longrightarrow A\omega_x^{-1}$ $\quad (\hat{\mathcal{O}}_x = \mathrm{funct}(X^3)/\text{ideal of diagonal})$
- how to define this : ω_x is subsheaf of $\Delta_*^{(J)}\omega_x$.
then $\mathcal{O}$ linear morphism $\Delta_*^{(J)}\omega_x \longrightarrow \Delta_*^{(J)} A$
will send $\underset{\omega_x}{\cup} \longrightarrow \check{A}$ – part killed by diagonal
– same as desired section $\mathrm{Hom}(\ )\otimes\hat{\mathcal{O}}_x \to A\omega_x^{-1}$
But this image is complete in adic topology – made out of δ-functions of certain order.... and has same fiber as $\hat{\Delta}_* A\omega_x^{-1}$ ... so we have an isomorphism
– So maps $\mathrm{Hom}(\Delta_* A, \Delta_* B) = \hat{\Delta}_*(BA^{-1})$:
formal completion is dual to δ-functions, so endos of 1 $\Leftrightarrow$ endos of other.

For next term of Cousin : $\mathrm{Hom}(\Delta_*^{J/I}\omega_{U,I}, \Delta_* A)$
$= \hat{\Delta}_*\Delta_* A\omega_x^{-1}$:

$\Delta_*^{J/I}\omega$

– $\omega_{U,I} = j_*^I\omega$ – extend by poles to ⟶ and then transversally by δ-functions.
Crucial remark : $j : X\setminus\{x\} \hookrightarrow X$, then U
$\mathrm{Hom}_{\mathcal{O}_x}(j_*\omega_U, j_*\mathbb{C}) = \tilde{j}_*\mathbb{C} = K_x$, Laurent series at x.

$f\in K_x \mapsto (\omega \mapsto \text{polar part of } f\cdot\omega)$

<u>Next</u> : This complex is <u>exact</u> (acyclic) : $\Delta_*^{(J)}A$ is <u>injective</u> in transverse direction – injective envelope of skyscraper at diagonal, while <u>parallel</u> to diagonal $\mathbb{C}$ is projective....

Now let's modify our complex $\mathrm{Hom}(C^\cdot, \Delta_* A)$:
tensor with $j_*\mathcal{O}_U$ .. kills first term (supp on diagonal).
T stays the same – $j_*\mathcal{O}_U$ already invertible.
flat $\mathcal{O}$-module
$\Delta \longrightarrow \hat{\Delta} \Longrightarrow$ shorter complex

$$0 \leftarrow T \leftarrow \bigoplus_I \Delta_*^{(J,I)}(A\omega_x^{-1}) \leftarrow \tilde{\Delta}_*^{(J)}(A\omega_x^{-1}) \leftarrow 0$$

<u>Return to operation</u> : $\lambda_J \otimes P_J^{ch}(A) \xleftarrow{\sim} Hom((A\omega_X^{-1})^{\boxtimes J}, T)$

– this arrow comes from projection $T \to \Delta_*^J(A\omega_X^{-1})$.

but T has $j_* \mathcal{O}_U$ invertible

so $Hom((A\omega_X^{-1})^{\boxtimes J}, T) = Hom((A\omega_X^{-1})^{\boxtimes J} \otimes j_* \mathcal{O}_{U^{(J)}}, T)$

tensor by ω_J : $\lambda_J \otimes Hom(j_* j^* A^{\boxtimes J}, T\omega_J)$ $\to$ $\Delta_* A$

again! – $P_J^{ch}(A) = Hom(j_* j^* A^{\boxtimes J}, \Delta_* A)$

$$= Hom((A\omega_X^{-1})^{\boxtimes J} \otimes_{\mathcal{O}_{X^J}} j_* \omega_J, \Delta_* A) \otimes \lambda_J$$

$$= \lambda_J \otimes Hom((A\omega_X^{-1})^{\boxtimes J}, Hom(j_*\omega, \Delta_* A))$$

$$= \underline{\lambda_J \otimes Hom_{\mathcal{D}}((A\omega_X^{-1})^{\boxtimes J}, T)}$$

$\Rightarrow$ Short exact from above $0 \leftarrow T \leftarrow \oplus \leftarrow \tilde{\Delta}_*^{J} \leftarrow 0$

(in reverse order) $0 \to \mathcal{O}_J(\{A\omega_X^{-1}\}, A\omega_X^{-1}) \to \bigoplus_I Hom((A\omega_X^{-1})^{\boxtimes J}, \Delta^{(J/I)}(A\omega_X^{-1}))$

$\to Hom((\,), T) \to 0$

ternary OPE ops

Chiral ops

Compositions of binary ops

ternary ops $\Leftrightarrow$ binarys which compose nicely...

Middle term numbers our three triple products.

• is associative $\Leftrightarrow$ all three sit in image of map from ternary ops $\Leftrightarrow$ projects to zero in chiral ops $\Leftrightarrow$ sum of three terms, corresponding compositions

= <u>Jacobi identity</u>!!! ■

$Ch(X) \longrightarrow Lie^*(X)$:

$$\begin{tikzcd} j_* j^* A \boxtimes A \arrow[r, "\mu"] & \Delta_* A \\ A \boxtimes A \arrow[u] \arrow[r, "\cdot"] \arrow[ur, dashed, "{[\,]}"] & \hat{\Delta}_* A \arrow[u] \end{tikzcd}$$

So $[\,]$ is polar part of OPE :- Chiral algebra is commutative if $* \to \hat{\Delta}_* A$

$\Rightarrow$ operation $A \otimes A \to A$

$$\begin{tikzcd} j_* j^* A \boxtimes A \arrow[r, "\mu"] & \Delta_* A \\ A \boxtimes A \arrow[u] \arrow[r, "\cdot"] \arrow[ur, "{[\,]}"] \arrow[dr, dotted] & \hat{\Delta}_* A \arrow[u] \\ & \hat{\Delta}_* A \arrow[u] \end{tikzcd}$$

Some Standard Constructions w/ Chiral Algebras 12/1

(1) Tensor product: $\mathcal{A}$ chiral, L_i $\mathcal{D}$-mod's, morphisms $L_1, L_2 \xrightarrow{f_1, f_2} \mathcal{A}$ commute if

$$L_1 \boxtimes L_2 \to \mathcal{A} \boxtimes \mathcal{A} \xrightarrow{[\,,\,]} \Delta_* \mathcal{A} \quad \text{compose to zero.}$$

Assume $\mathcal{B}_i$ a finite collection of chiral algebras.

consider $\mathrm{Hom}^{com}(\{\mathcal{B}_i\}, \mathcal{A}) = \{(\varphi_i : \mathcal{B}_i \to \mathcal{A}) \ \ni \varphi_i$ commute pairwise if $\neq\}$

Prop This functor is representable by a chiral algebra $\bigotimes_I \mathcal{B}_i$

— note (φ_i) defines a morphism of $\mathcal{D}$-mod $\otimes^! \mathcal{B}_i \to \mathcal{A}$

- so $\otimes^! \mathcal{B}_i$ underlies $\otimes \mathcal{B}_i$ as a $\mathcal{D}$-mod.

We have

$$\begin{array}{ccccc} B_1 \boxtimes B_2 & & \xrightarrow{\quad 0 \quad} & & \\ \downarrow & & & & \\ j_* j^* B_1 \boxtimes B_2 & \longrightarrow & j_* j^* A \boxtimes A & \longrightarrow & \Delta_* A \\ \downarrow & & & & \\ \Delta_* B_1 \otimes^! B_2 & & \longrightarrow & & \end{array}$$

— same by Kashiwara as maps $B_1 \otimes^! B_2 \to \Delta_* A$

This is parallel to construction of $\otimes$ for (unital) assoc. algebras.

(2) Enveloping algebras: Any chiral algebra $\in \mathcal{C}h(X) \to \mathcal{L}ie^*(X)$ gives a $*$ Lie algebra.

Prop This functor admits a left adjoint $L \mapsto U(L)$ — the vacuum rep.. Any module over a chiral alg is automatically a module over the corresp. L. To L we assign a sheaf of Lie algebras.. $x \in X$, $\mathcal{A}$ chiral, $\mathcal{M}(\mathcal{A})_x$ modules supp at x.

Lemma $\mathcal{A} \to j_{x*} j_x^* \mathcal{A}$ induces an equivalence $\mathcal{M}(j_{x*} j_x^* \mathcal{A})_x \xrightarrow{\sim} \mathcal{M}(\mathcal{A})_x$

$$j_* j^* A \boxtimes M \xrightarrow{\mu^M} \Delta_* M$$

— same to localize e.g. wrt diagonal and wrt say a horizontal line through p

$$\Rightarrow \quad j_* j^* A \boxtimes M = j_* j^* ((j_{x*} j_x^* A) \boxtimes M)$$

$j_{x*} j_x^* \mathcal{A} / \mathcal{A} = i_{x*} \mathcal{A}_x$, $\mathcal{A}_x$ is a $j_{x*} j_x^* h(A)$-mod

$\quad = h(j_{x*} j_x^* A)$

U.E.A. : $U(L)_x$ is an $h(L)$ (punct. disc at x) – module.

$h(L) \to h(j_{x*} j_x^* L)$

$h(L)_x \to h(\quad)_x$

coh. of a disc — coh. of a punct disc at x.

$U(L)_x$ has a distinguished vector – image of unit in $U(L)$.

This is killed by $h(L)_x$.

$\Rightarrow$ $\boxed{\text{Claim} \quad \mathrm{Ind}_{h(\text{disc})}^{h(\text{p.disc})}(\mathbb{C}) \xrightarrow{\sim} U(L)_x}$

– in K-M case get sheaf of vacuum representations

(3) Local associative algebras

Coisson at punctured disc $\to$ poisson.

Quantize: chiral $\longrightarrow$?

Consider topology on fiber of $j_{x*} j_x^* A \supset A'$ (agree outside x)

Complete wrt this topology – we get a canonical associative algebra structure. In case of chiral enveloping algebras $\rightsquigarrow$ local completion of UEA.

$M(A)_x$ has a fiber functor $\to$ Vect – as D-mods supp at point. $\Rightarrow$ faithful functor. Call this functor ϕ_x $\Rightarrow$ topological ~~ass~~ assoc alg $\mathrm{End}(\phi_x)$

Claim $M(A)_x \xrightarrow{\sim} \mathrm{End}(\phi_x)$-mod

(as topological modules. ...)

(4) Global chiral cohomology

$A \mapsto$ vector space $H_\nabla(X, A)$ .. assume X compact for now.

$x \in X$, A_x is a module over the Lie algebra $h(A)(X \setminus \{x\})$

$H_\nabla(X, A) = A_{x\, h(A)(X \setminus x)}$ – coinvariants

In commutative case the spectrum of this is precisely space of horizontal sections..

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Hecke chiral algebra

X curve, G semisimple group $\Rightarrow$ chiral algebra on X with extra symmetry: group actions by ${}^LG$, $G(\mathcal{O}_x)$ in commuting way

$\rightarrow \mathcal{H} = \mathcal{H}_{G,X} \supset U_k(\mathfrak{g}\otimes\mathcal{O}_X)$ with K-M extension given by $-\mathrm{Tr}(\mathrm{ad}_\mathfrak{g}, \mathrm{ad}_\mathfrak{g})\cdot k$

- twisted enveloping algebra of $0 \to \omega_X \to \mathfrak{g}\widetilde{\otimes\mathcal{O}_X} \to \mathfrak{g}\otimes\mathcal{O}_X \to 0$
- identify unit (ω_X) with extension ω_X in chiral enveloping algebra

fibers are vacuum reps with negative level = 2·critical.

Consider an indscheme $\overset{F}{\underset{\downarrow}{X}} \ni x$, $F_x = G(K_x)/G(\mathcal{O}_x)$ affine Grassmannian at x.

Fact F is a $\mathcal{D}_X$-indscheme: canonical connection along X.

- over affine piece $F_x = G(X \setminus \{x\})/G(X)$ at least set-theoretically
- can move x infinitesimally, fibers don't change $\Rightarrow$ connection

Ex Canonical section $e: X \to F$ to $G(\mathcal{O}_x)/G(\mathcal{O}_x)$.

Only horizontal section ....

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G semisimple $\Rightarrow$ Lie, K-M algebra, $\Rightarrow$ sheaf of vacuum reps Vac. When the cocycle is "positive, integral" Vacuum reps is reduced $\Rightarrow$ maximal integrable quotient - much smaller. For negative weights - shouldn't project to quotient, rather embed in larger algebra $\mathrm{Vac} \hookrightarrow \mathcal{H}$

Integrable - well for Vacuum get ∞-dim coinvariants, for integrable get f.d. space of conformal blocks.

Conjecturally should also have fin dim space of $\mathcal{H}$-coinvariants!

Torus case - Vacuum is Heisenberg rep (Fock)

For group case enlarge rep of Lie alg. to rep of group - tensor by group algebra of weight lattice or so..

For twice critical level, $\mathcal{H}$ should relate to geometric Langlands. (Negative - means smaller than critical.)

Construction of $\mathcal{H}_G$: $\underset{x}{\bullet}$ X $K_x \supset \mathcal{O}_x$

$\Rightarrow$ ind scheme $P = G(K_x)/G(\mathcal{O}_x)$. $G(\mathcal{O}_x)$ orbits here are double cosets

Double cosets $\Longleftrightarrow H(K_x)/H(\mathcal{O}_x) = X(H^\vee)$ dual torus (torus)

$X(H^\vee)$ = weights of ${}^LG$, so W-orbits are same as set of highest weights of dual groups, hence isom classes of irreps

so $G(\mathcal{O}_x)$-orbits on $\mathcal{P}$ = Irr ${}^LG$

Lifts to equivalence of categories $\mathcal{P}_G \overset{\sim}{\longleftrightarrow} Rep\ {}^LG$

$\mathcal{P}_G$ - perverse sheaves ~~...~~ constant along stratification

To define this equivalence, put tensor category structure on $\mathcal{P}_G$, get fiber functor, Tannakian theory produces ${}^LG$.

The fiber functor is cohomology with coeffs in the perverse sheaf — $\oplus H^i(\mathcal{P})$

Tensor structure: <u>First definition</u> (Lusztig) - the category $\mathcal{P}_G$ is semisimple (orbits simply connected.)

$\Longleftrightarrow$ <u>$H^0 j_*(\mathbb{C}_0) = j_{!*}(\mathbb{C}_0)$</u> $\quad \mathbb{C}_0$ constant on orbit O.

$\Rightarrow$ any sheaf loc constant along orbits is canonically equivariant.

Now consider $G(K_x)$ action on $P \times P$ - in fin dim case $G(K_x)$ equiv sheaves on $P \times P \Leftrightarrow G(\mathcal{O}_x)$ equiv sheaves on P..

Here we have <u>convolution</u> of perverse sheaves $\Rightarrow$ tensor product.

$\Rightarrow$ clearly associative, <u>but not clearly commutative</u>...

<u>Second definition</u> Idea (Drinfeld): Convolution $\Leftrightarrow$ composition of Hecke correspondences.. at different points x, y these commute, take limits as $x \to y$.

We have fibration $P_x \to \underset{\downarrow}{\underline{P}}$, pass to square $P^{(2)}$
$\qquad\qquad\qquad\qquad X \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad X \times X$

$\mathcal{F}_0$ - trivialized G-torsor on curve

$P^{(2)}$ is defined as follows: first redefine P,

$P_x = \{\mathcal{F}_0 \to \mathcal{F}_1\} = \{(\mathcal{F}, s) : \mathcal{F}\ G\text{-torsor on } X,$
s a section of $\mathcal{F}$ on $X \setminus \{x\}\}$

via taking transition functions - forgetting section $\{\ldots\ s' \in \mathcal{F}(\text{neigh. of } x)$

$$P^{(2)} = \{(\underset{\in X \times X}{\underline{x,y}}), \underline{s} \in \underline{\mathcal{F}}(X \setminus \{x,y\})\}$$.

Fiber over (x,y) = isom classes of $s, \mathcal{F}$.

along diagonal get P, away from diagonal get $P \times P$. But $P^{(2)}$ is formally smooth on $X \times X$..

Given ~~two~~ perverse sheaves $\in \mathcal{P}$. take exterior product to $P \times P$, away from diagonal this is same as $P^{(2)}$,
$\downarrow$
$X \times X$
take minimal extension from complement of Δ to $P^{(2)}$ – still get perverse sheaf

$$\begin{tikzcd} (P \times P)_U = P_U \arrow[r, hook] \arrow[d] & P^{(2)} \arrow[d] & P \arrow[l, hook'] \arrow[d] \\ U \arrow[r, hook] & X \times X & X \arrow[l, hook', "\Delta"'] \end{tikzcd}$$

Set $M \otimes N = \Delta_p^* j_{!*} (M \boxtimes N)_U$ – still ~~get~~ perverse sheaf!

Now switch from $\mathcal{P}_G$ to $\mathcal{D}$-modules, P has some line bundles $\Rightarrow$ twists, twist with high enough bundle the $\mathcal{D}$-mod is acyclic (Kashiwara – Tanisaki).

sheaf-theoretic pushforward $\leftarrow$ — Take $\mathcal{D}$-mod $M \in \mathcal{P}_G$, twist $M \otimes \mathcal{L}$, $\begin{matrix} P \\ \downarrow \pi \\ X \end{matrix}$

consider $\pi_* (M \otimes \mathcal{L})$ – which is a $\mathcal{D}$-mod on X ....

($\mathcal{L} \leftrightarrow$ level of rep). Why is this a $\mathcal{D}$-mod? the bundle ~~on~~ P carries canonical connection along X as before.

$\Rightarrow$ vector fields on X act on $M \otimes \mathcal{L}$ via this connection.

$$\mathcal{H} = \bigoplus_{\substack{V \in \mathrm{Irr}\, G \\ \Updownarrow \\ M_V \in \mathcal{P}_G}} V^* \otimes \pi_* (M_V \otimes \mathcal{L}) \quad \text{– depends only on isom class of } V.$$

Claim: $\mathcal{H}$ is canonically a chiral algebra.

V trivial $\rightarrow M_V$ is δ functions transverse to the unique flat section of P, get piece in $\mathcal{H}$ corresp to vacuum rep!

Why is $\mathcal{H}$ chiral? – should be canonical ~~pair~~ chiral pairing
$M, N \rightarrow M \otimes N$!
section of M, section of $N \Rightarrow$ section of $M \boxtimes N$, restrict to U, – mero section on $X \times X$, restrict to formal neigh get Laurent series transversal to diagonal

$\mathcal{H} \leftrightarrow$ $\mathcal{D}$-module of regular rep: ~~as~~ $\bigoplus$ rep $\otimes$ rep* ... which has a tensor product : $\mathcal{H} = \pi_* (\mathrm{Reg} \otimes \mathcal{L})$,
and Reg has a ring structure