CENTRE DE RECHERCHES MATHÉMATIQUES

Luminy

A. Beilinson · Topological Epsilon Factors 6/28/06

Motivation: X smooth projective $/\mathbb{F}_q$, $\mathcal{F}$ ℓ-adic sheaf on X
Global epsilon factor (constant in functional eqn)
(Grothendieck) $\varepsilon(X, \mathcal{F}) = \det(-\mathrm{Fr}\cdot t \,;\, R\Gamma(X,\mathcal{F}))$
... monomial of degree = Euler characteristic. (again)

ε-factorization $\varepsilon(X,\mathcal{F}) = \prod_{x\in X} \varepsilon_x(\mathcal{F})_\nu$ local factors

depending only on behavior near x --- need to choose global $\neq 0$ meromorphic form on curve (or additive character of ~~...~~ field).

Geometric meaning? rewrite in terms of graded lines $\det R\Gamma(X,\mathcal{F})$:

$$\varepsilon(X,\mathcal{F}) = \mathrm{Tr}(\mathrm{Fr}\cdot t, \det R\Gamma(X,\mathcal{F}))$$

supertrace in correct degree, don't need $-$ sign change

Q: what is local factorization on level of superlines ($\mathbb{Z}$-graded)?
... $\varepsilon_x(\mathcal{F})_\nu$ $\mathbb{Z}$-graded superline with Frob action depending on 1-form ν.

– & want $\det R\Gamma(X,\mathcal{F}) = \bigotimes_{x\in X} \varepsilon_x(\mathcal{F})_\nu$

... for almost all points local factor is trivialized canonically ... part of our hypotheses (when local geometry is trivial).

What is a superline? $\otimes$ ~~...~~ commutativity constraint corrected by signs --- e.g. want $\det(C^\cdot \oplus C'^\cdot) = \det C^\cdot \otimes \det C'^\cdot$
independent of ordering, so needed if want to tensor over all points on a curve

Today explain why such a factorization is reasonable ...
in model topological s. Function

From now on X = compact real analytic manifold
$\mathcal{F}$ = constructible complex
real analytic ... used only to have canonical
class of stratifications (subanalytic).

$R\Gamma(X,\mathcal{F})$ complex with fin dim cohomology $\Rightarrow$
get graded super line $\det R\Gamma(X,\mathcal{F})$.
(degree = Euler char). would like $\det R\Gamma = \bigotimes_{x \in X} \mathcal{E}_x$.
How to compute this?

Basic example of computation:
$X = S^1$, $\nu = d\theta$ differential form,
constructible sheaf $\mathcal{F}$ on S^1 : order the
points where $\mathcal{F}$ jumps in orientation given by ν:
$\{x_\alpha\}$ = jump points, $I_\alpha = (x_{\alpha-1}, x_\alpha)$,
$I_\alpha^+ = (x_{\alpha-1}, x_\alpha]$ (again orientation from ν).

$\det R\Gamma(X,\mathcal{F})$:

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to \bigoplus_\alpha \mathcal{F}_{x_\alpha} \to 0$$

j : inclusion of complements of points

sum of constant sheaves extended by zero

sum of skyscrapers

$$(*)^{\mathcal{F}} \quad \det R\Gamma(X,\mathcal{F}) = \Big(\bigotimes_\alpha \det \mathcal{F}_{x_\alpha}\Big) \otimes \Big(\bigotimes_\alpha \det R\Gamma_c(I_\alpha, \mathcal{F}|_{I_\alpha})\Big)$$

also have $0 \to j_\alpha{}_! \mathcal{F}|_{I_\alpha} \to \mathcal{F}|_{I_\alpha^+} \to \mathcal{F}_{x_\alpha} \to 0$

so $(*) = \bigotimes_\alpha \underbrace{\det R\Gamma_c (I_\alpha^+, \mathcal{F}|_{I_\alpha^+})}_{\mathcal{E}_{x_\alpha}(\mathcal{F})_{d\theta}} = \bigotimes_{x \in X} \mathcal{E}$

($\mathcal{E} = \mathbb{C}$ where sheaf is smooth) $\qquad$ epsilon factor

... behaves microlocally: depends only on microlocal behavior in x_α in $d\theta$ codirection

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<u>General situation</u> To each $\mathcal{F}$ assign singular support $SS(\mathcal{F}) \subset T^*X$ conical Lagrangian subvariety.

Characteristic cycle $CC(\mathcal{F})$ $\mathbb{Z}$-cycle supported on $SS(\mathcal{F})$ of dim $=$ dim X: sum of components of $SS(\mathcal{F})$

- case of local system: $SS(\mathcal{F}) = X \subset T^*X$
  $CC(\mathcal{F}) = \text{rk}\, \mathcal{F} \cdot X$
- case of skyscraper: $SS(\mathcal{F}) = T^*_x X$, $CC(\mathcal{F}) = \text{rk} \cdot T^*_x X$.

Dubson-Kashiwara formula:
$$\chi(X, \mathcal{F}) = \langle CC(\mathcal{F}), T^*_X X \rangle$$
$\to$ 0-section (conormal to X)

... euler char. given by intersection number.
For $\mathcal{F}$ local system just get rank times self intersection of 0-section $\Leftrightarrow$ of diagonal in $X \times X$.

... essentially obvious in $\mathcal{D}$-module theory, not obvious in topology!

Now <u>animate</u>: gradual hyperplanes form Picard groupoid

... $\mathbb{Z}$ = iso classes of objects here in this Picard groupoid. so want to animate RHS
... lift cc to cycle valued in Picard groupoid of super graded lines.

Generalize further to make obvious.

Consider K-theory spectrum of coefficients
... consider constructible sheaves with coeffs in any associative or dg algebra R,
& assume $\mathcal{F}$ is perfect

$\Rightarrow R\Gamma(X,\mathcal{F})$ perfect complex $\Rightarrow$
$[R\Gamma(X,\mathcal{F})] \in K(R)$ homotopy point of K-theory spectrum .... defined only up to homotopies.

On level of π_0 get Euler characteristic
det $R\Gamma$ comes from looking a little deeper:
(det ... must assume $\boxed{R \text{ commutative}}$)

Any spectrum K defines Picard groupoid
ΠK Poincaré groupoid is a Picard groupoid

... K is infinite loop space $\Rightarrow$ get addition, comm & assoc constraints come from delooping, (at least twice)

ΠK remembers π_0, π_1 :

ΠK carries the 1-truncated spectrum
Picard groupoids $\longleftrightarrow$ spectra with two adjacent homotopy groups

$K(R)$ K-theory spectrum $\rightsquigarrow \Pi K(R)$. Picard groupoid

$\xrightarrow{\det} \mathcal{L}_R$ Picard groupoid of graded R-superlines

So for any perfect complex get a lifts of its determinant line to homotopy point of $K(R)$.

So now $K = K(R)$

$K(X) = K(P(X,R))$ K-theory of perfect complexes on X.

have morphism (cohomology) $K(X) \xrightarrow{R\Gamma} K$

$[F] \longmapsto [R\Gamma(X,F)]$

(1) For topological space X, spectrum K can take cohomology of X with K coefficients $C(X,K)$, which is itself a spectrum -- eg for EM space $H(R)$ this is ~~KR~~ $C^\bullet(X,R)$ singular cohomology).

- generally $K = (K_i, \Sigma K_i \to K_{i+1})$
  $\Rightarrow C(X,K) = (X \wedge K_i, \ldots)$
- Functorial in X, K.

<u>Claim</u> $R\Gamma$ can be decomposed as

$$\begin{tikzcd} K(X) \arrow[r, "\varepsilon"] \arrow[rr, bend right, "{[R\Gamma]}"'] & C(X,K) \arrow[r, "tr"] & K \end{tikzcd}$$

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Definition of $\mathcal{E}$: use fact that $K(X)$ is <u>additive</u> wrt partitions of X, $X = \coprod X_\alpha$, with X_α subanalytic & <u>locally closed</u>.

$\Rightarrow$ get map $K(X) \xleftarrow[\sim]{(i_\alpha !)} \prod K(X_\alpha)$

for sheaf on X_α consider extensions by zero from each X_α.

Quillen $\Rightarrow$ $(i_\alpha !)$ is a homotopy equivalence.

Suppose also each $X_\alpha \subset Y_\alpha$ <u>contractible</u> subset (just make strata small enough)

Invert this arrow get

$$\begin{tikzcd}
K(X) \arrow[drr, "\mathcal{E}"'] & & \prod_\alpha K(X_\alpha) \arrow[ll, "(i_\alpha !)"', "\sim"] \arrow[rr, "R\Gamma_c"] & & \prod_\alpha K \\
& & C(X, K) & & \prod_\alpha (C(Y_\alpha, K)) \arrow[ll] \arrow[u, "s"', "tr"]
\end{tikzcd}$$

we choose X_α, Y_α but these data are directed & have canonical homotopies.....

.... universal home for local additive constructions?

Restrict to strata, take trace locally on X_α, put it on any point of contractible sets Y_α's, & then add it up:

$$\sum x_\alpha \cdot [R\Gamma_c(X_\alpha, F|_{X_\alpha})]$$

Microlocalization of $\mathcal{E}$

localize "singular homology" $C(X,k)$:
-- same as cohomology with coefficients in dualizing complex; defined as follows:

$$X \supset U \longmapsto \text{relative homology } \operatorname{Cone}(C(X\setminus U,k) \to C(X,k))$$

get presheaf of spectra $K^!(U)$.
This is flabby, so $K^!(U) \xrightarrow{\sim} R\Gamma(U,K^!)$

$U=X$ case: $C(X,k) \xrightarrow{\sim} R\Gamma(X,K^!)$
$$\big\downarrow s$$
$p: T^*X \to X$. $\qquad R\Gamma(T^*X, p^*K^!)$

Let $S \subset T^*X$ be closed & conical, $V = T^*X \setminus S$

Inside of category of complexes have dg subcategory of complexes with microsupport on S:

$P(X) \supset P(X)_S$. Thick subcategory, & quotient is the category of microlocal sheaves on V.

$$\Rightarrow \begin{tikzcd} K(X)_S \arrow[r] \arrow[drr, dashed, "\sim 0"] & K(X) \arrow[d, "\mathcal{E}"] & \\ & R\Gamma(T^*X, p^*K^!) \arrow[r, "\text{Restrict}"] & R\Gamma(V, p^*K^!) \end{tikzcd}$$

Claim composition is canonically homotopic to zero:
$$K(X)_S \xrightarrow{0} R\Gamma(V, p^*K^!)$$

K theory: cone of $K(X)_S \to K(X)$ is K theory of quotient category, $K^m(V)$

$$\begin{tikzcd}
K(X)_S \arrow[r] \arrow[d, "\varepsilon_S"] & K(X) \arrow[r, "\text{take cones}"] \arrow[d, "\varepsilon"] & K^\mu(V) \arrow[d, "\varepsilon^\mu"] \\
R\Gamma_S(T^*X, p^*K^!) \arrow[r] & R\Gamma(T^*X, p^*K^!) \arrow[r] & R\Gamma(V, p^*K^!)
\end{tikzcd}$$

Take cones: $K(X)_S \to R\Gamma(V, p^*K^!)$ homotopic to zero $\Rightarrow$ get commutative diagram as above, defining microlocalization of ε, ε^μ.

ε_S : animation of characteristic cycle:

replacing K by $\mathbb{Z}$ then $R\Gamma_S(T^*X, p^*K^!)$ becomes just cycles • supported on S, get usual characteristic cycle. ∴ $p^*\mathbb{Z}^!$ will take care of all orientation problems.

<u>ε - factorization</u> Suppose $U \subset X$ & ν 1-form on U with $\nu(U) \subset V$ doesn't intersect our singular support $S = SS(\mathcal{F})$ $V = T^*X \setminus S$.

$$\begin{tikzcd}
K(X)_S \arrow[r] & K(X) \arrow[d, "\varepsilon"] & \\
& C(X,K) \arrow[r] & \mathrm{Cone}(C(X \setminus U, K) \to C(X,K)) = K^!(U)
\end{tikzcd}$$

ν defines canonical homotopy to zero of composition $K(X)_S \to K^!(U)$: use ν to pull back to $R\Gamma(V, p^*K^!) \xrightarrow{\nu^*} R\Gamma(U, K^!)$

So just as before can complete to

$$\begin{tikzcd} K(X)_s \arrow[r] \arrow[d, "\varepsilon_v"] & K(X) \arrow[d, "\varepsilon"] & \\ C(X \setminus U, K^\bullet) \arrow[r] & C(X, k) \arrow[r] & K^!(U) \end{tikzcd}$$

~~Some~~ commutative diagram, so we get

$$[R\Gamma(X,\mathcal{F})] = \mathrm{tr} \cdot \underbrace{\varepsilon_v([\mathcal{F}])}_{\in C(X \setminus U, k)}$$

If $X \setminus U = \{x_\alpha\}$ fin many points, then

$C(X \setminus U, k) = \prod_\alpha K$, & trace is just summation

so for each x_α get ε-factor $\varepsilon(\mathcal{F})_{v, x_\alpha}$ & get factorization.

.... can now pass to det lines or Euler characteristic, get usual ε-factors.