IAS Character Sheaves & Real Groups

Note Title 2/25/2008

G reductive group $\rightsquigarrow$
finite Hecke algebra $\mathcal{H}_\alpha$
- family of monoidal (∞-) categories
parametrized by $\alpha \in T^\vee$

$\mathcal{H}_\alpha = \mathcal{D}(B \backslash_\alpha G /_\alpha B)$ (enriched version of)
derived category of $\mathcal{D}$-modules on $N \backslash G/N$
which are T-bimonodromic with generalized eigenvalue α : T-biequivariant as $\mathcal{O}$-modules with Lie T acting with generalized character $\log \alpha$
(monodromy $\alpha : \pi_1(T) \to \mathbb{C}$)

Three aspects!
- I. centers
- II. modules associated to real forms of G
- III. Langlands duality & its effect on I, II

Start with III : Langlands duality for $\mathcal{H}_\alpha$ - studied by Soergel (& Beilinson-Ginzburg) in mixed setting (Koszul duality).

Last time : approach to this via affine Hecke algebras
$$\mathcal{H}_\alpha \longleftrightarrow \mathcal{H}^\vee_\alpha \quad \text{(2-periodic version)}$$

<u>Thm</u> (Bezrukavnikov:)

$$\hat{\mathcal{H}}_\alpha = D\left(\hat{I}_\alpha \backslash LG / \hat{I}_\alpha\right) \xrightarrow[\sim]{\text{canonical monoidal}} \mathcal{O}(\mathcal{S}t^\vee_\alpha / G^\vee)$$

$$\widetilde{G}^\vee \times_{G^\vee} \widetilde{G}^\vee / G^\vee \Big|_\alpha = \left\{ (g, B_1, B_2) : \begin{array}{l} g \in B_1 \cap B_2, \\ [g] \in \widehat{\alpha, \alpha} \in T^\vee \times T^\vee \end{array} \right\} \Big/ G^\vee$$

<u>Prop.</u> $\hat{\mathcal{H}}_\alpha^{S^1, per} = \mathcal{H}_\alpha^{per}$ S^1-equivariant localization

<u>Theorem</u> $\mathcal{O}(\mathcal{S}t^\vee_\alpha / G^\vee)^{S^1, per} = \mathcal{H}^{\vee, per}_\alpha$ } nice family

$$\mathcal{H}^\vee_\alpha := D\big(\underbrace{B^\vee(\alpha) \backslash G^\vee(\alpha) / B^\vee(\alpha)}_{\text{bad family of spaces!}}\big)$$

equivariant derived category of semisimple centralizers.

- interpret formal completion of $\mathcal{S}t^\vee_\alpha / G^\vee$ in terms of small loops on $\mathcal{B}B^\vee(\alpha)$.

Real groups: Beilinson-Bernstein:

K = symmetric subgroup = G^σ

$G_{\mathbb{R}}$ = real form = $G^{\sigma \circ c}$ cartan involution

$\mathcal{D}(K \backslash G /_{\lambda} B) \cong HC(G_{\mathbb{R}})_\lambda$

Harish-Chandra modules with regular infinitesimal character $\log \lambda$

$\mathcal{H}_\lambda$ intertwining functors

Goal (Soergel conjecture): describe Langlands dual for this $\mathcal{H}_\lambda$ module. More precisely η quasisplit involution

$\{ g : g\eta(g) = 1 \} /_{\lambda} B$

$= \coprod K_i \backslash G /_{\lambda} B \rightsquigarrow$ Harish-Chandra modules for real forms in inner class of η

(see all real forms this way)

Belief: this $\mathcal{H}_\lambda$ module determined canonically in terms of character theory χ_λ

$\Longrightarrow$ Langlands dual description

[BZ-Francis-Nadler]

<u>Centers</u> $\mathcal{C}, *$ monoidal category

$\Rightarrow$ Drinfeld center $Z(\mathcal{C})$ braided tensor category : $Z(\mathcal{C}) = \{ M \in \mathcal{C} + M * \_\_ \xrightarrow{\sim} \_\_ * M \}$

<u>example</u>: G finite group, $Coh\, G$ = vector bundles on G (G-graded vector spaces) with convolution (group algebra of G in Vect) $= \mathbb{C}[G]$-mod

$\Rightarrow$ $Z(Shv_G, *) = Coh(G/G)$ G-equivariant sheaves

$=$ modules for Drinfeld double $\mathbb{C}[G] \rtimes \mathbb{C}G$

<u>example</u> : $Rep\, G, \otimes = Coh(BG), \otimes$

$Z(Rep\, G) = \underbrace{Coh\left(\frac{G}{G}\right)}$ inertia stack of BG

To apply to categories like $\mathcal{H}_k$ need notion of Drinfeld center for monoidal (cohen E_1 or A_∞) ∞-categories,

e.g monoidal dg categories : (ie include all higher homotopies)

$Z(\mathcal{C}) = Hom_{\mathcal{C} \otimes \mathcal{C}^{op}}(\mathcal{C}, \mathcal{C})$

$= End(Id_{\mathcal{C}\text{-mod}})$ Hochschild cohomology

(exact, colimit preserving functors)

"Deligne conjecture" : $Z(\mathcal{C})$ is an E_2-category
– homotopical version of braided $\otimes$ category.

We calculate $Z(\mathcal{C})$ explicitly for categories of geometric origin:

Let $\begin{matrix} X \\ \pi\downarrow \\ Y \end{matrix}$ be a quasicompact quasiseparated morphism of derived stacks with affine diagonal, & let $QC(-)$ be the stable ∞-category of quasicoherent sheaves.

$\mathcal{C} := QC(X \times_Y X)$ is monoidal for convolution

$$\begin{tikzcd} & X \times_Y X \times_Y X \arrow[dl] \arrow[d] \arrow[r] & X \times_Y X \\ X \times_Y X & X \times_Y X & \end{tikzcd}$$

<u>Theorem</u> $Z(\mathcal{C}, *) \simeq QC(\mathcal{L}Y)$

<u>Examples</u>: • $X = Y$ a [derived] scheme
$\Rightarrow Z(QC(Y), \otimes) = QC(T_Y[-1])$
interesting derived structure

• $X = Y$ a [derived] stack with affine diagonal,
eg X/G for G affine group
[or representing spectrum for tmf...]

$$\Rightarrow \quad Z(QC(Y), \otimes) = QC(\mathcal{L}Y)$$

derived inertia stack $Map(S^1, Y)$

$$= QC(Y) \otimes_{QC(Y \times Y)} QC(Y) \quad \text{Hochschild homology}$$

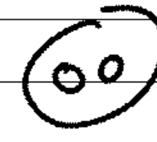
eg $Y = BG \rightsquigarrow \mathcal{L}Y = \frac{G}{G}$

[<u>Hinich</u>: underived version of this for D-M stacks with affine atlas]

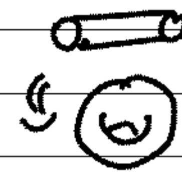
Monoidal structure: convolution over composition of loops.

Better picture: Σ topological surface with boundary

$$\begin{tikzcd} & Map(\Sigma, Y) \arrow[dl] \arrow[dr] & \\ (\mathcal{L}Y)^{\times in} & & (\mathcal{L}Y)^{out} \end{tikzcd}$$

get topological field theory operations
– in particular E_2 structure 
(cyclic Deligne conjecture in geometric case).

$$TFT_{X\to Y}: \quad \bullet \longmapsto QC(X\times_Y X)\text{-mod}$$
$$S^1 \longmapsto QC(\mathcal{L}Y)$$
$$\Sigma \longmapsto \mathcal{O}(Map(\Sigma, Y))$$

<u>Example</u> $Z(QC(St^\vee/G^\vee), *) =$ 
$= QC(\mathcal{L}(G^\vee/G^\vee))$
$= QC(Map(S^1\times S^1, G^\vee))$ commuting variety

<u>More generally</u>: $H\backslash G/H = \bullet/H \times_{\bullet/G} \bullet/H$
$\Rightarrow Z(QC(H\backslash G/H)) = QC(G/G)$
independent of H (Müger, <u>Ostrik</u> in nonderived setting for G finite)

How does $Z(QC(X \times_Y X)) = QC(\mathcal{L}Y)$ work?

$$\begin{tikzcd} & X \times_Y \mathcal{L}Y \arrow[dl] \arrow[dr] & \\ \mathcal{L}Y & & X \times_Y X \end{tikzcd}$$

$QC(X \times_Y X) = \mathrm{Fun}_Y(QC(X), QC(X))$ – sheaves are Y-linear matrices acting on X

What about $\mathcal{D}$-modules?

$\mathcal{D}(X) = QC(X^{dR})$ deRham stack – not affine diagonal!

$\mathcal{D}(\mathcal{L}Y) \rightleftarrows \mathcal{D}(X \times_Y X)$ adjoint factoring through center

Corollary: $Z(\mathcal{D}(X \times_Y X))$ is span in $\mathcal{D}(\mathcal{L}Y)$ of $\mathcal{D}(X \times_Y X)$.

<u>Example</u>

$$\begin{tikzcd} & \frac{G}{B} \arrow[dl] \arrow[dr] & \\ \frac{G}{G} & & B \backslash G / B \end{tikzcd}$$

twisted Springer correspondence: for $w \in {}_B\backslash G/_B$

this is $\{g, B : (B, gB) \text{ in rel position } w\}$

(Deligne-Lusztig)

$$\downarrow$$

$$\{g\}$$

<u>Lusztig</u>: <u>unipotent character sheaves</u> are the simple constituents of the application of this correspondence to simple perverse sheaves on ${}_B\backslash G/B$. More generally character sheaves for central character $\alpha \in T^\vee$ come from α-bimonodromic sheaves.

<u>Mirkovic-Vilonen</u>: $\iff$ $M \in D\left(\frac{G}{G}\right)$ has characteristic variety = nilpotent cone.

<u>Corollary</u> $Z(\mathcal{H}_\alpha) = \mathcal{C}har_\alpha$

character sheaves with central character α.

<u>Corollary</u> $\mathcal{C}har_\alpha \simeq \mathcal{C}har^\vee_\alpha$

character sheaves on $\dfrac{G^\vee(\alpha)}{G^\vee(\alpha)}$

(image of <u>equivariant</u> rather than <u>monodromic</u>)

Real Groups

θ quasisplit involution of G, want to describe

$\bigoplus \mathcal{D}(G^\sigma \backslash G / B)$ via Langlands dual data.

$\{\sigma \; \theta(\sigma)=1\}/_G$ $\quad \circlearrowright fl_\alpha$

- on simples this is Langlands' classification of irreducibles
- on K-groups this is Vogan's character duality
- on (mixed) categories this is Soergel's conjecture.

To describe this duality we resort to the "4d" (affine) origin of the duality:

Observation $\coprod G^\sigma \backslash G / B = (Bun_\theta)^{S'}$

$Bun_\theta = LG_{\theta,\mathbb{R}} \backslash LG / I$ = G-bundles on $\mathbb{P}^1$ with flags at $0, \infty$ & real structure with respect to $\theta \circlearrowright G$ & $\mathbb{Z} \circlearrowright \mathbb{P}^1$ antipodal

Fixed points : trivial bundle locus

What are the corresponding Langlands parameters?

$\Theta \rightsquigarrow \sigma \circlearrowright G^\vee$ holomorphic involution determined by Dynkin diagram data - or Tannakianly ...

$St^L/G^\vee = \{ g, B \;:\; g\cdot\sigma(g) \in B \}/G^\vee$
Langlands parameter space

$\simeq$ local systems on $\mathbb{P}^1$ tamely ramified at $0, \infty$ & invariant under involution σ, z

__Theorem__ $\mathrm{Coh}\left(St^L_\alpha/G^\vee\right)^{s, per} \simeq$

$$\bigoplus_{\eta\sigma(\eta)=\alpha} \mathcal{D}\left(K^\vee(\alpha,\eta) \backslash G^\vee(\alpha) / B^\vee(\alpha)\right)^{per} \circlearrowleft \mathfrak{ll}^\vee_\alpha$$

Soergel's conjecture: this is the $\mathfrak{ll}^\vee_\alpha$ module corresponding to our real module.

How to prove this?

<u>Geometric base change</u> - an instance of (geometric) Langlands functoriality.

$St^L/G^\vee = (St/G^\vee)^{\mathbb{Z}/2}$ homotopy fixed points

$\tilde{C} \xrightarrow{\Gamma} C$ covering map $\Rightarrow$ $\mathcal{L}oc_{G^\vee} C = (\mathcal{L}oc_{G^\vee} \tilde{C})^\Gamma$ homotopy fixed points.

Can describe coherent sheaves on fixed points Tannakianly (Barr-Beck):

$Coh(\mathcal{L}oc_{G^\vee} C)$ are sheaves on $\mathcal{L}oc_{G^\vee} \tilde{C}$ on which we coequalize the $Rep\, G^\vee$ actions at Γ-conjugate points

<u>Geometric base change conjecture</u>:

$\mathcal{D}(Bun_G C)$ is coequalizer of Hecke actions at Γ conjugate points of $\tilde{C}$ on $\mathcal{D}(Bun_G \tilde{C})$.

In our case $D(\mathrm{Bun}_G \tilde{\mathcal{E}}) \longleftrightarrow \hat{\mathcal{H}}$ affine Hecke (or rather completed regular bimodule):

discrete stack with understood stabilizers, as is Bun_θ & the relation between the two is under control...

so get affine Vogan duality / real Kazhdan-Lusztig!

<u>Theorem</u> $$K(Sh(\mathrm{Bun}_\theta)_\alpha)$$
$$\longleftrightarrow$$
$$K(\mathfrak{St}^L_\alpha / G^\vee) \quad \text{as Hecke-modules}$$

$\Rightarrow$ on S^1 fixed points get Vogan duality

$$\oplus K(HC(G^{\theta_i})_\alpha) \longleftrightarrow \oplus K(HC((G^\vee_\alpha)^{\theta_i}))$$