# Oxford Geometric Function Theory

Note Title 3/15/2007

The Geometric Langlands program is a <u>Fourier theory</u> for <u>sheaves</u> on <u>moduli spaces of bundles on Riemann surfaces</u>

- The Fourier transform & Pontrjagin duality

$G$ locally compact abelian group
e.g $\mathbb{R}$, $S^1$, $\mathbb{Z}$

A unitary character of $G$ is a ($\mathbb{C}$) function $\chi$ on $G$ valued in $U(1) \subset \mathbb{C}$
such that $\chi(x+y) = \chi(x)\chi(y)$

Characters form a locally compact abelian group $G^\vee$ (under pointwise product)

e.g $S^1$ : characters are $e^{inx}$, $(S^1)^\vee = \mathbb{Z}$
$\mathbb{R}$: characters are $e^{itx}$, $\mathbb{R}^\vee = \mathbb{R}$

Fourier theory: characters span functions on $G$.

- ie functions are <u>integrals</u> of characters

$$f(x) = \int_{G^\vee} \hat{f}(t)\, \chi_t(x)\, dt = \int_{\pi_1} \pi_2^*(\hat{f})\, \chi(x,t)\, dt$$

$\chi(x,t) = \chi_t(x)$

$$\begin{tikzcd}
 & G \times G^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
f(x) \quad G & & G^\vee \quad \hat{f}(t) \\
x & & t
\end{tikzcd}$$

Key feature: Fourier transform diagonalizes action of $G$ on functions:

convolution / translation / differentiation $\longrightarrow$ multiplication

$*g(x) \qquad T_\lambda \qquad \frac{d}{dx} \qquad \longmapsto \qquad \cdot\hat{g}(t),\ \cdot e^{i\lambda t},\ \cdot t$

"spectral decomposition"

More precisely: $\mathcal{F}: L^2(G) \longleftrightarrow L^2(G^\vee)$

+ many variants for different function spaces:

$$e^{\lambda x} \longleftrightarrow \delta_\lambda(t)$$

characters $\longleftrightarrow$ points

## Geometric Function Theory:

We seek a version of harmonic analysis in algebraic geometry. First obstacle: find a rich analog of function spaces.

Our objects:

- $X \subset \mathbb{A}^n$ affine algebraic variety, have $\mathcal{O}(X) = \mathbb{C}[X]$ polynomial functions on $X$, very meager from Fourier POV

- More general spaces: all obtained by gluing affine varieties in various ways

$$\underbrace{\coprod \left(U_{ij} = U_i \times_X U_j\right)}_{\text{relations}} \longrightarrow \underbrace{\coprod U_i}_{\text{affine}} \xrightarrow[\text{not nec. open}]{} X$$

- e.g. $X \subset \mathbb{P}^n$ projective.
Don't tend to find any nonconstant functions from gluing polynomials $\mathcal{O}(U_i)$

What can we assign to $X$?

- Sections of line or vector bundles (twisted functions)
- functions on subvarieties (eg on pts $x \in X$)

- notions of generalized functions share:
  - Locality: determined by their restriction to each $U_i$
  - Linearity: Can multiply by polynomial fns: On each $U_i$ restrictions form an $\mathcal{O}(U_i)$-module.

So: replace generalized functions by the $\mathcal{O}(U_i)$-modules they generate

$$f \longmapsto M_f = \{\mathcal{O}(U_i)\cdot f\}_i$$

$M_f$ is an $\underline{\mathcal{O}_X\text{-module}}$ ("coherent sheaf")

Leap: Spaces of functions $\rightsquigarrow$ Categories of modules (sheaves): have maps $\mathrm{Hom}(M_1, M_2) \in \mathrm{Vect}_{\mathbb{C}}$.

A ($\mathbb{C}$-linear) category is an <u>associative</u> (noncommutative) partially defined algebra

$$\begin{tikzcd} M_1 \arrow[r] \arrow[dr] & M_2 \arrow[d] \\ & M_3 \end{tikzcd}$$

can only compose some arrows

... e.g. loops $\mathrm{Hom}(M, M)$ is an associative unital algebra

$\rightarrow$ NC geometry!

In fact our category of $\mathcal{O}_X$-modules is just A-mod, modules over an NC "matrix" algebra made out of $\mathcal{O}(U_i)$ & the $(ij)$ gluing maps — a Connes-style substitute for functions!

<u>Modification</u>: We also need to <u>integrate</u> & no measures around ..... so really need <u>forms</u> or <u>cochains</u>, not just functions.

$\Rightarrow$ vector spaces replaced by <u>cochain complexes</u> $\to C^i \xrightarrow{d} C^{i+1} \xrightarrow{d} \cdots\cdots$

quasi-isomorphism: up to "<u>refinement</u>": if $C^\bullet \longrightarrow D^\bullet$ gives isomorphism $H^\bullet(C^\bullet) \longrightarrow H^\bullet(D^\bullet)$ then should consider it an isomorphism (comes from refinement of triangulations, or forms $\Omega^\bullet$ vs singular cochains, etc)

Thus algebras $\rightsquigarrow$ differential graded algebras

categories $\rightsquigarrow$ dg categories (= partially defined dgas)

sheaves $\rightsquigarrow$ complexes of sheaves

$$\mathcal{F}^\bullet = \{ \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \}$$

Our substitute for functions: [Day 4: some? answer from TFT]

$D(X, \mathcal{O}) =$ [dg] derived category of $\mathcal{O}$-modules

$\Rightarrow$ get theory of integration:

e.g. $\pi : X \longrightarrow pt$

$\pi_* \mathcal{F} \quad = \quad C^*(X, \mathcal{F}) \sim H^*(X, \mathcal{F})$

cochains on $X$ with coefficients in $\mathcal{F}$.

... e.g. Čech cochains $\bigoplus \mathcal{F}(U_{ij\ldots k})$

or Dolbeault cochains $\mathcal{F}\, d\bar{z}_1 \wedge \ldots d\bar{z}_k$

or simplicial cochains for triangulation or.....

[general $\pi_*$: integrate along fibers]

## Convolution

Matrix product: $Z$ finite set, $\mathbb{C}[Z] \simeq \mathbb{C}^n$

$\mathbb{C}[Z \times Z] \simeq Mat_{n \times n} = End\, \mathbb{C}[Z]$

$$\begin{tikzcd} & \overset{A}{Z \times Z} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \overset{v}{Z} & & Z \end{tikzcd}$$

$A \cdot v = \pi_{2*}(A \cdot \pi_1^* v)$

$(A \cdot v)_i = \sum_j A_{ij} v_j$

Matrix multiplication given by similar diagram:

$$\begin{tikzcd} & Z\times Z\times Z \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ Z\times Z & Z\times Z & Z\times Z \end{tikzcd}$$

$f * g = \pi_{13 *}(\pi_{12}^{*} f \cdot \pi_{23}^{*} g)$

$(f*g)_{ij} = \sum_k f_{ik}\, g_{kj}$

Can replace finite set by space of some kind, if we have a <u>measure</u>:

$\pi_{13 *} = \int_{\pi_{13}}$ integration along fiber

Operators from functions on $Z$ to functions on $Z'$ given by <u>kernel functions</u> of some kind on $Z \times Z'$.

Can also replace <u>functions</u> by any theory in which we have pullback, product & pushforward
- eg $H^*(Z)$, $K^*(Z)$ for $Z$ compact.

Künneth => $X, Y$ manifolds,

$\mathrm{Hom}(H^*(X), H^*(Y)) = H^*(X\times Y)$, $f \mapsto \int_{\pi_2} \pi_1^* f \cup c$ $\quad X \leftarrow X\times Y \rightarrow Y$, $c \in H^*(X\times Y)$

Our setting: $D(Z,0)$ has same property:

<u>Theorem</u> (Toën) Functors $D(X,0) \to D(Y,0)$ for $X, Y$ algebraic varieties given by $D(X \times Y, 0)$: any [continuous] functor has a kernel $K \to X \times Y$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

$$F \longmapsto \pi_{2*}(\pi_1^* F \otimes K)$$

# The Fourier-Mukai Transform

$A$ abelian variety:

Complex torus $\sim \mathbb{C}^g/\Lambda$ $\quad \Lambda \simeq \mathbb{Z}^{2g}$ lattice

which is also an algebraic variety

$\Longleftrightarrow$ connected projective variety with group structure

$\mu: A \times A \to A \qquad x, y \mapsto x+y$

<u>Def</u> A geometric character of $A$ (character sheaf)

is a line bundle $\mathcal{L}$ on $A$

~~$[x \mapsto \mathcal{L}_x$ complex line, varying holomorphically$]$~~

$+$ isomorphisms $\mathcal{L}_{x+y} \xrightarrow{\sim} \mathcal{L}_x \otimes \mathcal{L}_y \quad x, y \in A$

varying holomorphically, ie.

$$\mu^*\mathcal{L} \xrightarrow{\sim} \mathcal{L} \boxtimes \mathcal{L} \quad \text{on } X \times X$$

$$= \pi_1^*\mathcal{L} \otimes \pi_2^*\mathcal{L}$$

ie holomorphic homomorphism

$$A \longrightarrow \{\mathbb{C}\text{lines}, \otimes\} = B\mathbb{C}^*$$

Another POV: $x \in A$, $\mu_x : A \to A$ translation

We're asking for $\mu_x^*\mathcal{L} \simeq \mathcal{L}_x \otimes \mathcal{L}$

ie $\mathcal{L}$ is transformed by multiplication by complex line $\mathcal{L}_x$ : eigensheaf, with eigenvalue $\mathcal{L}_x$.

Proposition The characters of $A$ form the points of an abelian variety $A^\vee$, (group structure = $\otimes$) -- the dual abelian variety

$\bar{V}^* / \Lambda^* = \mathrm{Pic}^\circ A$ for $A = V/\Lambda$

$\Rightarrow$ construct analog of $e^{xt}$, the Poincaré bundle

$$\mathcal{P}|_{x,\mathcal{L}} = \mathcal{L}|_x$$

$$\begin{tikzcd}
 & \mathcal{P} \arrow[d] & \\
 & A \times A^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
A & & A^\vee
\end{tikzcd}$$

Fourier-Mukai functor:

$$\mathbb{F} : D(A^\vee, \mathcal{O}) \longrightarrow D(A, \mathcal{O})$$

$$\mathcal{F} \longmapsto \pi_{1*}(\pi_2^* \mathcal{F} \otimes \mathcal{P})$$

eg skyscraper $\mathcal{O}_{\{\mathcal{L}\}} \longmapsto \pi_{1*}(\mathcal{P}|_{\pi_2^{-1}\{\mathcal{L}\}}) = \mathcal{L}$

Theorem (Mukai) $\mathbb{F}$ is an equivalence

$\mathbb{F}^{-1} = \mathbb{F}$ up to "signs".

- ie any sheaf on $A$ is an integral of character sheaves (eigensheaves)

• Moreover if $\mathcal{F} * \mathcal{G} := \mu_*(\mathcal{F} \boxtimes \mathcal{G})$ is convolution - eg $\mathcal{O}_{\mathcal{L}} * \mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\mathcal{L} \otimes \mathcal{M}}$ -

then $$\boxed{(\mathcal{F} * \mathcal{G})^\vee = \mathcal{F}^\vee \otimes \mathcal{G}^\vee}$$

# $\mathcal{D}$-modules

Important variant on $\mathcal{O}$-modules:
look at functions + their derivatives together!

$\mathcal{D}_X$ = associative algebra generated
by alg. functions $\mathcal{O}_X$ and vector fields $T_X$
with relations
- $\partial f - f \partial = f'$
- $\partial_1 \partial_2 - \partial_2 \partial_1 = [\partial_1, \partial_2]$

e.g. $\mathcal{D}(\mathbb{A}^1)$ = Weyl algebra $\mathbb{C}\langle x, \partial_x \rangle / \partial x - x \partial = 1$

What are $\mathcal{D}$-modules?

- Generalized functions:

$f \longmapsto \mathcal{D} \cdot f \subset \mathrm{Fun}\, X$ (left) $\mathcal{D}$-module.

examples:
- $\mathcal{D} e^{\lambda x} \simeq \mathcal{D} / \mathcal{D}(\partial_x - \lambda)$

convert $\partial$ to $\lambda \Rightarrow \mathcal{D} e^{\lambda x} = \mathbb{C}[x] . e^{\lambda x}$.

The diff eq determines $e^{\lambda x}$ up to scalar
- great algebraic substitute. [holonomic]

- $\mathcal{D}\cdot\delta_\lambda \subset$ distributions :

  $\rightsquigarrow \mathcal{D}/\mathcal{D}(x-\lambda) \cong \mathbb{C}[\partial]\cdot\delta_\lambda$ (holonomic)

  So we have exponentials & $\delta$-functions!

$X = \mathbb{A}^1$ $\quad \mathcal{D} =$ Weyl algebra has involution

$x \longmapsto \partial_x \ , \quad \partial_x \longmapsto -x$

$\Rightarrow \quad \mathcal{D}_{\mathbb{A}^1}\text{-mod} \xleftrightarrow{\ \mathbb{F}\ } \mathcal{D}_{\mathbb{A}^1}\text{-mod}$ Geometric Fourier transform

"$e^{\lambda x}$" $\longleftrightarrow$ "$\delta_\lambda(t)$"

Exponentials $\leftrightarrow$ pts.

Can write any $\mathcal{D}$-module "in this basis"

In fact concretely:

$$\begin{tikzcd} & \mathbb{A}^1\times\mathbb{A}^1 \ \ \mathcal{P} = \text{"}e^{ixt}\text{"} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{A}^1 & & \mathbb{A}^1 \end{tikzcd}$$

$$M \rightsquigarrow \mathbb{F}(M) = \text{"}\int M(x)\, e^{ixt}\, dt\text{"}$$

$$= \pi_{2*}(\pi_1^* M \otimes \mathcal{P})$$

- Sheaves on quantized cotangent bundle:

Throw $\hbar$ into relations!

$$\mathcal{D}_\hbar = \langle \mathcal{O}, T \rangle / \begin{array}{l} \partial f - f\partial = \hbar f' \\ \partial_1\partial_2 - \partial_2\partial_1 = \hbar[\partial_1, \partial_2] \end{array}$$

For $\hbar \neq 0$ can rescale away

For $\hbar = 0$ get commutative algebra

$\mathcal{D}_0 = \mathrm{Sym}\, T = \mathcal{O}_{T^*X}$ functions on cotangent bundle (symbols of diff ops)

So $\mathcal{D}$ deforms nicely (flatly) to $\mathcal{O}_{T^*X}$

$\mathcal{D}$-modules are a deformed ("quantized") version of $\mathcal{O}$-modules on $T^*X$.

[Heisenberg uncertainty: can't pin down both $X$ & $T^*$ directions completely.. best we can do is holonomic $\mathcal{D}$-modules: its shadow on $T^*X$ is Lagrangian]

- Sheaves with flat connection:

E vector bundle with flat connection

$\Rightarrow$ can act on sections by functions ($\mathcal{O}$-module) & by vector fields: covariant derivatives of sections.

Flatness $\Longleftrightarrow$ relations in $\mathcal{D}$ are satisfied!

$\mathcal{O}$-mod : vector bundles :: $\mathcal{D}$-modules : flat vector bundles

Typical $\mathcal{D}$-modules built out of flat vector bundles on subvarieties (throw in derivatives in normal directions for free), glued together. eg $x \in X$ $\delta_x$
$\mathcal{D}$-module generated by skyscraper at a point

- Representations of Lie algebras : <u>Lecture 5</u>

$G$ Lie group, $\mathfrak{g} = \operatorname{Lie} G$, $U\mathfrak{g}$ enveloping alg.

$G \circlearrowright X \implies \mathfrak{g} \longrightarrow T(X)$ $\mathfrak{g}$ acts by vector fields

$U\mathfrak{g} \longrightarrow \mathcal{D}(X)$ $U\mathfrak{g}$ acts by differential operators

$M$ rep. of $\mathfrak{g} \iff U\mathfrak{g} \implies \mathcal{M} = \mathcal{D} \underset{U\mathfrak{g}}{\otimes} M$ $\mathcal{D}$-module

functor $\mathfrak{g}$-mod $\longrightarrow \mathcal{D}_X$-mod, often very rich!

Concretely : $M$ has generators $e_i$ and relations $r_i$ $\implies$

$\mathcal{M} = \bigoplus \mathcal{D} e_i / \mathcal{D} r_i$ system of diffeqs.

# Fourier - Mukai for $\mathcal{D}$-modules

$A$ abelian variety, can look for flat characters: $L$ flat line bundle, $\mu^* L \xrightarrow{\sim} L \boxtimes L$ flat.

$A^\natural$ = flat characters $\xrightarrow[\text{connection}]{\text{forget}}$ $A^\vee$ = characters
= $\mathrm{Pic}^\natural A$ = $\mathrm{Pic}^0 A$
line bundles + connection    degree zero

Fibers of map: $H^0(A, \Omega^1)$ one-forms on $A$ (difference of connections)
[all are constant coeff. $\Rightarrow$ closed]

<u>Theorem</u> (Laumon, Rothstein)
The Fourier-Mukai transform induces
$$D(A, \mathcal{D}) \simeq D(A^\natural, \mathcal{O})$$
$L$ character    $\mathcal{O}_L$ skyscraper

i.e any $\mathcal{D}$-module on $A$ is an integral of flat characters = flat line bundles.

Geometrically:

$$\begin{tikzcd} \mathcal{A} \arrow[dr] & & \mathcal{A}^\vee \arrow[dl] \\ & B & \end{tikzcd}$$

family of abelian varieties over base,

$\mathcal{A}^\vee$ = fiberwise characters $\Rightarrow$ perform Fourier fiberwise

$$D(\mathcal{A}, \mathcal{O}) \xrightarrow{\sim} D(\mathcal{A}^\vee, \mathcal{O})$$

Take $\mathcal{A} = T^*A = A \times T_0^*A \longrightarrow B = T_0^*A$

$$\mathcal{A}^\vee = A^\vee \times B = A^\vee \times H^0(A, \Omega^1)$$

$$\begin{tikzcd} D(T^*A, \mathcal{O}) \arrow[r, "\sim"] \arrow[d, squiggly, "\text{quantize}"] & D(A^\vee \times H^0(A,\Omega^1), \mathcal{O}) \arrow[d, squiggly, "\text{deform}"] \\ D(A, \mathcal{D}) \arrow[r, "\sim"] & D\left(\begin{matrix} A^\natural \\ \downarrow \\ A^\vee \end{matrix}, \mathcal{O}\right) \end{tikzcd}$$