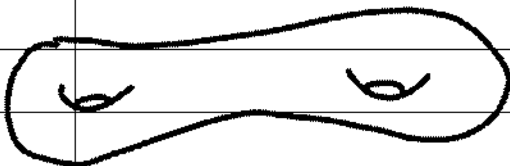
Oxford Moduli of Bundles

Note Title 3/13/2007

X compact Riemann surface

$\Longleftrightarrow$ complex projective smooth algebraic curve

connected, genus g



Picard group $\mathrm{Pic}\,X = \left\{ \begin{matrix}\text{holomorphic} \\ \text{line bundles on } X\end{matrix} \right\}$

- This is a <u>complex manifold</u> (algebraic variety)

  - because we know what a holomorphic family of line bundles on X is, & can parametrize nicely.

- $\mathrm{Pic}\,X$ is an abelian group under $\otimes$ of line bundles.

Line bundles have a <u>degree</u> $(c_1) \in \mathbb{Z}$

$$\mathrm{Pic}\,X = \mathrm{Pic}^0 X \times \mathbb{Z}$$

$$\mathrm{Jac}\,X = \mathrm{Pic}^0 X = H^0(X,\Omega^1)^* / H_1(X,\mathbb{Z}) \simeq \mathbb{C}^g / \mathbb{Z}^{2g}$$

Jacobian is a g-$\dim_{\mathbb{C}}$ torus (abelian variety)

$T^*_L \mathrm{Pic}\,X = H^0(X,\Omega^1)$ cotangent space = holomorphic forms

[follows from $T_L \mathrm{Pic} = H^{0,1}(X)$: deformations of line bundles]

Abel-Jacobi map $AJ_{x_0} : X \longrightarrow \mathrm{Jac}\, X$

$x \longmapsto \int_{x_0}^{x}$: functional on one forms up to integration on cycles $H_1(\mathbb{Z})$.

$\Rightarrow$ induces $\pi_1(X)^{ab} = H_1(X, \mathbb{Z}) \xrightarrow{\sim} \pi_1(\mathrm{Jac}\, X)$

so abelian covers of X & $\mathrm{Jac}\, X$ correspond. (geometric class field theory)

Likewise $\left\{ \begin{array}{c} \text{flat line bundles} \\ \text{on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{flat line} \\ \text{bundles on Jac} \end{array} \right\}$

- both given by monodromy maps

$\pi_1(X)^{ab} = \pi_1(\mathrm{Jac}) \longrightarrow \mathbb{C}^*$

Extend to a Fourier transform:

- $\mathrm{Jac}\, X$ is a self-dual abelian variety:

$$\mathrm{Pic}^0(\mathrm{Jac}\, X) \xrightarrow[AJ^*]{\sim} \mathrm{Pic}^0(X)$$

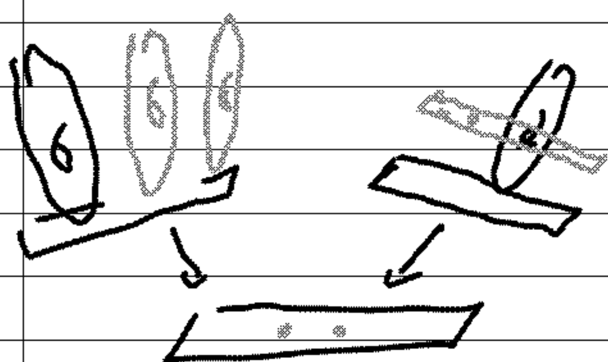
so we have a Fourier-Mukai transform

$$\mathbb{F} : \underset{\mathcal{L}}{D(\mathrm{Jac}\, X, \mathcal{O})} \xleftrightarrow{\sim} \underset{\mathcal{O}_{\{\mathcal{L}\}}}{D(\mathrm{Jac}\, X, \mathcal{O})}$$

Now pass to $T^*\mathrm{Jac} = \mathrm{Jac} \times H^0(X,\Omega^1)$ trivial torus fibration

$$\downarrow$$

$B = H^0(X,\Omega^1)$



proper, Lagrangian torus fibration

[Hitchin integrable system]

[Promo: all functions come from base
$\mathbb{C}[T^*\mathrm{Jac}] = \mathbb{C}[B] = \mathrm{Sym}(T_0\,\mathrm{Jac})$]

$$D(T^*\mathrm{Jac}, \mathcal{O}) \longrightarrow D(T^*\mathrm{Jac}, \mathcal{O})$$

trivial line bundle on fiber over ω $\longleftrightarrow$ skyscraper at $\{\text{trivial}\} \times \omega$

<u>Quantize</u>: $D(T^*\mathrm{Jac}, \mathcal{O}) \rightsquigarrow D(\mathrm{Jac}, \mathcal{D})$

noncommutative cotangent bundle

$\mathrm{Jac}^\natural = \mathrm{Conn}_{GL_1} X \longrightarrow \mathrm{Jac}\, X$: flat line bundles on X

$$D(\mathrm{Jac}\,X, \mathcal{D}) \xleftrightarrow{\ \sim\ } D(\mathrm{Conn}_1 X, \mathcal{O})$$

L $\qquad\qquad$ $\mathcal{O}_{\{L\}}$

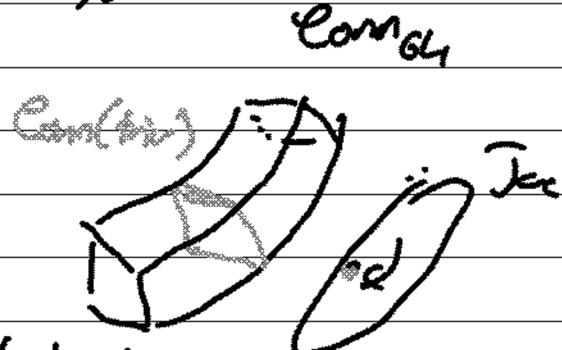
flat line bundle on Jac $\longleftrightarrow$ flat line bundle on X

Concrete description of slice of this transform:

$$D(\mathrm{Conn}_{GL_1}, \mathcal{O}) \xrightarrow{\sim} D(\mathrm{Jac}, \mathcal{D})$$

$\cup$

$\mathcal{O}$-modules on $\left\{ \begin{matrix} \text{connections on trivial} \\ \text{line bundle} \end{matrix} \right\}$



$\omega \in B = H^0(X, \Omega) \simeq$ connections on trivial bundle on X
$d + \omega$

Fourier transform on this slice is very explicit, since global differential operators $H^0(\mathrm{Jac}, \mathcal{D})$ are all <u>constant coefficient</u> $= \mathrm{Sym}\, T_0 \mathrm{Jac} = \mathbb{C}[B]$ just like their symbols $\mathbb{C}[T^*B]$:

So have $D(B, \mathcal{O}) \longrightarrow D(\mathrm{Jac}, \mathcal{D})$

$$M \longmapsto \mathcal{D} \otimes_{\mathbb{C}[B]} M$$

$$\omega \in B \longmapsto \mathcal{D} / \mathcal{D}(\partial_i - \langle \partial_i, \omega \rangle) = \mathcal{D} \otimes_{\mathbb{C}[B]} \mathcal{O}_{\{\omega\}}$$

$=$ trivial line bundle on Jac with connection $d + \omega$

quantum analog of fiber over ω

Moduli of Bundles

G reductive complex algebraic group
- eg $GL_n\mathbb{C}$, $SL_n\mathbb{C}$, $SO_n\mathbb{C}$, $Sp_n\mathbb{C}$, $Spin_n\mathbb{C}$...

A principal G-bundle on X (holomorphic) is
$P \to X$ holomorphic with simply transitive
G-action on fibers.

e.g. V holomorphic vector bundle, rk n $\Leftrightarrow$
$Fr(V)$ frames of V is a principal GL_n bundle
($\mathcal{L}$ line bundle $\leftrightarrow$ $\mathcal{L}^*$ principal $\mathbb{C}^* = GL_1$ bundle)

$Bun_G X$ = moduli space of G-bundles
- parametrizes all G-bundles on X.

eg $Bun_{GL_n} X =: Bun_n X$ moduli of vector bundles
$Bun_1 X = Pic X$.

This is an algebraic / geometric space:
have notion of <u>family</u> of G-bundles
$\Longleftrightarrow$ map into $Bun_G X$.
In fact $Bun_G X$ obtained by gluing affine
schemes $\{\amalg U_i\} \to Bun_G X$:
Can parametrize open neighborhood of any rk n bundle
as quotients of some fixed rank $N \gg n$ bundle.

- looks like a quotient of a Grassmannian by an algebraic group action.

Extreme case: $G = SL_n$, $X = \mathbb{P}^1$

Grothendieck-Birkhoff: any $V \cong \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_n)$

$\sum k_i = 0$. But space is connected:

$(\mathcal{O} \oplus \mathcal{O} \quad (\mathcal{O}(1) \oplus \mathcal{O}(-1) \quad (\mathcal{O}(2) \oplus \mathcal{O}(-2)) \subset \dots$

like $\mathbb{P}^n$/upper triangulars.

[Technically: <u>smooth</u> algebraic stack]

$\Rightarrow$ can talk about tangent & cotangent, $\mathcal{O}$-modules, $\mathcal{D}$-modules, ...: all defined by gluing <u>local</u> notions (i.e. good <u>noncommutative space</u>)

<u>Cotangents</u> $\quad P \in Bun_G X$

$$T^*_P Bun_G X = H^0(X, \mathrm{ad}\, P \otimes \Omega^1)$$

adjoint 1-forms - Higgs fields

$$T^*_V Bun_n X = H^0(X, \mathrm{End}\, V \otimes \Omega^1)$$

matrix valued 1-forms

$$\mathrm{Higgs}_G X = T^* Bun_G X \quad \text{total space symplectic}$$

$$\downarrow$$

$$Bun_G X$$

— follows from $T_P \mathrm{Bun}_G X = H^{0,1}(X, \mathrm{ad}\, P)$:
deforming a bundle $\longleftrightarrow$ adding adjoint (0,1) form
to $\bar\partial_P$ $\longleftrightarrow$ deforming transition functions
of P by infinitesimal automorphisms on
overlaps $\in H^1(X, \mathrm{ad}\, P)$.

Note that $\eta \in H^0(X, \mathrm{ad}\, P \otimes \mathcal{L})$ matrix one form
is very close to a holomorphic connection on P
$\nabla_1 + \eta = \nabla_2$ connections form affine space
for $T^* \mathrm{Bun}_G X$:

$$\begin{array}{ccc} T^*\mathrm{Bun}_G X & \rightsquigarrow & \mathrm{Conn}_G X \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G X & & \mathrm{Bun}_G X \end{array}$$

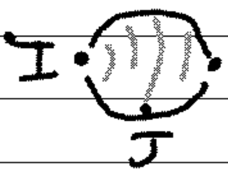
(note also all holomorphic
connections are flat $\Longrightarrow$
$\mathrm{Conn}_G X \underset{\text{analytically}}{\longleftrightarrow} \pi_1(X) \longrightarrow G$ monodromy)

Nonabelian Hodge theory: there's a good
approximation to "nice" (semistable) part
of $T^*\mathrm{Bun}_G$: $\mathcal{M}_H$ Hitchin moduli space
(solutions of Hitchin's equations
reduction of Yang-Mills equations to 2d.)

$\mathcal{M}_H$ is a hyperkähler manifold!
$\mathbb{P}^1 \ni \{I, J, K\}$ of Kähler structures.

$(\mathcal{M}_H, I) \approx T^*\mathrm{Bun}^s_G X \subset T^*\mathrm{Bun}_G X$

$(\mathcal{M}_H, J) \approx \mathcal{C}onn^s_G X \subset \mathcal{C}onn_G X$



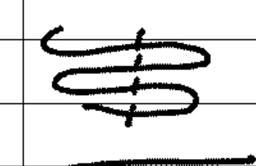
$\mathbb{C}^*$-symmetric family.

As we rescale J towards I we're rescaling the affine bundle $\mathcal{C}onn_G \to \mathrm{Bun}_G$ to the associated vector bundle $T^*\mathrm{Bun}_G \to \mathrm{Bun}_G$

Abelianization

Easiest way to construct vector bundles:

take $Y \xrightarrow[\pi]{n:1} X$ branched cover



& $\mathcal{L} \in \mathrm{Pic}\, Y$

$\Rightarrow V = \pi_* \mathcal{L} \in \mathrm{Bun}_n X$

(add up lines in fiber)

Hitchin discovered a beautiful relation between $T^* \mathrm{Bun}_G$ & abelianization.

$(V, \eta) \in T^* \mathrm{Bun}_n$ Higgs bundle

$\eta \in \mathrm{End}\, V \otimes \Omega^1$

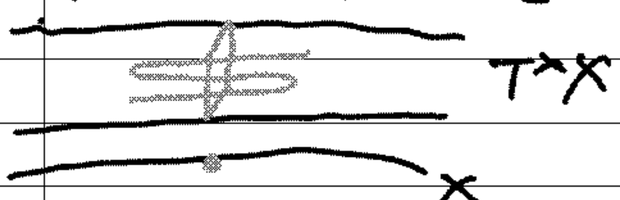
$\Leftrightarrow \quad V \longrightarrow V \otimes \Omega^1$

$\Leftrightarrow \quad T \otimes V \longrightarrow V$

$\Leftrightarrow \quad \mathrm{Sym}\, T \otimes V \longrightarrow V \quad$ (since $\dim_{\mathbb{C}} X = 1$)

$\Leftrightarrow$ lift $\mathcal{O}_X$-module V to $\mathcal{O}_{T^*X}$-module $\mathcal{V}$, $\pi_* \mathcal{V} = V$

Let $Y \subset T^*X$ be the support of $\mathcal{V}$: Riemann surface mapping $n:1$ to X



Fiber of Y over $x \in X$ = eigenvalues of matrix of one-forms η_x.

fiber of $\mathcal{V}$ at an eigenvalue = eigenspace!

Equation for $Y \subset T^*X$ $\Leftrightarrow$ characteristic polynomial of η

Hitchin system

$$\begin{array}{ccc} T^*\mathrm{Bun}_n X = \{V,\ \text{Hff } V \text{ to } T^*X\} & & \\ \downarrow & \searrow^{H} & \\ \mathrm{Bun}_n X & & B = \left\{ \begin{matrix}\text{spectral} \\ \text{curves}\end{matrix} \; Y \subset T^*X \downarrow X,\ Y \xrightarrow{n:1} X \right\} \end{array}$$

B = Hitchin base = space of characteristic polynomials

$$= H^0(X, \Omega^1) \oplus H^0(X, \Omega^{\otimes 2}) \oplus \cdots \oplus H^0(X, \Omega^{\otimes n})$$

$$\text{char } \eta = t^n - (\mathrm{tr}\,\eta)\, t^{n-1} + \cdots \cdots + (-1)^n \det \eta$$

$$B \supset B^{\mathrm{reg}} = \{ \underline{\text{smooth}}\ Y \}$$

$H^{-1}(Y) = \mathrm{Pic}\, Y$ for Y smooth

$H^{-1}(0) = \mathrm{Bun}_n X \cup$ other irred components

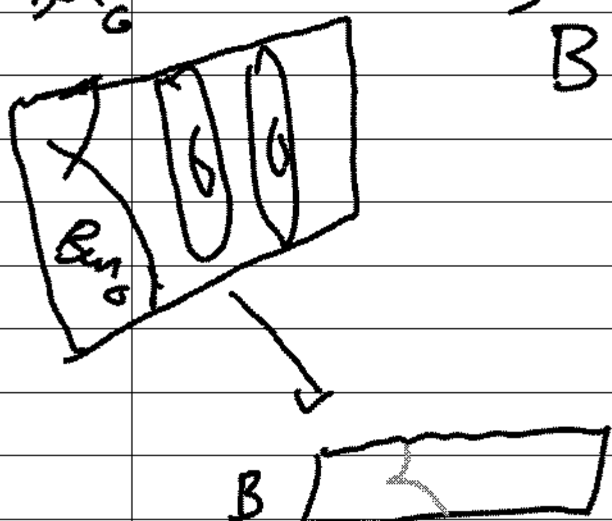
- $\dim B = \dim \mathrm{Bun}_n X$, & H is a <u>Lagrangian</u> projection (generically $\pitchfork$ to Bun_n)
- All functions on (each component of) $T^*\mathrm{Bun}_n$ come from B: $\mathbb{C}[T^*\mathrm{Bun}_n] \simeq \mathbb{C}[B]$ (polynomial ring), & they all Poisson commute (algebraically completely integrable system)

Same story for any G reductive:
replace characteristic polynomial by (basis of) invariant polynomials $\mathbb{C}[\mathfrak{g}]^G$

$$\mathcal{B} = \bigoplus_{i=1}^{\mathrm{rk}\, G} H^0(X, \Omega^{\otimes d_i})$$

$$\begin{tikzcd}
& T^*\mathrm{Bun}_G \arrow[dl] \arrow[dr] & \supset & \mathcal{A} \arrow[d] \\
\mathrm{Bun}_G & & B & \supset_{\substack{\text{open}\\ \text{dense}}} \; B^{reg}
\end{tikzcd}$$

family of abelian varieties roughly — <u>Prym</u> varieties



$\Longrightarrow$ can do Fourier-Mukai!

$$\begin{tikzcd}
T^*\mathrm{Bun}_G \supset \mathcal{A} \arrow[dr] & & \mathcal{A}^\vee \arrow[dl] \\
& \bullet &
\end{tikzcd}$$

<u>characters</u> (dual abelian varieties)

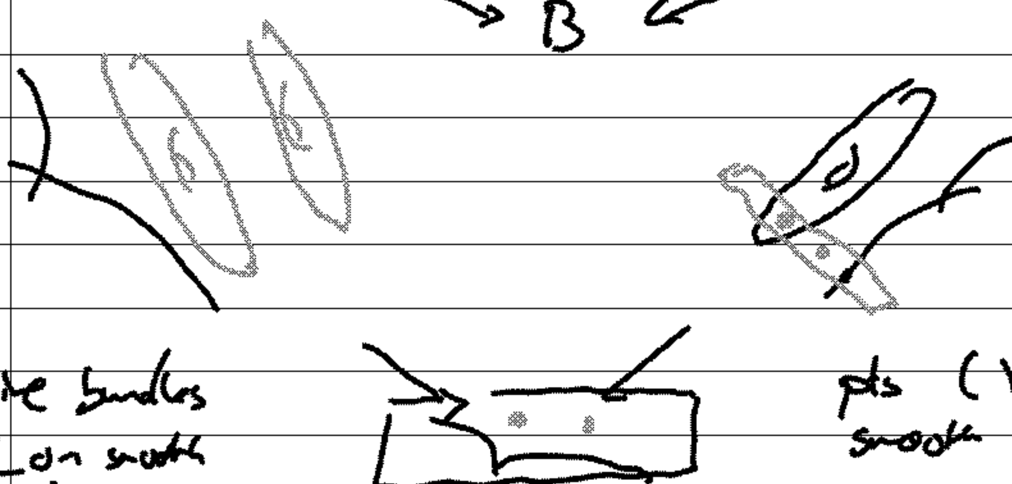
GL_n: $\mathcal{A}$ is family of Jacobians, <u>self-dual</u>
ie $\mathcal{A} = \mathcal{A}^\vee \subset T^*\mathrm{Bun}_n$.

__Theorem__ (Donagi - Pantev, following Thaddeus-Hausel, ...)

For any reductive G the dual fibration $\mathcal{A}^\vee$ (= characters on $\mathcal{A}$ over B^{reg}) is again the Hitchin fibration $\mathcal{A}_{G^\vee}$ for another reductive group, the <u>Langlands dual group</u> $G^\vee$, & we have an equivalence

$$D(\mathcal{A},\mathcal{O}) \longleftrightarrow D(\mathcal{A}^\vee,\mathcal{O})$$

$$\begin{tikzcd}
T^*\mathrm{Bun}_G \arrow[ddrr] & \supset \mathcal{A} \arrow[dr] & & \mathcal{A}^\vee \arrow[dl] & \subset T^*\mathrm{Bun}_{G^\vee} \arrow[ddll] \\
& & B^{reg} \arrow[d, phantom, sloped, "\subset"] & & \\
& & B & &
\end{tikzcd}$$



line bundles L on smooth fibers

pts $(V, \eta^\vee)$ on smooth fibers

Geometric Langlands Conjecture (rough form)

There is an equivalence

$$D(\mathrm{Bun}_G X, \mathcal{D}) \longleftrightarrow D(\mathrm{Conn}_{G^\vee} X, \mathcal{O})$$

deforming $\Uparrow$ $\Uparrow$

$$D(T^*\mathrm{Bun}_G X, \mathcal{O}) \longleftrightarrow D(T^*\mathrm{Bun}_{G^\vee} X, \mathcal{O})$$

"classical limit conjecture"

... Fourier-Mukai for $\mathcal{D}$-modules on $\mathrm{Bun}_G X$. taking skyscrapers $\mathcal{O}_L$ at a $G^\vee$-connection L to "characters"...

[Hecke eigensheaves]

Beilinson-Drinfeld:

Construct a quantization of Hitchin's hamiltonians $\Longleftrightarrow$ quantized analog of structure sheaves of Hitchin fibers:

- $B \simeq$ Fiber of $\mathrm{Conn}_{G^\vee} \longrightarrow \mathrm{Bun}_{G^\vee}$ at a particular bundle, P_{oper} (these connections are called opers)

[PSL_2 : these are projective structures on X. GL_n : correspond to n^{th} order diffops ..... KdV phase space]

- $H^0(\mathrm{Bun}_G X, \mathcal{D}') \simeq \mathbb{C}[\mathcal{B}] \simeq \mathbb{C}[T^*\mathrm{Bun}_G]$: deform all of Hitchin's hamiltonians to global differential operators (up to spin structure twist)

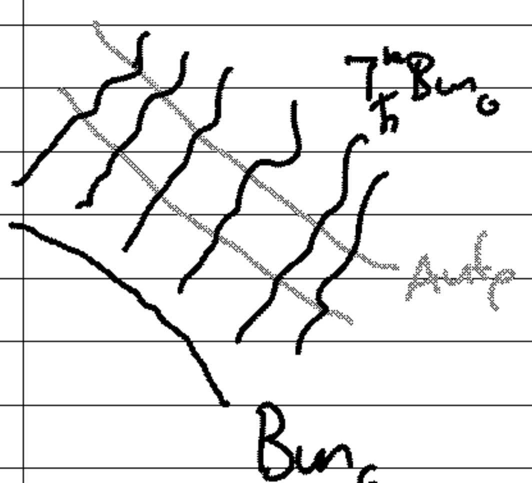
$$\Rightarrow \quad \rho \in \mathcal{B} \simeq Op_{G^\vee} X \subset Conn_{G^\vee} X$$

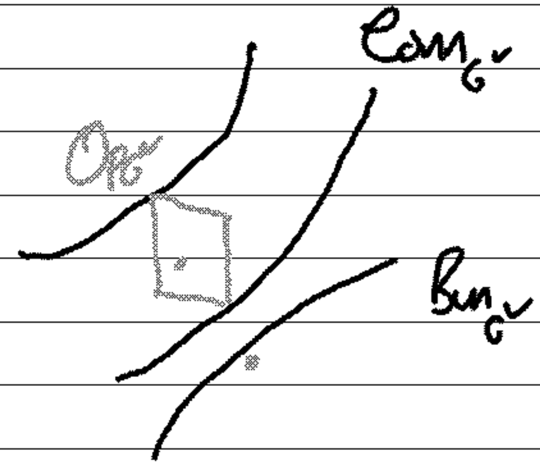
$$\longmapsto \quad Aut_\rho = \mathcal{D}_{\mathrm{Bun}_G} / \mathcal{D}(\partial_i - \langle \partial_i, \rho \rangle)$$

$$= \mathcal{D}_{\mathrm{Bun}} \otimes_{\mathbb{C}[\mathcal{B}]} \mathcal{O}_\rho$$ Hecke eigensheaf

& more generally this gives the desired transform for $M \in D(Op_{G^\vee}, \mathcal{O}) \subset D(Conn_{G^\vee}, \mathcal{O})$:

$$M \longmapsto \mathcal{D} \otimes_{\mathbb{C}[\mathcal{B}]} M$$





Main technique:
Construct $\mathcal{D}$-modules on $Bun_G X$
by Beilinson–Bernstein localization
from representations of the loop algebra
Lg (affine Kac-Moody algebra) of G.
Easy to construct $\mathcal{D}_{Bun_G}$ itself this way
– comes from the vertex algebra of Lg.

Can then import a result of Feigin–Frenkel
describing the center of this algebra
in terms of opers to describe the
decomposition of $\mathcal{D}_{Bun_G}$.

- Frenkel–Gaitsgory–Vilonen:
Construct the geometric Langlands
transform over locus $\mathrm{Conn}^{irr}_{GL_n} \subset \mathrm{Conn}_{GL_n}$
of irreducible rank n connections.