Oxford Hecke Algebras

Note Title 3/13/2007

Line Bundles & Divisors

Punctured disc $\odot$ $K_x \simeq \mathbb{C}((z))$

$\cap \qquad \cup$

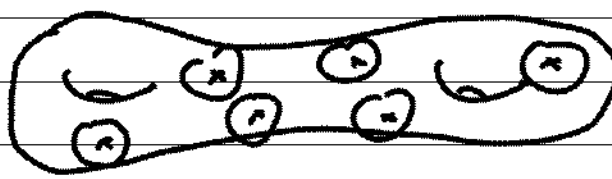
disc $\bigcirc$ $\mathcal{O}_x \simeq \mathbb{C}[[z]]$

"Local" description of $\mathrm{Pic}\, X$:

any $\mathcal{L}$ admits a meromorphic section s:

away from finitely many zeros & poles $\{x_i\}$

this trivializes $\mathcal{L}$



$$\mathrm{Div}\, s = \sum_{x_i} \mathrm{ord}_{x_i}(s) = \sum_{x\in X} \mathrm{ord}_x(s) \in \prod_{x\in X}{}' \mathbb{Z}$$

- only finitely many nonzero orders

Origin of divisor: can trivialize $\mathcal{L}$ in small disc around each x $\Rightarrow$ s gives nonzero Laurent series in each trivialization $\in K_x^* \simeq \mathbb{C}((z))\setminus 0$, almost everywhere $\in \mathcal{O}_x^* \simeq \mathbb{C}[[z]]^*$

$$\{\mathcal{L}, s, \text{local trivs}\} \leftrightarrow \prod_{x_i} K_{x_i}^* \quad \text{or} \quad \prod_{x\in X}{}' K_x^* = \mathbb{A}_X^*$$

adeles

Divisor comes by forgetting local trivializations:

change of trivialization $\in \mathcal{O}_x^*$ unit Taylor series,

$K_x^*/\mathcal{O}_x^* = \mathbb{Z}$ remember only order of pole

$$\{\mathcal{L}, s\} \in \prod{}' K^*/\mathcal{O}^* = \prod{}' \mathbb{Z} = \mathrm{Div}\, X$$

Changing section s: ratio of any two

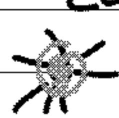
$s/s' \in \mathbb{C}(X)^*$ nonzero rational function

Div s/s' is a <u>principal divisor</u>

$$\Rightarrow \quad \mathrm{Pic}\, X = \{\mathcal{L}\} = \mathbb{C}(X)^* \backslash \mathrm{Div}\, X = \mathbb{C}(X)^* \backslash {\prod}' K_x^* / \mathcal{O}_x^*$$

$$= GL_1(\mathbb{C}(X)) \backslash GL_1(\mathbb{A}_X) / GL_1(\mathcal{O}_{\mathbb{A}})$$

General group G: seek symmetries of Bun_G

$\mathcal{P}$ G-bundle, trivialize away from finitely many $x_i \in X$ & near each $x_i \Rightarrow$ transition functions  in <u>loop group</u>

- topologically take $LG = \{S^1 \to G\}$ 
- algebraically Laurent series in G

$G(K) = G(\mathbb{C}((z)))$:

$=$ <u>invertible</u> matrices of Laurent series

Change of local trivialization $\Longleftrightarrow$

hol. map from disc to G : $L_+ G \subset LG$

or Taylor series in G $\quad G(\mathcal{O}_x) \subset G(K)$

Change of meromorphic trivialization $\Longleftrightarrow G(X \setminus \{x_i\})$
or in extreme setting $G(\mathbb{C}(X))$
meromorphic matrices.

So $\mathrm{Bun}_G X = G(\mathbb{C}(X)) \backslash \prod' G(K_x) / G(\mathcal{O}_x)$

If G is semisimple, (eg SL_n, not GL_n)
<u>or</u> we work analytically, only need <u>one</u> point:

$\mathrm{Bun}_G X = G(X \setminus x) \backslash LG / L_+ G$

Thus <u>local data</u> at x is

<u>Affine Grassmannian</u> $\mathcal{G}r = LG/L_+G = G(K_x)/G(\mathcal{O}_x)$
$= G$-bundle + trivialization on $X \setminus x$
-- generalizes $\mathbb{Z} = \mathcal{G}r_{GL_1}$.

However: $\mathbb{Z}$ is a <u>group</u> & <u>acts</u> on $\mathrm{Pic}\, X$:

$\mathcal{L} \longmapsto \mathcal{L}(x)$: add to x divisor of $\mathcal{L}$

$\Longleftrightarrow$ new sections allowed to have pole at x.

Abel-Jacobi map : $X \longrightarrow \mathrm{Pic}^1 X$
$x \longmapsto \mathcal{O}(x)$

all these points <u>generate</u> $\mathrm{Pic} = {}_{\mathbb{C}(X)^*}\backslash \prod' \mathbb{Z}$.

Hecke algebras G finite group

$$\begin{tikzcd} & G \times G \arrow[dl, "\pi_1"'] \arrow[d, "\mu"] \arrow[dr, "\pi_2"] & \\ G & G & G \end{tikzcd}$$

$\Rightarrow$ functions on G have convolution $f * g = \mu_*(\pi_1^* f \cdot \pi_2^* g)$

--- group algebra $\mathbb{C}G$.

$K \subset G$ subgroup $\Rightarrow$

Hecke algebra $\mathcal{H}_{G,K}$ = K-bi invariant fns on G

$\mathbb{C}G^{K \times K} = \mathbb{C}[K \backslash G / K] \subset \mathbb{C}G$

Why subalgebra: given by same diagram

$$\begin{tikzcd} & K \backslash G \times_K G / K \arrow[dl] \arrow[d] \arrow[dr] & \\ K \backslash G / K & K \backslash G / K & K \backslash G / K \end{tikzcd}$$

$G \times_K G = \{(g_1, g_2) \sim (g_1 k^{-1}, k g_2)\}$

$= \{(g_1, k g_2) \sim (g_1 k, g_2)\}$

What's its meaning? Q: What acts on K-invariants?

$V_{G,K} = \mathbb{C}[G/K] = \mathbb{C}G \otimes_{\mathbb{C}K} \mathbb{C}1$:

induced representation, generated by trivial K-representation

$\Rightarrow \mathrm{Hom}_G(V_{G,K}, W) = W^K$ K-invariants

(determined by image of 1, must be K-invariant)

$$\Rightarrow \quad \mathrm{End}_G(V_{G,K}) = \mathrm{Hom}_G(V_{G,K}, V_{G,K}) =$$
$$= (V_{G,K})^K = \mathbb{C}[K \backslash G/K] = \mathcal{H}_{G,K}$$

$$\boxed{\begin{array}{l}\mathcal{H}_{G,K} \text{ acts on } K\text{-invariants } W^K \\ \text{in any representation } W\end{array}}$$

Geometrically: $X \circlearrowleft G$ G acting
take quotient by K, ask what remains of G symmetry:

$$\begin{tikzcd}[column sep=small] & X \times G \arrow[dl, "\pi"'] \arrow[dr, "a"] & \\ X & & X \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd}[column sep=small] & X \times_K G/K \arrow[dl] \arrow[dr] & \\ X/K & & X/K \end{tikzcd}$$

get fiber bundle over X/K with fiber G/K & structure group K, acting on X/K

e.g. $\mathcal{H}_{G,K}$ acts on $\mathbb{C}[X/K] = \mathbb{C}[X]^K$

Also get integration $\mathbb{C}[X \times_K G] \longrightarrow \mathbb{C}[X/K]$
from $X \times_K G \longrightarrow X/K \times X/K$ $\circlearrowleft \mathcal{H}_{G,K} \circlearrowleft$
X/K

Same picture holds for any <u>theory of functions</u>
- e.g. $D(X, \mathcal{D})$ $\mathcal{D}$-modules.

Symmetry of $\mathrm{Bun}_G X = {}_{G(X\setminus x)}\backslash \prod' G(K_x)/G(O_x)$:

at each $x \in X$ have action of
$G(O_x)\backslash G(K_x)/G(O_x) = L_+G \backslash LG / L_+G$

Hecke correspondence : $x \in X$

$$\begin{tikzcd}[column sep=small] & \mathrm{Hecke}_x = \{ P_1, P_2 \;\&\; P_1|_{X\setminus x} \xrightarrow{\sim} P_2|_{X\setminus x} \} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \end{tikzcd}$$

-- fiber over P_1 = trivial is
$\{ P + \text{trivialization on } X \setminus x \} = LG/LG_+ = Gr$

-- Hecke_x is a Gr_x bundle over Bun_G
with structure group LG_+.

<u>Corollary</u> • $D({}_{LG_+}\backslash LG/LG_+, \mathcal{D}) = D_{LG_+}(Gr, \mathcal{D})$ equivariant
is monoidal (associative $*$) & acts on $D(\mathrm{Bun}_G, \mathcal{D})$
for every $x \in X$.

— remnant of loop group symmetries
of bundles with trivialization.

$$\begin{tikzcd}[column sep=small] & GL_1 : \{\mathcal{L} + \text{modifications at } x\} \simeq \mathrm{Pic} \times \mathbb{Z} \arrow[dl] \arrow[dr] & \\ \{\mathcal{L}\} = \mathrm{Pic} & & \{\mathcal{L}(nx)\} = \mathrm{Pic} \end{tikzcd}$$

$\parallel$ graph of action of $\mathbb{Z}$

Character sheaves get multiplied by a line under corresponding operators.

GL_n: instead of $\mathbb{Z}$ components find particular closed strata: elementary modifications

e.g. $0 < k < n$

$$\begin{tikzcd}[column sep=small] & \{V + \text{rank } k \text{ subspace } W \subset V_x\} \arrow[dl, "Gr_{k,n}"'] \arrow[dr] & \\ V \in Bun_n & & Bun_n \end{tikzcd}$$

$\ni$ elementary transform of V: sections are sections of V with value in W at x.

... closed LG_+ orbit $Gr_{k,n} \subset \mathcal{G}r$

... these generate Hecke correspondences for GL_n. Note: $H^*(Gr_{k,n}) \simeq \Lambda^k \mathbb{C}^n$ representation of GL_n ....

The affine Grassmannian

GL_n: $LG/LG_+ = GL_n(\mathbb{C}((z)))/GL_n(\mathbb{C}[[z]])$

$= \{$ lattices $V \subset \mathbb{C}((z))^{\oplus n}$: preserved by z & commensurable with $\mathbb{C}[[z]]^{\oplus n}\}$

Can likewise model with any kind of loops, Hilbert space Grassmannian, etc.

- union of finite dimensional orbits of LG_+: $\bigcup$ lattices further & further from H_+.

Orbits (double cosets) $LG_+ \backslash LG / LG_+$:
Laurent matrices up to invertible Taylor row & column operations

$$\rightsquigarrow \begin{pmatrix} z^{k_1} & & 0 \\ & \ddots & \\ 0 & & z^{k_n} \end{pmatrix} \qquad k_1 \le k_2 \le \dots \le k_n$$

normal form.

General $G \supset T$ maximal torus:
$\lambda \in \Lambda$ = cocharacter lattice $= \mathrm{Hom}(\mathbb{C}^*, T)$
Interpret as loop in T, get special double cosets $[\lambda] \in Gr$.

LG_+ orbits $\longleftrightarrow W\cdot[\lambda]$ Weyl group orbit

$\longleftrightarrow \Lambda/W$

But $\Lambda = \mathrm{Hom}(T^\vee, \mathbb{C}^*)$ = characters of dual torus $(= \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{C}^*)$

So $\Lambda/W \longleftrightarrow$ dominant characters of $T^\vee$

$\longleftrightarrow$ irreducible representations of group $G^\vee$ with $T^\vee$ as its maximal torus!

Langlands dual group: dualize root data of G.

$\pi_1(G^\vee) = Z(G)^\vee$

simply connected form $\longleftrightarrow$ adjoint form

G	$G^\vee$	
GL_n	GL_n	type A
SL_n	PGL_n	
Sp_n	SO_{2n+1}	types $B \longleftrightarrow C$
$Spin_{2n+1}$	$Sp_n/\mathbb{Z}/2$	
$Spin_{2n}$	$SO_{2n}/\mathbb{Z}/2$	type D

Let's look at the classical analog of $\mathcal{D}$-modules on $LG_+ \backslash Gr$: functions on Gr constant along LG_+ orbits (& compactly supported) - in fact this is the K-group $K(_{LG_+}\backslash Gr, \mathcal{D})$.

To define convolution directly: model $LG_+ \backslash LG / LG_+$ by replacing $\mathbb{C}((z)) \rightsquigarrow \mathbb{F}_q((z))$: doesn't affect this combinatorial question, but now have a nice locally compact field & can integrate!

$\Rightarrow$ spherical Hecke algebra

$$\mathbb{C}_c[LG_+ \backslash LG / LG_+]$$

<u>Satake Theorem</u> The spherical Hecke algebra is isomorphic to the representation ring <u>$\mathrm{Rep}\, G^\vee$</u> $(\simeq \mathbb{C}[T^\vee/W])$, in particular it's <u>commutative</u>!

Honest definition of $G^\vee$: Tannakian theory

Given a group H we have a category Rep H
of (f.d.) representations on $\mathbb{C}$-vector spaces.

Structures:
- $\mathbb{C}$-linear category
- monoidal category: $V, W \longmapsto V \otimes W$ (associative)
- commutativity: $\underbrace{V \otimes W \xrightarrow{\sim} W \otimes V \xrightarrow{\sim} V \otimes W}_{=}$
- unit $\underline{1}$, duals V^*
- Fiber functor: $\mathrm{Rep}\, H \longrightarrow \mathrm{Vect}$
  faithful $\otimes$ functor (H-maps $\subset \mathbb{C}$-maps)

Def A Tannakian category is a category with all above:
$\mathcal{C}, \otimes, F : \mathcal{C} \longrightarrow \mathrm{Vect}$ etc.

Tannakian reconstruction: Given $\mathcal{C} \Longrightarrow$ group H
$H = \mathrm{Aut}(F)$ natural isomorphisms $\mathcal{C} \overset{F}{\underset{F}{\rightrightarrows}} \mathrm{Vect}$
ie automorphisms of "underlying vector spaces" of every $V \in \mathcal{C}$ compatibly.

Theorem $\mathcal{C} = \mathrm{Rep}\, H$. ie
Tannakian categories $\longleftrightarrow$ groups

e.g. $\mathcal{C}$ = $\mathbb{Z}$-graded vector spaces, $\otimes$
$\Longrightarrow$ $H = \mathbb{C}^*$ multiplicative group
$\mathcal{C}$ = flat vector bundles on X,
F = take fiber at $x \in X$
$H = \pi_1(X, x)$ (or pro-algebraic completion.....)

Origin of $G^\vee$:
Geometric Satake Theorem (Mirkovic-Vilonen, following Lusztig, Drinfeld, Ginzburg)
LG_+-equivariant $\mathcal{D}$-modules on LG/LG_+ are a Tannakian category; $G^\vee :=$ Tannakian group is a reductive group with dual root data.

Example: GL_1: $Gr = \mathbb{Z}$,
$\mathcal{D}$-modules $= \mathbb{Z}$-graded vect, Tannakian group $=$
$\mathbb{C}^* = GL_1^\vee$

Consequences $D(Bun_G, \mathcal{D})$ carries lots of commuting operators!
$x \in X$ $V \in Rep\, G^\vee$ $\longmapsto$ $H_{x,V} : D(Bun_G, \mathcal{D}) \circlearrowleft$
$\Longrightarrow$ try to simultaneously diagonalize
– find analog of characters on Pic.

Spectral decomposition (Fourier transform)

Want an equivalence between "functions" on Bun_G & "functions" on some other space, so that Hecke operators $H_{x,V}$ become multiplication operators:

i.e. find space Z & bundles $W_{x,V}$ on Z s.t.

$$\begin{array}{ccc} D(\mathrm{Bun}_G, \mathcal{D}) & \simeq & D(Z, \mathcal{O}) \\ \circlearrowleft & & \circlearrowleft \\ H_{x,V} & & \otimes W_{x,V} \end{array}$$

Universal guess for such a Z is "spectrum of Hecke algebra $\bigotimes_{x \in X} \mathrm{Rep}\, G^\vee$"

= machines that turn representations of $G^\vee$ into flat vector bundles on X

(ie flatly varying vectorspaces for $x \in X$)

$= \mathrm{Conn}_{G^\vee} X$ moduli of flat $G^\vee$ bundles: $\mathcal{L}$

$: V \longmapsto \mathcal{V}_\mathcal{L}$ associated vector bundle

$$W_{x,V}|_\mathcal{L} := \mathcal{V}_\mathcal{L}|_x$$

So our optimistic hope is that functions on Bun_G have multiplicity one (no degeneracy) over spectrum:

<u>Geometric Langlands Conjecture</u> There is an equivalence of categories

$$D(\mathrm{Bun}_G X, \mathcal{D}) \longleftrightarrow D(\mathrm{Conn}_{G^\vee} X, \mathcal{O})$$

intertwining Hecke actions $\underset{x\in X}{\otimes} \mathrm{Rep}\, G^\vee$

[really need more care because Conn is singular, & Bun is big.... needs some modification "along the fringes"]

Hecke eigensheaves: $\mathcal{L} \in \mathrm{Conn}_{G^\vee}$

$\mathcal{O}_{\mathcal{L}}$ skyscraper, $\mathrm{Aut}_{\mathcal{L}}$ corresponding $\mathcal{D}$-module

$$\Rightarrow \quad H_{x,V}\, \mathrm{Aut}_{\mathcal{L}} = \mathcal{L}_V|_x \otimes \mathrm{Aut}_{\mathcal{L}}$$

"eigenvector" for Hecke operators with eigenvalues determined by the $G^\vee$ connection $\mathcal{L}$.

— analog of character sheaf in abelian setting.

Geometric Satake

Idea of proof: LG_+ orbits are simply connected $\Rightarrow$ only trivial flat vector bundles on orbits
- orbits even relative dim $\Rightarrow$ no extensions of $\mathcal{D}$-modules [$\Rightarrow G^\vee$ reductive]
- Fiber functor: global cohomology.
- Mirkovic-Vilonen: canonical Λ-grading on this cohomology, labelled by fixed points $[\lambda] \in Gr$: "perverse cell decomposition" with one cell through each $[\lambda]$
  $\Rightarrow$ find dual torus $T^\vee \subset$ Tannakian group

. . .

Most interesting aspect: commutativity
Why is this convolution algebra commutative??

Drinfeld's answer: operator product expansion or fusion picture, coming from conformal field theory.

Example: $G = GL_1$, $Gr = \mathbb{Z}$. Why is $\mathbb{Z}$ abelian?
Addition of divisors: $\overset{5}{\bullet} \leftarrow \overset{-3}{\bullet} \Rightarrow \overset{2}{\bullet}$

$\frac{1}{z^5}(z-\lambda)^3 \Rightarrow \frac{1}{z^2}$

commutativity comes from colliding points in different orders : 

Recall $\mathcal{G}r_x = LG/LG_+ = G(K_x)/G(O_x)$
$=$ G- bundles with trivialization on $X \setminus x$.

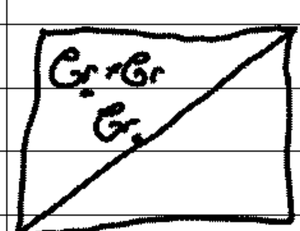
As x varies get $\mathcal{G}r$-bundle
$$\begin{tikzcd} \mathcal{G}r_x \arrow[d] \arrow[r, phantom, "\subset"] & \mathcal{G}r_X \arrow[d] \\ x \arrow[r, phantom, "\in"] & X \end{tikzcd}$$

Let
$$\begin{tikzcd} \mathcal{G}r_{X\times X} \arrow[d] \arrow[r, phantom, "\supset"] & \{\text{G-bundles on } X + \text{triv on } X \setminus \{x,y\}\} \arrow[d] \\ X \times X \arrow[r, phantom, "\ni"] & \{x,y\} \end{tikzcd}$$

<u>Claim</u> 1. $\mathcal{G}r_{X\times X}$ is a <u>fibration</u> (flat) over $X \times X$.

2. $\mathcal{G}r_{X\times X}|_{x\neq y} \simeq \mathcal{G}r_x \times \mathcal{G}r_y$ two copies

$\mathcal{G}r_{X\times X}|_{x=y} \simeq \mathcal{G}r_x$ one copy!



Analog of addition of divisors as points collide.

Fusion of sheaves: $\mathcal{F}, \mathcal{G}$ D-modules on Gr
put $\mathcal{F} \boxtimes \mathcal{G}$ on $Gr \times Gr$
$\Rightarrow$ on $Gr_{X \times X}|_{X \times X - \Delta}$. Now collide:
take limit as we approach diagonal
[Gaitsgory: nearby cycles]

$$\begin{array}{ccc} \mathcal{F}\cdot & & \cdot\mathcal{G} \\ & \searrow\ \swarrow & \\ & \cdot\,\mathcal{F} * \mathcal{G} & \end{array}$$

Claim This agrees with convolution!
$\Longrightarrow$ convolution is commutative.

Beilinson-Drinfeld: geometric theory of vertex algebras based on this picture.
— a vertex algebra is a vector bundle V on X & a vector bundle V_2 on $X \times X$
(quasicoherent sheaf flat along Δ)
which glues $V \boxtimes V$ off the diagonal to V on the diagonal (plus unit section)

- gluing law along Δ gives OPE of sections.
Linearizing Gr along basepoint gives
Kac-Moody vertex algebra — version of enveloping algebra of loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((z))$.